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ENERGIES OF HIGHER ORDER IN ADVANCED DYNAMICS OF SYSTEMS

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Abstract: *The dynamical study of the current and sudden motions of the rigid body and multibody systems (MBS), as example the mechanical robot structure, and in accordance with differential principles typical to analytical dynamics of systems, is based, among others, on the advanced notions, such as momentum and angular momentum, kinetic energy, acceleration energies of different orders and their absolute time derivatives of higher order. Advanced notions are developed in the direct connection with the generalized variables, also named independent parameters corresponding to holonomic mechanical systems. But, the expressions of definition of the advanced notions contain on the one hand kinematical parameters and their differential transformations, corresponding to absolute motions, on the other hand the mass properties. By means of the especially researches of the author, in this paper will be presented reformulations and formulations concerning the classical notions and theorems from dynamics. In the following the study will be extended on the energies of higher order. So, the expressions of definition in explicit form will be presented for the acceleration energy of first, second and third order. They are corresponding to the current and sudden motions of rigid body and multi body systems. These formulations will also contain the absolute time derivatives of higher order of the advanced notions, according to differential equations of higher order, typically to analytical dynamics of systems.*

Key words: *analytical dynamics, mechanics, advanced notions, dynamics equations, robotics.*

1. ADVANCED KINEMATICS NOTIONS

The dynamical study of current and sudden motions of the rigid body and multibody systems (MBS), example in Fig.1 the mechanical robot structure, and in accordance with differential principles typical to analytical dynamics of systems, is based on the advanced notions of dynamics: momentum and angular momentum, kinetic energy, acceleration energies of different orders and their absolute time derivatives of higher order, in accordance with [3] – [12].

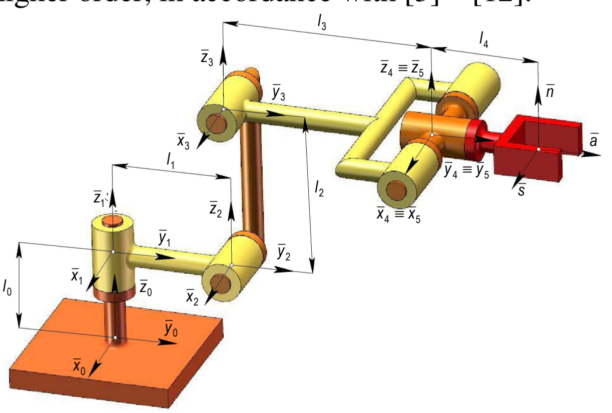

Fig.1 Robot Mechanical Structure (MBS)

Advanced notions are developed in the direct connection with the generalized variables. These are also named independent parameters (d.o.f.). These univocally characterize the absolute motions for any holonomic mechanical systems.

The dynamical study from this paper is oriented on mechanical structure with opened kinematical chain. So, the kinetic ensembles $i = 1 \rightarrow n$ are physically linked by driving joints of fifth order [4] – [6]. As example is considered mechanical structure of serial robot, see Fig.1. This is characterized by $(n\ d.o.f.)$, according to:

$$\bar{\theta} \neq \bar{\theta}^{(0)}; \quad \bar{\theta}(t) = \left[q_i(t); \quad i = 1 \rightarrow n\right]^T, \tag{1}$$

where $q_i(t)$ is the generalized coordinate from every driving axis. But, considering the current and sudden motions, the generalized variables of higher order are developed as follows:

$$\left\{ \begin{array}{l} \left\{ \bar{\theta}(t); \dot{\bar{\theta}}(t); \ddot{\bar{\theta}}(t); \cdots; \overset{(m)}{\bar{\theta}}(t) \right\} = \\ = \left\{ \begin{array}{c} q_i(t); \dot{q}_i(t); \ddot{q}_i(t); \cdots; \overset{(m)}{q_i}(t) \\ i = 1 \rightarrow n,\ m \geq 1 \end{array} \right\} \end{array} \right\}, \tag{2}$$

and (m) represents the time deriving order. The main objective of this section consists in the establishment of time variation law for (1) / (2).

Analyzing all above parameters of advanced kinematics, it results that they are functions of generalized variables (1) / (2), as well their time derivatives. So, according to author researches they can be developed as time functions, using polynomial interpolating functions [4] and [11]. It proposes following functions of higher order:

$$\left\{\begin{array}{l} \overset{(m-p)}{q}_{ji}(\tau)=(-1)^p\cdot\dfrac{(\tau_i-\tau)^{p+1}}{t_i\cdot(p+1)!}\cdot\overset{(m)}{q}_{ji-1}+ \\ +\dfrac{(\tau-\tau_{i-1})^{p+1}}{t_i\cdot(p+1)!}\cdot\overset{(m)}{q}_{ji}+\delta_p\cdot\displaystyle\sum_{k=1}^{p}\dfrac{\tau^{p-k}}{(p-k)!}\cdot a_{jik} \end{array}\right\};\quad(3)$$

$$\left\{\begin{array}{c} where\quad p=0\rightarrow m \\ m-deriving\ order,\ m\geq 2,\ m=2,3,4,5,\ldots \\ \delta_p=\{(0,\ p=0);(1;p\geq 1)\} \\ j=1\rightarrow n\quad degrees\ of\ freedom-(d.o.f.) \\ i=1\rightarrow s\quad intervals\ of\ motion\ trajectories \\ \tau-actual\ time\ variable \\ t_i=\tau_i-\tau_{i-1}\ (time\ to\ each\ trajectory\ interval) \end{array}\right\};\quad(4)$$

For every trajectory interval $(i=1\rightarrow s)$, number of unknowns is $(m+1)$, and their significance is:

$$\left\{\begin{array}{c} (a_{jik})\ for\ k=1\rightarrow m;\ and\ \left(\overset{(m)}{q}_{ji-1}\right)\ for\ i=2\rightarrow s \\ where\ (a_{jik})-integration\ constants,\ and \\ \left(\overset{(m)}{q}_{ji-1}\right)-generalized\ accelerations\ of\ (m)\ order \end{array}\right\}\quad(5)$$

The determination the unknowns (5) requires, in accordance with [4] – [11], the application of the geometrical and kinematical constraints as:

$$\left\{\begin{array}{l} (\tau_0)\Rightarrow\overset{(m-p)}{q}_{j0},\quad p=0\rightarrow m;\quad(\tau_s)\Rightarrow\left\{\overset{(m)}{q}_{js},\ q_{js}\right\} \\ (\tau_i)\Rightarrow\left\{\begin{array}{c} \overset{(2)}{q}_{ji}-generalized\ accelerations \\ \left\{\begin{array}{c}\overset{(m-p)}{q}_{ji}(\tau^+)=\overset{(m-p)}{q}_{ji+1}(\tau^-),\quad p=0\rightarrow m \\ continuity\ conditions\end{array}\right\} \\ all\ conditions\ are\ applied\ to\ each\ (\tau_i) \\ where\quad i=1\rightarrow s-1 \end{array}\right\} \end{array}\right\}\quad(6)$$

Finally, the results (3) will be substituted in the advanced notions of kinematics and dynamics.

2. CLASSICAL NOTIONS IN DYNAMICS

Fundamental theorems of classical dynamics are: motion theorem of the mass center, theorem of the angular momentum and theorem of the kinetic energy in differential and integral form. These are known, in accordance with scientific literature, for example [4] and [5]. The objective of this section consists in a few reformulations of the fundamental notions and theorems, in consonance with general motion (Fig.2 / Fig.3).

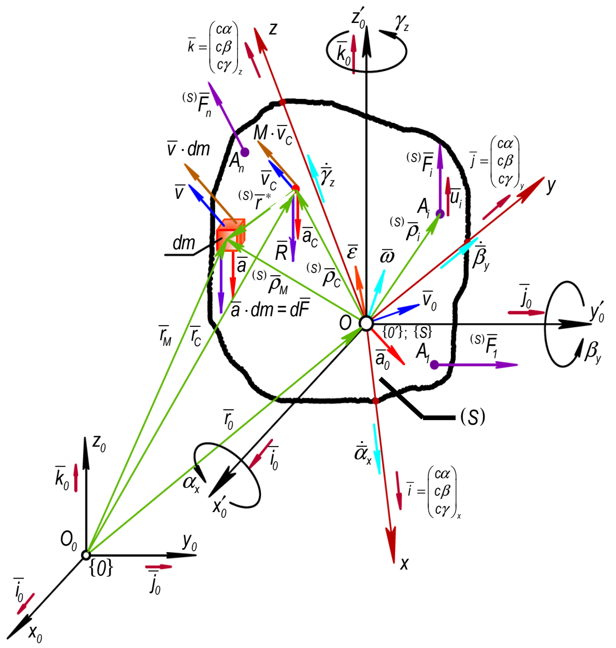

Fig.2 Rigid Body Free in Cartesian Frame

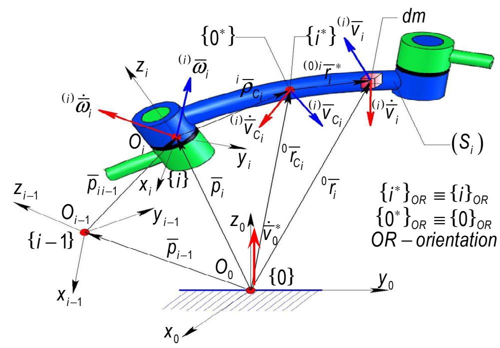

Fig. 3 Kinetic Ensemble from MBS

Every fundamental notion and theorem will be defined for rigid body, and then for multibody systems. In the view of this input parameters of advanced kinematics are applied [3] – [12].

- At beginning, the momentum and theorem of the momentum is developed. The definition equation, according to [4] and [5], is written as:

$$\left\{\begin{array}{l} \bar{H}=\int d\bar{H}=\int\bar{v}_M\cdot dm=\int(\bar{v}_0+\bar{\omega}\times\bar{\rho}_M)\cdot dm \\ \quad=\bar{v}_0\cdot\int dm+\bar{\omega}\times\int\bar{\rho}_M\cdot dm= \\ \quad=M\cdot(\bar{v}_0+\bar{\omega}\times\bar{\rho}_C)=M\cdot\bar{v}_C \end{array}\right\};\quad(7)$$

First time derivative of momentum is defined:

$$\frac{d}{dt}(d\bar{H})=\frac{d}{dt}(\bar{v}_M\cdot dm)=\bar{a}_M\cdot dm;\quad(8)$$

$$\frac{d}{dt}\left(d\overline{H}\right)=d\left(\frac{d\overline{H}}{dt}\right)=d\dot{\overline{H}}\cdot \quad (9)$$

Applying the time integral on (8) / (9) it obtains:

$$\dot{\overline{H}}=\int d\dot{\overline{H}}=\int \overline{a}_M\cdot dm=\int d\overline{F}=\overline{R}\,. \quad (10)$$

This is named theorem of momentum, where $\overline{R}$ represents resultant vector of the active forces. Substituting $\overline{a}_M$ in (10), according to [12], and performing the mass integral, time derivative of the momentum becomes the next expression:

$$\dot{\overline{H}}=\int \overline{a}_M\cdot dm=\int\left(\overline{a}_0+\overline{\varepsilon}\times\overline{\rho}_M+\overline{\omega}\times\overline{\omega}\times\overline{\rho}_M\right)\cdot dm=$$

$$\left\{\begin{matrix}=\overline{a}_0\cdot\int dm+\overline{\varepsilon}\times\int\overline{\rho}_M\cdot dm+\overline{\omega}\times\overline{\omega}\times\int\overline{\rho}_M\cdot dm\\ =M\cdot\left(\overline{a}_0+\overline{\varepsilon}\times\overline{\rho}_C+\overline{\omega}\times\overline{\omega}\times\overline{\rho}_C\right)=M\cdot\overline{a}_C\end{matrix}\right\} \quad (11)$$

As result, *the motion theorem of the mass center* is characterized by means of the next equation:

$$M\cdot\overline{a}_C=M\cdot\dot{\overline{v}}_C=M\cdot\ddot{\overline{r}}_C=\overline{R}\,. \quad (12)$$

For the multibody systems, *momentum* (7) and *the motion theorem of the mass center* (12) are changed according to following equations:

$$\overline{H}_i=M_i\cdot\overline{v}_{C_i}=M_i\cdot\dot{\overline{r}}_{C_i}\,; \quad (13)$$

$$M_i\cdot\overline{a}_{C_i}=M_i\cdot\dot{\overline{v}}_{C_i}=M_i\cdot\ddot{\overline{r}}_{C_i}=\overline{F}_i^*\,; \quad (14)$$

where $\overline{F}_i^*$ is the resultant vector of active forces applied on the kinetic ensemble (i), see Fig.3. Substituting linear acceleration of mass center with expression from [8] – [12], theorem (14) is:

$$M_i\cdot\sum_{j=1}^{k^*=n}\left[\frac{\partial \overset{(m)}{\overline{r}_{C_i}}}{\partial \overset{(m)}{q_j}}\cdot\ddot{q}_j+\frac{1}{m+1}\cdot\frac{\partial \overset{(m+1)}{\overline{r}_{C_i}}}{\partial \overset{(m)}{q_j}}\cdot\dot{q}_j\right]=\overline{F}_i^*\,. \quad (15)$$

- Next classical notion is angular momentum. In accordance with [5], the starting equation is:

$$\overline{K}_O(t)=\int\overline{r}_M\times\overline{v}_M\cdot dm= \quad (16)$$

$$=\int\left[\overline{r}_0(t)+\overline{\rho}_M(t)\right]\times\left[\overline{v}_0(t)+\overline{\omega}(t)\times\overline{\rho}_M(t)\right]\cdot dm\,.$$

After a few transformations, on mass integral, the fourth terms from (16) are written as follows:

$$\left\{\begin{matrix}\overline{r}_0(t)\times M\cdot\overline{v}_0(t)+\overline{r}_0(t)\times M\cdot\overline{\omega}(t)\times\overline{\rho}_C(t)=\\ =\overline{r}_0(t)\times M\cdot\left[\overline{v}_0(t)+\overline{\omega}(t)\times\overline{\rho}_C(t)\right]=\\ =\overline{r}_0(t)\times M\cdot\overline{v}_C(t)\end{matrix}\right\}; \quad (17)$$

$$\left\{\begin{matrix}\int\overline{\rho}_M(t)\times\overline{v}_0(t)\cdot dm=\int\overline{\rho}_M(t)\cdot dm\times\overline{v}_0(t)=\\ =\overline{\rho}_C(t)\times M\cdot\overline{v}_0(t)\end{matrix}\right\}; \quad (18)$$

$$\left\{\begin{matrix}\int\overline{\rho}_M(t)\times\overline{\omega}(t)\times\overline{\rho}_M(t)\cdot dm=\\ =\left[\int(\overline{\rho}_M\times)\cdot(\overline{\rho}_M\times)^T\cdot dm\right]\cdot\overline{\omega}(t)=\end{matrix}\right\};$$

$$\left\{\begin{matrix}\int\overline{\rho}_M(t)\times\overline{\omega}(t)\times\overline{\rho}_M(t)\cdot dm=\\ =I'_S(t)\cdot\overline{\omega}(t)={}^0_S[R](t)\cdot{}^SI_S\cdot{}^0_S[R]^T(t)\cdot\overline{\omega}(t)\end{matrix}\right\}; \quad (19)$$

$$I'_S=\int(\overline{\rho}_M\times)\cdot(\overline{\rho}_M\times)^T\cdot dm={}^0_S[R](t)\cdot{}^SI_S\cdot{}^0_S[R]^T(t)\,; \quad (20)$$

where (20) is inertial tensor axial and centrifugal of the body (S), relative to frame $\{0'\}$ (Fig.2). Substituting (17) – (19) in (16), the definition equation of the angular momentum is obtained:

$$\left\{\begin{matrix}\overline{K}_O(t)=\overline{r}_0(t)\times M\cdot\overline{v}_C(t)+\\ +\overline{\rho}_C(t)\times M\cdot\overline{v}_0(t)+I'_S(t)\cdot\overline{\omega}(t)=\\ =\overline{r}_0(t)\times M\cdot\overline{v}_C(t)+\overline{\rho}_C(t)\times M\cdot\overline{v}_0(t)+\\ +{}^0_S[R](t)\cdot{}^SI_S\cdot{}^0_S[R]^T(t)\cdot\overline{\omega}(t)\end{matrix}\right\}. \quad (21)$$

On the equation (16) the first time derivative is applied. After a few transformations it obtains:

$$\left\{\begin{matrix}\frac{d}{dt}\left(d\overline{K}_O\right)=d\left(\frac{d\overline{K}_O}{dt}\right)=d\dot{\overline{K}}_O=\\ =\frac{d}{dt}\left(\overline{r}_M\times\overline{v}_M\cdot dm\right)=\overline{r}_M\times\overline{a}_M\cdot dm\end{matrix}\right\}; \quad (22)$$

$$\dot{\overline{K}}_O=\int d\dot{\overline{K}}_O=\int\overline{r}_M\times\overline{a}_M\cdot dm=\overline{M}_O\,; \quad (23)$$

where (23) is theorem of the angular momentum. The time derivative of the angular momentum is developed in the following. Substituting $\overline{a}_M$ in (23), according to [5], the starting equation is:

$$\left\{\begin{matrix}\dot{\overline{K}}_O=\int\overline{r}_M\times\overline{a}_M\cdot dm=\\ \int\left(\overline{r}_0+\overline{\rho}_M\right)\times\left(\overline{a}_0+\overline{\varepsilon}\times\overline{\rho}_M+\overline{\omega}\times\overline{\omega}\times\overline{\rho}_M\right)\cdot dm\end{matrix}\right\} \quad (24)$$

Developing the integrand and applying the mass integral, the fourth terms from (24) become:

$$\left\{\begin{matrix}\overline{r}_0\times M\cdot\overline{a}_0+\overline{r}_0\times M\cdot\overline{\varepsilon}\times\overline{\rho}_C+\overline{r}_0\times M\cdot\overline{\omega}\times\overline{\omega}\times\overline{\rho}_C\\ =\overline{r}_0\times M\cdot\left(\overline{a}_0+\overline{\varepsilon}\times\overline{\rho}_C+\overline{\omega}\times\overline{\omega}\times\overline{\rho}_C\right)=\overline{r}_0\times M\cdot\overline{a}_C\end{matrix}\right\} \quad (25)$$

$$\int\overline{\rho}_M\times\overline{a}_0\cdot dm=\int\overline{\rho}_M\cdot dm\times\overline{a}_0=\overline{\rho}_C\times M\cdot\overline{a}_0\,. \quad (26)$$

The fifth term from (24) is below developed as:

$$\left\{\begin{matrix}\int\overline{\rho}_M\times\overline{\varepsilon}\times\overline{\rho}_M\cdot dm=\\ =\left[\int(\overline{\rho}_M\times)\cdot(\overline{\rho}_M\times)^T\cdot dm\right]\cdot\overline{\varepsilon}=\\ =I'_S(t)\cdot\overline{\varepsilon}(t)={}^0_S[R](t)\cdot{}^SI_S\cdot{}^0_S[R]^T(t)\cdot\overline{\varepsilon}(t)\end{matrix}\right\} \quad (27)$$

On the last term from (24) are performed a few matrix transformations and mass integrals, thus:

$$\left\{\begin{matrix}\int\overline{\rho}_M\times\overline{\omega}\times\overline{\omega}\times\overline{\rho}_M\cdot dm=\\ =\int\overline{\rho}_M\times\left[\left(\overline{\omega}^T\cdot\overline{\rho}_M\right)\cdot\overline{\omega}-\omega^2\cdot\overline{\rho}_M\right]=\\ =\int\overline{\rho}_M\times\left(\overline{\omega}^T\cdot\overline{\rho}_M\right)\cdot\overline{\omega}-\int\overline{\rho}_M\times\omega^2\cdot\overline{\rho}_M=\\ =\int\overline{\rho}_M\times\left(\overline{\omega}^T\cdot\overline{\rho}_M\right)\cdot\overline{\omega}\end{matrix}\right\}; \quad (28)$$

$$\int \overline{\rho}_M \times \left(\overline{\omega}^T \cdot \overline{\rho}_M\right) \cdot \overline{\omega} = -\int \overline{\omega} \times \overline{\rho}_M \cdot \overline{\rho}_M^T \cdot \overline{\omega}; \quad (29)$$

$$\overline{\rho}_M \cdot \overline{\rho}_M^T = \overline{\rho}_M^T \cdot \overline{\rho}_M \cdot I_3 - \left(\overline{\rho}_M \times\right) \cdot \left(\overline{\rho}_M \times\right)^T; \quad (30)$$

$$\begin{Bmatrix} -\int \overline{\omega} \times \overline{\rho}_M \cdot \overline{\rho}_M^T \cdot \overline{\omega} = -\int \overline{\omega} \times \overline{\rho}_M^T \cdot \overline{\rho}_M \cdot \overline{\omega} + \\ + \overline{\omega} \times \int \left(\overline{\rho}_M \times\right) \cdot \left(\overline{\rho}_M \times\right)^T \cdot dm \cdot \overline{\omega} = \\ = \overline{\omega} \times \int \left(\overline{\rho}_M \times\right) \cdot \left(\overline{\rho}_M \times\right)^T \cdot dm \cdot \overline{\omega} \end{Bmatrix}. \quad (31)$$

$$\begin{Bmatrix} \int \overline{\rho}_M \times \overline{\omega} \times \overline{\omega} \times \overline{\rho}_M \cdot dm = \\ = \overline{\omega} \times \int \left(\overline{\rho}_M \times\right) \cdot \left(\overline{\rho}_M \times\right)^T \cdot dm \cdot \overline{\omega} = \\ = \overline{\omega} \times I'_S(t) \cdot \overline{\omega}(t) \\ = \overline{\omega}(t) \times [R](t) \cdot {}^S I_S \cdot {}^0_S[R]^T(t) \cdot \overline{\omega}(t) \end{Bmatrix}. \quad (32)$$

Substituting (25), (26), (27) and (32) in (24), the definition equation of first time derivative for angular momentum is obtained, according to:

$$\begin{Bmatrix} \int \left(\overline{r}_0 + \overline{\rho}_M\right) \times \left(\overline{a}_0 + \overline{\varepsilon} \times \overline{\rho}_M + \overline{\omega} \times \overline{\omega} \times \overline{\rho}_M\right) \cdot dm \\ = \overline{r}_0 \times M \cdot \overline{a}_C + \overline{\rho}_C \times M \cdot \overline{a}_0 + \\ = I'_S(t) \cdot \overline{\varepsilon}(t) + \overline{\omega} \times I'_S(t) \cdot \overline{\omega}(t) = \\ = \overline{r}_0 \times M \cdot \overline{a}_C + \overline{\rho}_C \times M \cdot \overline{a}_0 + \\ + {}^0_S[R](t) \cdot {}^S I_S \cdot {}^0_S[R]^T(t) \cdot \overline{\varepsilon}(t) + \\ + \overline{\omega}(t) \times [R](t) \cdot {}^S I_S \cdot {}^0_S[R]^T(t) \cdot \overline{\omega}(t) = \dot{\overline{K}}_O \end{Bmatrix}. (33)$$

The equation (33) can be obtained, applying the time derivative on (21). The starting equation is:

$$\begin{Bmatrix} \dot{\overline{K}}_O(t) = \frac{d}{dt}\left[\overline{r}_0(t) \times M \cdot \overline{v}_C(t) + \right. \\ \left. + \overline{\rho}_C(t) \times M \cdot \overline{v}_0(t) + I'_S(t) \cdot \overline{\omega}(t)\right] = \\ = \frac{d}{dt}\left[\overline{r}_0(t) \times M \cdot \overline{v}_C(t) + \overline{\rho}_C(t) \times M \cdot \overline{v}_0(t) + \right. \\ \left. + {}^0_S[R](t) \cdot {}^S I_S \cdot {}^0_S[R]^T(t) \cdot \overline{\omega}(t)\right] \end{Bmatrix}. \quad (34)$$

After application the time derivative and performing a few transformations, the first two terms from (34) take the following expressions:

$$\begin{Bmatrix} \frac{d}{dt}\left[\overline{r}_0(t) \times M \cdot \overline{v}_C(t)\right] = \\ = \overline{v}_0(t) \times M \cdot \overline{v}_C(t) + \overline{r}_0(t) \times M \cdot \overline{a}_C(t) \end{Bmatrix}; \quad (35)$$

$$\frac{d}{dt}\left[\overline{\rho}_C(t) \times M \cdot \overline{v}_0(t)\right] = \quad (36)$$

$$\begin{Bmatrix} = \dot{\overline{\rho}}_C(t) \times M \cdot \overline{v}_0(t) + \overline{\rho}_C(t) \times M \cdot \overline{a}_0(t) = \\ = \left[\overline{\omega}(t) \times \overline{\rho}_C(t)\right] \times M \cdot \overline{v}_0(t) + \overline{\rho}_C(t) \times M \cdot \overline{a}_0(t) \end{Bmatrix};$$

$$\begin{Bmatrix} \overline{v}_0(t) \times M \cdot \overline{v}_C(t) + \left[\overline{\omega}(t) \times \overline{\rho}_C(t)\right] \times M \cdot \overline{v}_0(t) = \\ = \overline{v}_0(t) \times M \cdot \left[\overline{v}_C(t) - \overline{\omega}(t) \times \overline{\rho}_C(t)\right] = \end{Bmatrix}$$

$$= \overline{v}_0(t) \times M \cdot \overline{v}_0(t) = 0; \quad (37)$$

$$\begin{Bmatrix} \frac{d}{dt}\left[\overline{r}_0(t) \times M \cdot \overline{v}_C(t) + \overline{\rho}_C(t) \times M \cdot \overline{v}_0(t)\right] = \\ = \overline{r}_0(t) \times M \cdot \overline{a}_C(t) + \overline{\rho}_C(t) \times M \cdot \overline{a}_0(t) \end{Bmatrix}. \quad (38)$$

The last term from (34) is below developed as:

$$\begin{Bmatrix} \frac{d}{dt}\left[{}^0_S[R](t) \cdot {}^S I_S \cdot {}^0_S[R]^T(t) \cdot \overline{\omega}(t)\right] = \\ = \frac{d}{dt}\left\{{}^0_S[R](t)\right\} \cdot {}^S I_S \cdot {}^0_S[R]^T(t) \cdot \overline{\omega}(t) + \\ + {}^0_S[R](t) \cdot {}^S I_S \cdot \frac{d}{dt}\left\{{}^0_S[R]^T(t) \cdot \overline{\omega}(t)\right\} \end{Bmatrix}. \quad (39)$$

Considering matrix transformations and using differential properties, according to [8] – [12], the time derivatives from (39) are changed thus:

$$\begin{Bmatrix} \frac{d}{dt}\left\{{}^0_S[R](t)\right\} \cdot {}^S I_S \cdot {}^0_S[R]^T(t) \cdot \overline{\omega}(t) = \\ \left\{{}^0_S\left[\dot{R}\right](t) \cdot {}^0_S[R]^T(t)\right\} \cdot {}^0_S[R](t) \cdot {}^S I_S \cdot {}^0_S[R]^T(t) \cdot \overline{\omega}(t) \end{Bmatrix}$$

$$= \overline{\omega}(t) \times I'_S(t) \cdot \overline{\omega}(t); \quad (40)$$

$$\frac{d}{dt}\left\{{}^0_S[R]^T(t) \cdot \overline{\omega}(t)\right\} =$$

$$\left\{ = \frac{d\,{}^S\overline{\omega}(t)}{dt} = \frac{\partial\,{}^S\overline{\omega}(t)}{\partial t} = {}^S\overline{\varepsilon}(t) \equiv {}^0_S[R]^T(t) \cdot \overline{\varepsilon}(t) \right\};$$

$$\begin{Bmatrix} {}^0_S[R](t) \cdot {}^S I_S \cdot \frac{d}{dt}\left\{{}^0_S[R]^T(t) \cdot \overline{\omega}(t)\right\} = \\ {}^0_S[R](t) \cdot {}^S I_S \cdot {}^0_S[R]^T(t) \cdot \overline{\varepsilon}(t) = I'_S(t) \cdot \overline{\varepsilon}(t) \end{Bmatrix}; \quad (41)$$

So, the last term from (34) takes the final form:

$$\begin{Bmatrix} \frac{d}{dt}\left[I'_S(t) \cdot \overline{\omega}(t)\right] = \\ = \frac{d}{dt}\left[{}^0_S[R](t) \cdot {}^S I_S \cdot {}^0_S[R]^T(t) \cdot \overline{\omega}(t)\right] = \end{Bmatrix}$$

$$= I'_S(t) \cdot \overline{\varepsilon}(t) + \overline{\omega}(t) \times I'_S(t) \cdot \overline{\omega}(t). \quad (42)$$

Substituting (35) – (42) in (34), (33) is obtained. Usually, the mobile frame has the origin in mass center (Fig. 2). The particularities are following:

$$O = C;\ \overline{\rho}_C = 0;\ \overline{r}_0 = \overline{r}_C;\ \overline{v}_0 = \overline{v}_C;\ \overline{a}_0 = \overline{a}_C;\ I'_S = I^*_S \quad (43)$$

Angular momentum and its time derivative are:

$$\overline{K}_C(t) = \overline{r}_C(t) \times M \cdot \overline{v}_C(t) + I^*_S(t) \cdot \overline{\omega}(t) = \quad (44)$$

$$\left\{ = \overline{r}_C(t) \times M \cdot \overline{v}_C(t) + {}^0_S[R](t) \cdot {}^S I^*_S \cdot {}^0_S[R]^T(t) \cdot \overline{\omega}(t) \right\};$$

$$\dot{\overline{K}}_O = \overline{r}_C \times M \cdot \overline{a}_C + I^*_S(t) \cdot \overline{\varepsilon}(t) + \overline{\omega} \times I^*_S(t) \cdot \overline{\omega}(t) =$$

$$\begin{Bmatrix} = \overline{r}_C \times M \cdot \overline{a}_C + {}^0_S[R](t) \cdot {}^S I^*_S \cdot {}^0_S[R]^T(t) \cdot \overline{\varepsilon}(t) + \\ + \overline{\omega}(t) \times [R](t) \cdot {}^S I^*_S \cdot {}^0_S[R]^T(t) \cdot \overline{\omega}(t) \end{Bmatrix} (45)$$

Considering (12) and the variation of resultant moment of active forces [4], (45) is changed as:

$$\overline{M}_O - \overline{r}_C \times M \cdot \overline{a}_C = \overline{M}_O - \overline{r}_C \times \overline{R} = \overline{M}_C ; \qquad (46)$$

$$I_S^*(t) \cdot \overline{\varepsilon}(t) + \overline{\omega} \times I_S^*(t) \cdot \overline{\omega}(t) = \overline{M}_C ; \qquad (47)$$

where (47) is named *the theorem of the angular momentum with respect to mass center* [4] – [5].

In the case of the multibody systems, *angular momentum* (44) and *the theorem of the angular momentum* (47) are changed, in accordance with [3] – [9]. So, the following equations are obtained:

$$\left\{\begin{array}{c} \overline{K}_i^*(t) = \overline{r}_{C_i}(t) \times M \cdot \overline{v}_{C_i}(t) + I_i^*(t) \cdot \overline{\omega}_i(t) = \\ = \overline{r}_{C_i} \times M \cdot \overline{v}_{C_i}(t) + {}^0_i[R](t) \cdot {}^iI_i^* \cdot {}^0_i[R]^T(t) \cdot \overline{\omega}_i(t) \end{array}\right\};(48)$$

$$\left\{\begin{array}{c} I_i^*(t) \cdot \overline{\varepsilon}_i(t) + \frac{d}{dt}\left(I_i^*\right) \cdot \overline{\omega}_i(t) = \\ I_i^*(t) \cdot \overline{\varepsilon}_i(t) + \overline{\omega}_i(t) \times I_i^*(t) \cdot \overline{\omega}_i(t) = \overline{N}_i^* \end{array}\right\}. \quad (49)$$

Substituting angular velocity and acceleration, according to [12], the theorem (49) is changed:

$$\left\{\begin{array}{c} I_i^* \cdot \sum_{j=1}^{k^*=n}\left[\frac{\partial \overset{(m)}{\overline{\psi}_i}}{\partial \overset{(m)}{q_j}} \cdot \Delta_j \cdot \ddot{q}_j + \frac{1}{m+1} \cdot \frac{\partial \overset{(m+1)}{\overline{\psi}_i}}{\partial \overset{(m)}{q_j}} \cdot \Delta_j \cdot \dot{q}_j\right] + \\ \sum_{j=1}^{k^*=n}\sum_{p=1}^{k^*=n}\left[\frac{\partial \overset{(m)}{\overline{\psi}_i}}{\partial \overset{(m)}{q_j}}\right] \times \left[I_i^* \cdot \frac{\partial \overset{(m)}{\overline{\psi}_i}}{\partial \overset{(m)}{q_p}}\right] \cdot \Delta_j \cdot \Delta_p \cdot \dot{q}_j \cdot \dot{q}_p = \overline{N}_i^* \end{array}\right\} (50)$$

$${}^iI_i^* = \int\left[{}^i\overline{r}_i^* \times\right] \cdot \left[{}^i\overline{r}_i^* \times\right]^T \cdot dm = \begin{bmatrix} {}^iI_x^* & -{}^iI_{xy}^* & -{}^iI_{xz}^* \\ -{}^iI_{yx}^* & {}^iI_y^* & -{}^iI_{yz}^* \\ -{}^iI_{zx}^* & -{}^iI_{zy}^* & {}^iI_z^* \end{bmatrix} (51)$$

$$where\ \Delta_{j(p)} = \left\{\left(0,\ q_{j(p)} \in \overline{r}_{C_i}\right); \left(1,\ q_{j(p)} \in \overline{\psi}_i\right)\right\}, (52)$$

and $\overline{N}_i^*$ is resultant moment of active forces, I_i^* is inertia tensor axial and centrifugal of the kinetic ensemble(i), the both terms are relative to $\{i^*\}$ whose origin is the mass center (see Fig.3).

- For understanding mechanical significances of the energies of higher order, at beginning *the kinetic energy* is defined, according to [4] – [5]. Similarly with above notions, first of all is taken in study the rigid body in general motion (Fig.2). The starting equations of the kinetic energy are:

$$\left\{\begin{array}{c} E_C = \frac{1}{2} \cdot \int v_M^2 \cdot dm = \frac{1}{2} \cdot \int \overline{v}_M^T \cdot \overline{v}_M \cdot dm = \\ = \frac{1}{2} \cdot \int Trace\left[\overline{v}_M \cdot \overline{v}_M^T\right] \cdot dm \end{array}\right\}. \qquad (53)$$

$$E_C = \frac{1}{2} \cdot \int \left(\overline{v}_0 + \overline{\omega} \times \overline{\rho}_M\right)^T \cdot \left(\overline{v}_0 + \overline{\omega} \times \overline{\rho}_M\right) \cdot dm . \quad (54)$$

On the equation (54) are performed a few matrix transformations. They are below presented thus:

$$\frac{1}{2} \cdot \int \overline{v}_0^T \cdot \overline{v}_0 \cdot dm = \frac{1}{2} \cdot M \cdot \overline{v}_0^T \cdot \overline{v}_0 = \frac{1}{2} \cdot M \cdot v_0^2 ; \quad (55)$$

$$\left\{\begin{array}{c} \frac{1}{2} \cdot \int \overline{v}_0^T \cdot \left(\overline{\omega} \times \overline{\rho}_M\right) \cdot dm = \frac{1}{2} \cdot \int \left(\overline{\omega} \times \overline{\rho}_M\right)^T \cdot \overline{v}_0 \cdot dm \\ = \frac{1}{2} \cdot \overline{v}_0^T \cdot M \cdot \left(\overline{\omega} \times \overline{\rho}_C\right) \end{array}\right\} (56)$$

$$\left\{\begin{array}{c} \frac{1}{2} \cdot \int \left(\overline{\omega} \times \overline{\rho}_M\right)^T \cdot \left(\overline{\omega} \times \overline{\rho}_M\right) \cdot dm = \\ = \frac{1}{2} \cdot \overline{\omega}^T \cdot \left\{\int \left(\overline{\rho}_M \times\right) \cdot \left(\overline{\rho}_M \times\right)^T \cdot dm\right\} \cdot \overline{\omega} = \\ = \frac{1}{2} \cdot \overline{\omega}^T \cdot I_S' \cdot \overline{\omega} \end{array}\right\}. \quad (57)$$

Substituting (55) – (57) in (54), the definition equation of kinetic energy for general motion is:

$$E_C = \frac{1}{2} \cdot M \cdot v_0^2 + M \cdot \overline{v}_0^T \cdot \left(\overline{\omega} \times \overline{\rho}_C\right) + \frac{1}{2} \cdot \overline{\omega}^T \cdot I_S' \cdot \overline{\omega} . \ (58)$$

Applying the particularities (43), (58) is changed:

$$E_C = \frac{1}{2} \cdot M \cdot v_C^2 + \frac{1}{2} \cdot \overline{\omega}^T \cdot I_S^* \cdot \overline{\omega} . \qquad (59)$$

This is known as König's theorem of the kinetic energy under the explicit form, as well devoted to general motion. In the case of the multibody systems the above theorem is modified as [7]:

$$\left\{\begin{array}{c} (-1)^{\Delta_M} \cdot \frac{1-\Delta_M}{1+3\cdot\Delta_M} \cdot \left\{\frac{1}{2} \cdot M_i \cdot {}^i\overline{v}_{C_i}^T \cdot {}^i\overline{v}_{C_i}\right\} + \\ + \Delta_M^2 \cdot \frac{1}{2} \cdot {}^i\overline{\omega}_i^T \cdot {}^iI_i^* \cdot {}^i\overline{\omega}_i = E_C^i\left[\overline{\theta}(t); \dot{\overline{\theta}}(t)\right] \end{array}\right\}. (60)$$

To this, the operator is added with significance:

$\Delta_M = \{(-1; general\ motion); (0; translation); (1; rotation)\}$

Considering the notions from others papers of the author [4] – [11], the total kinetic energy of MBS is written by means of the components as follows:

$$\left\{\begin{array}{c} E_C\left[\overline{\theta}(t); \dot{\overline{\theta}}(t)\right] = \\ = \sum_{i=1}^{n} E_C^{iTR}\left[\overline{\theta}(t); \dot{\overline{\theta}}(t)\right] + \sum_{i=1}^{n} E_C^{iROT}\left[\overline{\theta}(t); \dot{\overline{\theta}}(t)\right] \end{array}\right\}; \ (61)$$

The translational and rotation components are changed due to substitution of linear and angular velocities, in accordance with [9]. They become:

$$\left\{\begin{array}{c} \sum_{i=1}^{n} E_C^{iTR}\left[\overline{\theta}(t); \dot{\overline{\theta}}(t)\right] = \\ (-1)^{\Delta_M} \cdot \frac{1-\Delta_M}{1+3\cdot\Delta_M} \cdot \frac{1}{2} \cdot \sum_{i=1}^{n} M_i \cdot \sum_{j=1}^{k^*=n} \frac{1}{m+1} \cdot \frac{\partial \overset{(m+1)}{\overline{r}_{C_i}}}{\partial \overset{(m)}{q_j}} \cdot \dot{q}_j \end{array}\right\} (62)$$

$$\left\{\begin{array}{c}\sum_{i=1}^{n} E_C^{iROT}\left[\bar{\theta}(t);\dot{\bar{\theta}}(t)\right]= \\ \frac{\Delta_M^2}{2}\cdot\sum_{i=1}^{n}\left[\sum_{j=1}^{k^*=n}\frac{\partial\overset{(m)}{\bar{\psi}_i}}{\partial\overset{(m)}{q_j}}\cdot\Delta_j\cdot\dot{q}_j\right]\cdot I_S^*\cdot\left[\sum_{p=1}^{k^*=n}\frac{\partial\overset{(m)}{\bar{\psi}_i}}{\partial\overset{(m)}{q_p}}\cdot\Delta_p\cdot\dot{q}_p\right]\end{array}\right\} \quad (63)$$

In the dynamic equations of higher order, the kinetic energy is included by means of the time derivative of higher order. It shows as follows:

$$\left\{\overset{(k)}{E_C^{iTR}}=\frac{1}{2}\cdot M_i\cdot\sum_{j=1}^{k^*=n}\frac{d^k}{dt^k}\left[\frac{1}{m+1}\cdot\frac{\partial\overset{(m+1)}{\bar{r}_{C_i}}}{\partial\overset{(m)}{q_j}}\cdot\dot{q}_j\right]\right\}, \quad (64)$$

$$\left\{\begin{array}{c}X=\{u;v\};\ v\neq u \\ u=\left\{\frac{1}{m+1}\cdot\frac{\partial\overset{(m+1)}{\bar{r}_{C_i}}}{\partial\overset{(m)}{q_j}};\dot{q}_j\right\};\ v=\left\{\dot{q}_j;\frac{1}{m+1}\cdot\frac{\partial\overset{(m+1)}{\bar{r}_{C_i}}}{\partial\overset{(m)}{q_j}}\right\}\end{array}\right\},$$

$$\left\{\begin{array}{c}\frac{d^k}{dt^k}\left[\frac{1}{m+1}\cdot\frac{\partial\overset{(m+1)}{\bar{r}_{C_i}}}{\partial\overset{(m)}{q_j}}\cdot\dot{q}_j\right]= \\ =\sum_{\{u;v\}}\left\{\overset{(k)}{u}\cdot v\right\}+\frac{k}{0!}\cdot\sum_{\{u;v\}}\left\{\overset{(k-1)}{u}\cdot\dot{v}\right\}+ \\ +\left(k-\Delta_k\right)\cdot\left[k-\left(j+1-\Delta_k\right)\right]\cdot\sum_{\{u;v\}}\left\{\overset{(k-2)}{u}\cdot\ddot{v}\right\}\cdot\delta_k \\ +k\cdot\left[k-\left(2-\Delta_k\right)\right]\cdot\sum_{\{u;v\}}\left\{\overset{(k-2)}{u}\cdot\overset{(3)}{v}\right\}\cdot\delta_{kk} \\ +(k-1)\cdot\left[k-2\cdot(1-\delta_{kk})\right]\cdot\sum_{\{u;v\}}\left\{\overset{(k-j)}{u}\cdot\overset{(k-j)}{v}\right\}\cdot\Delta_k\end{array}\right\}, \quad (65)$$

$$\left\{\begin{array}{c}\text{where } 1\leq k\leq 8;\ \text{for } j\geq k\Rightarrow\overset{(k-j)}{u}=0;\ \overset{(k-j)}{v}=0 \\ \Delta_k=\left\{(1;\ k=2\cdot j);\ (0;\ k\neq 2\cdot j),\ \text{and } j=1,2,3,4,\ldots\right\} \\ \delta_{k(k)}=\left\{\left(0;\ k\leq 4(6)\right);\ \left(1;\ k\geq 4(6)\right)\right\}\end{array}\right\};$$

$$\overset{(k)}{E_C^{iROT}}=\frac{1}{2}\cdot\frac{d^k}{dt^k}\left\{\left[\sum_{j=1}^{k^*=n}\frac{\partial\overset{(m)}{\bar{\psi}_i}}{\partial\overset{(m)}{q_j}}\cdot\Delta_j\cdot\dot{q}_j\right]\cdot I_i^*\cdot\left[\sum_{p=1}^{k^*=n}\frac{\partial\overset{(m)}{\bar{\psi}_i}}{\partial\overset{(m)}{q_p}}\cdot\Delta_p\cdot\dot{q}_p\right]\right\}$$

$$\left\{\begin{array}{c}\frac{1}{2}\cdot\left\{\sum_{\{u;v;w\}}\left\{\overset{(k)}{u}\cdot v\cdot w\right\}+\frac{k}{(4-k)!}\cdot\sum_{\{u;v;w\}}\left\{\overset{(k-1)}{u}\cdot(\dot{v}\cdot w+v\cdot\dot{w})\right\}\right\} \\ +\frac{1}{2}\cdot\frac{k!}{4\cdot(k-4)!}\cdot\sum_{\{u;v;w\}}\left\{\overset{(k-2)}{u}\cdot\overset{(k-2)}{v}\cdot w\right\}+\end{array}\right\}$$

$$+\frac{1}{2}\cdot\frac{k!}{(6-k)\cdot(k-3)!}\cdot\sum_{\{u;v;w\}}\left\{\overset{(k-2)}{u}\cdot\dot{v}\cdot\dot{w}\right\}=\overset{(k)}{E_C^{iROT}}; \quad (66)$$

where $k\leq 4$, and position of terms from (66) is:

$$\left\{\begin{array}{c}\left\{\begin{array}{c}X=\{u;v;w\} \\ u=\{a;b;c\} \\ v=\{b;c;a\} \\ w=\{c;b;a\} \\ v\neq u;w\neq v;u\neq v\end{array}\right\};\ a=\sum_{j=1}^{k}\frac{\partial\overset{(m)}{\bar{\psi}_i}}{\partial\overset{(m)}{q_j}}\cdot\Delta_j\cdot\dot{q}_j \\ b=I_i^*={}_i^0[R]\cdot{}^iI_i^*\cdot{}_i^0[R]^T;\ c=\sum_{p=1}^{k}\frac{\partial\overset{(m)}{\bar{\psi}_i}}{\partial\overset{(m)}{q_p}}\cdot\Delta_p\cdot\dot{q}_p\end{array}\right\}. \quad (67)$$

The theorem of the kinetic energy in differential form is considered the most general theorem of dynamics. Its equation of definition is written as:

$$\left\{\begin{array}{c}dE_C=\sum_{j=1}^{k^*=n}M_i\cdot\bar{a}_{C_i}^T\cdot\frac{\partial\bar{r}_{C_i}}{\partial q_j}\cdot dq_j+ \\ +\sum_{j=1}^{k^*=n}\left(I_i^*\cdot\bar{\varepsilon}_i+\bar{\omega}_i\times I_i^*\cdot\bar{\omega}_i\right)^T\cdot\frac{\partial\bar{\psi}_i}{\partial q_j}\cdot dq_j\cdot\Delta_j\equiv dL\end{array}\right\} \quad (68)$$

$$\left\{\begin{array}{c}dE_C\equiv dL=\bar{F}_i^{*T}\cdot d\bar{r}_{C_i}+\bar{N}_i^{*T}\cdot d\bar{\psi}_i \\ \sum_{j=1}^{k^*=n}\bar{F}_i^{*T}\cdot\frac{\partial\bar{r}_{C_i}}{\partial q_j}\cdot dq_j+\sum_{j=1}^{k^*=n}\bar{N}_i^{*T}\cdot\frac{\partial\bar{\psi}_i}{\partial q_j}\cdot dq_j\cdot\Delta_j\end{array}\right\}; \quad (69)$$

According to [4] – [11], it contains differential expression of the kinetic energy and elementary work. Therefore, the theorem of the kinetic energy (69) is mathematically reformulated.

In the case of the multibody systems (MBS), with holonomic mechanical significance, a few conditions are applied on (68) and (69), thus:

$$\left\{\begin{array}{c}q_j\neq 0,\ dq_j\neq 0,\ j=1\rightarrow n \\ q_i=0,\ dq_i=0,\ i=1\rightarrow n,\ i\neq j\end{array}\right\}. \quad (70)$$

They are referring to independent parameters in in the both finite and elementary displacements. After a few transformations on the differential of the theorem of the kinetic energy it obtains:

$$\left\{\begin{array}{c}\sum_{i=1}^{n}\left\{\left[\bar{F}_i^{*T}-M_i\cdot\bar{a}_{C_i}^T\right]\cdot\frac{\partial\bar{r}_{C_i}}{\partial q_j}+\right. \\ \left.\sum_{i=1}^{n}\left[\bar{N}_i^{*T}-\left(I_i^*\cdot\bar{\varepsilon}_i+\bar{\omega}_i\times I_i^*\cdot\bar{\omega}_i\right)^T\right]\cdot\frac{\partial\bar{\psi}_i}{\partial q_j}\cdot\Delta_j\right\}=0\end{array}\right\} \quad (71)$$

$$\text{where } \frac{\partial\bar{r}_{C_i}}{\partial q_j}=\frac{\partial\overset{(m)}{\bar{r}_{C_i}}}{\partial\overset{(m)}{q_j}};\quad \text{and } \frac{\partial\bar{\psi}_i}{\partial q_j}=\frac{\partial\overset{(m)}{\bar{\psi}_i}}{\partial\overset{(m)}{q_j}}. \quad (72)$$

According to author researches [8] – [11], for the multibody rigid system, expression (71) is considered the differential generalized principle (generalization of the D'Alembert — Lagrange principle) in analytical dynamics of systems.

3. ENERGIES OF HIGHER ORDER

The phrase, *"advanced notions"* founded in the analytical dynamics, is focused in this paper on the motion energies whose central functions are the accelerations of higher order. They are developing in any sudden and transitory motion of the mechanical systems. Leading to Appell's function, highlighted in 1899, [1] and [2], also named "kinetic energy of accelerations" [13], author has been developed new mathematical formulations on the expressions for acceleration energies of first, second, third and fourth order [3] – [10] and [11]. In this section they will be again presented, but only in their explicit form.

Considering papers [6] – [11], in following *the acceleration energies of order* $(p \geq 1)$ will be defined. The starting equation shows as follows:

$$\overset{(k-1)}{E}{}_A^{(p)}\left[\bar{\theta}(t);\dot{\bar{\theta}}(t);\cdots;\overset{(p+k)}{\bar{\theta}}(t)\right]= \tag{73}$$

$$\left\{\begin{array}{l} =\dfrac{1}{2}\cdot\sum\limits_{i=1}^{n}Trace\left\{\overset{(p+k)}{{}_i^0[R]}\cdot\left[\int{}^i\bar{r}_i^{*}\cdot{}^i\bar{r}_i^{*T}\cdot dm+\right.\right.\\ \left.\left.+{}^i\bar{r}_{C_i}\cdot{}^i\bar{r}_{C_i}^{T}\cdot\int dm\right]\cdot\overset{(p+k)}{{}_i^0[R]^T}\right\}+\\ +\dfrac{1}{2}\cdot\sum\limits_{i=1}^{n}Trace\left\{\dfrac{d^{k-1}}{dt^{k-1}}\left[\overset{(p+1)}{\bar{p}_i}\cdot\overset{(p+1)}{\bar{p}_i^T}\right]\right\}\cdot\int dm=\\ \dfrac{1}{2}\cdot\sum\limits_{i=1}^{n}Trace\dfrac{d^{k-1}}{dt^{k-1}}\left\{\overset{(p+1)}{{}_i^0[R]}\cdot\left[{}^iI_{pi}^{*}+M_i\cdot{}^i\bar{r}_{C_i}\cdot{}^i\bar{r}_{C_i}^{T}\right]\cdot\overset{(p+1)}{{}_i^0[R]^T}\right\}\\ +\dfrac{1}{2}\cdot\sum\limits_{i=1}^{n}Trace\left\{\dfrac{d^{k-1}}{dt^{k-1}}\left[\overset{(p+1)}{\bar{p}_i}\cdot\overset{(p+1)}{\bar{p}_i^T}\right]\right\}\cdot M_i \end{array}\right\}$$

$$\left\{\begin{array}{l} where\ p\geq 1,\ \ k\geq 1,\ \{p;\,k\}=\{1;2;3;4;5;.....\},\\ and\ \ \overset{(0)}{E}{}_A^{(p)}\left[\bar{\theta}(t);\dot{\bar{\theta}}(t);\cdots;\overset{(p+k)}{\bar{\theta}}(t)\right]=\\ =E_A^{(p)}\left[\bar{\theta}(t);\dot{\bar{\theta}}(t);\cdots;\overset{(p+1)}{\bar{\theta}}(t)\right] \end{array}\right\}. \tag{74}$$

The expression (73) includes the inertia tensor planar and centrifugal, relative to the frame $\{i^*\}$:

$${}^iI_{pi}^{*}=\int{}^i\bar{r}_i^{*}\cdot{}^i\bar{r}_i^{*T}\cdot dm=\begin{bmatrix}{}^iI_{xx}^{*} & {}^iI_{xy}^{*} & {}^iI_{xz}^{*}\\ {}^iI_{yx}^{*} & {}^iI_{yy}^{*} & {}^iI_{yz}^{*}\\ {}^iI_{zx}^{*} & {}^iI_{zy}^{*} & {}^iI_{zz}^{*}\end{bmatrix}.$$

The acceleration energies will be also defined for rigid body, and then for multibody systems. In the view of this input parameters of advanced kinematics and mass properties become [4], [12]. According to papers [6] – [11], the author was established the acceleration energy in the generalized form, corresponding to rigid body founded in the general motion. This was named acceleration energy of first order, as follows:

$$\left\{\begin{array}{l} E_A^{(1)}=\dfrac{1}{2}\cdot\int a_M^2\cdot dm=\dfrac{1}{2}\cdot\int\dot{\bar{v}}_M^T\cdot\dot{\bar{v}}_M\cdot dm=\\ \quad=\dfrac{1}{2}\cdot\int Trace\left[\dot{\bar{v}}_M\cdot\dot{\bar{v}}_M^T\right]\cdot dm \end{array}\right\}; \tag{75}$$

$$where\ \ \bar{a}_M=\dot{\bar{v}}_M=\left(\bar{a}_0+\varepsilon\times\bar{\rho}_M+\bar{\omega}\times\bar{\omega}\times\bar{\rho}_M\right).$$

Developing (75), the first three terms are defined:

$$\frac{1}{2}\cdot\int\bar{a}_0^T\cdot\bar{a}_0\cdot dm=\frac{1}{2}\cdot M\cdot\bar{a}_0^T\cdot\bar{a}_0=\frac{1}{2}\cdot M\cdot a_0^2; \tag{76}$$

$$\left\{\begin{array}{l} \dfrac{1}{2}\cdot\int\bar{a}_0^T\cdot\left(\bar{\varepsilon}\times\bar{\rho}_M\right)\cdot dm=\\ =\dfrac{1}{2}\cdot\int\left(\bar{\varepsilon}\times\bar{\rho}_M\right)^T\cdot\bar{a}_0\cdot dm=\dfrac{1}{2}\cdot M\cdot\bar{a}_0\cdot\left(\bar{\varepsilon}\times\bar{\rho}_C\right) \end{array}\right\} \tag{77}$$

$$\left\{\begin{array}{l} \dfrac{1}{2}\cdot\int\bar{a}_0^T\cdot\left(\bar{\omega}\times\bar{\omega}\times\bar{\rho}_M\right)\cdot dm=\\ =\dfrac{1}{2}\cdot\int\left(\bar{\omega}\times\bar{\omega}\times\bar{\rho}_M\right)^T\cdot\bar{a}_0\cdot dm=\\ =\dfrac{1}{2}\cdot M\cdot\bar{a}_0\cdot\left(\bar{\omega}\times\bar{\omega}\times\bar{\rho}_C\right) \end{array}\right\}. \tag{78}$$

The following three components are devoted to resultant rotation. Among of these, the first two contain the angular velocity and acceleration. So, their expressions of definition are below shown:

$$\left\{\begin{array}{l} \dfrac{1}{2}\cdot\int\left(\bar{\varepsilon}\times\bar{\rho}_M\right)^T\cdot\left(\bar{\varepsilon}\times\bar{\rho}_M\right)\cdot dm=\\ =\dfrac{1}{2}\cdot\bar{\varepsilon}^T\cdot\left\{\int\left(\bar{\rho}_M\times\right)\cdot\left(\bar{\rho}_M\times\right)^T\cdot dm\right\}\cdot\bar{\varepsilon}=\\ =\dfrac{1}{2}\cdot\bar{\varepsilon}^T\cdot I_S'\cdot\bar{\varepsilon} \end{array}\right\}; \tag{79}$$

$$\left\{\begin{array}{l} E_A^{(1\omega\varepsilon)}=\dfrac{1}{2}\cdot\int\left(\bar{\varepsilon}\times\bar{\rho}_M\right)^T\cdot\left(\bar{\omega}\times\bar{\omega}\times\bar{\rho}_M\right)\cdot dm\\ =\dfrac{1}{2}\cdot\int\left(\bar{\omega}\times\bar{\omega}\times\bar{\rho}_M\right)^T\cdot\left(\bar{\varepsilon}\times\bar{\rho}_M\right)\cdot dm \end{array}\right\}. \tag{80}$$

Components from (80) are below demonstrated:

$$\left(\bar{\varepsilon}\times\bar{\rho}_M\right)^T=\left[\left(\bar{\rho}_M\times\right)^T\cdot\bar{\varepsilon}\right]=\bar{\varepsilon}^T\cdot\left(\bar{\rho}_M\times\right); \tag{81}$$

$$\bar{\omega}\times\left(\bar{\omega}\times\bar{\rho}_M\right)=\left(\bar{\omega}^T\cdot\bar{\rho}_M\right)\cdot\bar{\omega}-\omega^2\cdot\bar{\rho}_M; \tag{82}$$

$$\left(\bar{\varepsilon}\times\bar{\rho}_M\right)^T\cdot\left(\bar{\omega}\times\bar{\omega}\times\bar{\rho}_M\right)= \tag{83}$$

$$=\bar{\varepsilon}^T\cdot\left(\bar{\rho}_M\times\right)\cdot\left(\bar{\omega}^T\cdot\bar{\rho}_M\right)\cdot\bar{\omega}-\bar{\varepsilon}^T\cdot\left(\bar{\rho}_M\times\right)\cdot\omega^2\cdot\bar{\rho}_M;$$

$$\bar{\varepsilon}^T\cdot\left(\bar{\rho}_M\times\right)\cdot\omega^2\cdot\bar{\rho}_M=\omega^2\cdot\bar{\varepsilon}^T\cdot\left(\bar{\rho}_M\times\bar{\rho}_M\right)=0; \tag{84}$$

$$\left\{\begin{array}{l}\bar{\varepsilon}^T\cdot(\bar{\rho}_M\times)\cdot(\bar{\omega}^T\cdot\bar{\rho}_M)\cdot\bar{\omega}=\\=\bar{\varepsilon}^T\cdot(\bar{\rho}_M\times\bar{\omega})\cdot(\bar{\omega}^T\cdot\bar{\rho}_M)=\\=\bar{\varepsilon}^T\cdot(\bar{\omega}\times)^T\cdot\left[\bar{\rho}_M\cdot\bar{\rho}_M^T\right]\cdot\bar{\omega}\end{array}\right\}; \quad (85)$$

$$\left[\bar{\rho}_M\cdot\bar{\rho}_M^T\right]=\bar{\rho}_M^T\cdot\bar{\rho}_M\cdot I_3-(\bar{\rho}_M\times)\cdot(\bar{\rho}_M\times)^T; \quad (86)$$

$$\left\{\begin{array}{l}\bar{\varepsilon}^T\cdot(\bar{\omega}\times)^T\cdot\left[\bar{\rho}_M^T\cdot\bar{\rho}_M\cdot I_3\right]\cdot\bar{\omega}=\\=\bar{\rho}_M^T\cdot\bar{\rho}_M\cdot\bar{\varepsilon}^T\cdot(\bar{\omega}\times)^T\cdot\bar{\omega}=0\end{array}\right\}; \quad (87)$$

$$\left\{\begin{array}{l}\bar{\varepsilon}^T\cdot(\bar{\omega}\times)^T\cdot\left[-(\bar{\rho}_M\times)\cdot(\bar{\rho}_M\times)^T\right]\cdot\bar{\omega}=\\=\bar{\varepsilon}^T\cdot(\bar{\omega}\times)\cdot\left[(\bar{\rho}_M\times)\cdot(\bar{\rho}_M\times)^T\right]\cdot\bar{\omega}\end{array}\right\}. \quad (88)$$

Replacing (81) – (88) in (80), next expression is:

$$\left\{\begin{array}{c}\frac{1}{2}\cdot\int(\bar{\varepsilon}\times\bar{\rho}_M)^T\cdot(\bar{\omega}\times\bar{\omega}\times\bar{\rho}_M)\cdot dm=\\\frac{1}{2}\cdot\bar{\varepsilon}^T\cdot(\bar{\omega}\times)\cdot\left[\int(\bar{\rho}_M\times)\cdot(\bar{\rho}_M\times)^T\cdot dm\right]\cdot\bar{\omega}\\=\frac{1}{2}\cdot\bar{\varepsilon}^T\cdot(\bar{\omega}\times I'_S\cdot\cdot\bar{\omega})=E_A^{(1\omega\varepsilon)}\end{array}\right\}. \quad (89)$$

The last component corresponding to rotation motion contains in exclusivity angular velocity:

$$E_A^{(1\omega^4)}=\frac{1}{2}\cdot\int(\bar{\omega}\times\bar{\omega}\times\bar{\rho}_M)^T\cdot(\bar{\omega}\times\bar{\omega}\times\bar{\rho}_M)\cdot dm\ . \quad (90)$$

The integrand from (90) is below developed thus:

$$\left\{\begin{array}{l}\bar{\omega}\times(\bar{\omega}\times\bar{\rho}_M)=\left[(\bar{\omega}\times\bar{\rho}_M)\times\right]^T\cdot\bar{\omega}=\\=-\left[(\bar{\rho}_M\times)^T\cdot\bar{\omega}\right]\times\bar{\omega}\end{array}\right\}; \quad (91)$$

$$\left\{\begin{array}{l}\left\{\left[(\bar{\omega}\times\bar{\rho}_M)\times\right]^T\cdot\bar{\omega}\right\}^T=\\=\bar{\omega}^T\cdot\left[(\bar{\omega}\times\bar{\rho}_M)\times\right]=\bar{\omega}^T\cdot\left[(\bar{\rho}_M\times)^T\cdot\bar{\omega}\times\right]\end{array}\right\}; \quad (92)$$

$$\left\{\begin{array}{l}-\bar{\omega}^T\cdot\left[(\bar{\rho}_M\times)^T\cdot\bar{\omega}\times\right]\cdot\left[(\bar{\rho}_M\times)^T\cdot\bar{\omega}\times\bar{\omega}\right]=\\=-\bar{\omega}^T\cdot\left[(\bar{\rho}_M\times)^T\cdot\bar{\omega}\cdot\bar{\omega}\right]\cdot\left[(\bar{\rho}_M\times)^T\cdot\bar{\omega}\right]+\\+\bar{\omega}^T\cdot\left[(\bar{\rho}_M\times)^T\cdot\bar{\omega}\cdot(\bar{\rho}_M\times)^T\cdot\bar{\omega}\right]\cdot\bar{\omega}\end{array}\right\}; \quad (93)$$

$$\left[(\bar{\rho}_M\times)^T\cdot\bar{\omega}\cdot\bar{\omega}\right]=\bar{\omega}\cdot(\bar{\omega}\times\bar{\rho}_M)=0; \quad (94)$$

$$\left\{\begin{array}{l}\bar{\omega}^T\cdot\left[(\bar{\rho}_M\times)^T\cdot\bar{\omega}\cdot(\bar{\rho}_M\times)^T\cdot\bar{\omega}\right]\cdot\bar{\omega}=\\=\bar{\omega}^T\cdot\left[\bar{\omega}^T\cdot(\bar{\rho}_M\times)\cdot(\bar{\rho}_M\times)^T\cdot\bar{\omega}\right]\cdot\bar{\omega}\end{array}\right\}. \quad (95)$$

The integrand (95) is replaced in (90). It obtains:

$$\left\{\begin{array}{l}E_A^{(1\omega^4)}=\frac{1}{2}\cdot\int(\bar{\omega}\times\bar{\omega}\times\bar{\rho}_M)^T\cdot(\bar{\omega}\times\bar{\omega}\times\bar{\rho}_M)\cdot dm\\=\frac{1}{2}\cdot\bar{\omega}^T\cdot\left\{\bar{\omega}^T\cdot\left[\int(\bar{\rho}_M\times)\cdot(\bar{\rho}_M\times)^T\cdot dm\right]\cdot\bar{\omega}\right\}\cdot\bar{\omega}\end{array}\right\} \quad (96)$$

So, the component (90) takes the final form as:

$$\left\{\begin{array}{c}E_A^{(1\omega^4)}=\frac{1}{2}\cdot\bar{\omega}^T\cdot\left[\bar{\omega}^T\cdot I'_S\cdot\bar{\omega}\right]\cdot\bar{\omega}=\\\frac{1}{2}\cdot\bar{\omega}^T\cdot\left[\bar{\omega}^T\cdot Trace(I'_{pS})\cdot\bar{\omega}-\bar{\omega}^T\cdot I'_{pS}\cdot\bar{\omega}\right]\cdot\bar{\omega}\end{array}\right\}. \quad (97)$$

Replacing (76) – (79), (89) and (97) in (75), the acceleration energy of first order takes final form:

$$\left\{\begin{array}{c}E_A^{(1)}=\frac{1}{2}\cdot M\cdot\bar{a}_0^T\cdot\bar{a}_0+M\cdot\bar{a}_0^T\cdot(\bar{\varepsilon}\times\bar{\rho}_C)+\\+M\cdot\bar{a}_0^T\cdot(\bar{\omega}\times\bar{\omega}\times\bar{\rho}_C)+\frac{1}{2}\cdot\bar{\varepsilon}^T\cdot I'_S\cdot\bar{\varepsilon}+\\+\bar{\varepsilon}^T\cdot(\bar{\omega}\times I'_S\cdot\bar{\omega})+\frac{1}{2}\cdot\bar{\omega}^T\cdot\left[\bar{\omega}^T\cdot I'_S\cdot\bar{\omega}\right]\cdot\bar{\omega}\end{array}\right\}. \quad (98)$$

When $O\equiv C,\ \bar{\rho}_C=0,\ and\ I'_S\equiv I^*_S$, (98) becomes:

$$\left\{\begin{array}{c}E_A^{(1)}=\frac{1}{2}\cdot M\cdot\bar{a}_C^T\cdot\bar{a}_C+\frac{1}{2}\cdot\bar{\varepsilon}^T\cdot I^*_S\cdot\bar{\varepsilon}+\\+\bar{\varepsilon}^T\cdot(\bar{\omega}\times I^*_S\cdot\bar{\omega})+\frac{1}{2}\cdot\bar{\omega}^T\cdot\left[\bar{\omega}^T\cdot I^*_S\cdot\bar{\omega}\right]\cdot\bar{\omega}\end{array}\right\}. \quad (99)$$

In the case of the multibody systems (MBS), the definition equation of the acceleration energy of first order (99) is changed considering [3] – [11]:

$$\left\{\begin{array}{c}E_A^{(1)}\left[\bar{\theta}(t);\dot{\bar{\theta}}(t);\ddot{\bar{\theta}}(t)\right]=\\(-1)^{\Delta_M}\cdot\frac{1-\Delta_M}{1+3\cdot\Delta_M}\sum_{i=1}^{n}\left[\frac{1}{2}\cdot M_i\cdot{}^{(i)}\dot{\bar{v}}_{C_i}^T\cdot{}^{(i)}\dot{\bar{v}}_{C_i}\right]\\+\Delta_M^2\cdot\sum_{i=1}^{n}\frac{1}{2}\cdot{}^{(i)}\dot{\bar{\omega}}_i^T\cdot{}^{(i)}I_i^*\cdot{}^{(i)}\dot{\bar{\omega}}_i+\\+\Delta_M^2\cdot\sum_{i=1}^{n}\left[{}^{(i)}\dot{\bar{\omega}}_i^T\cdot\left({}^{(i)}\bar{\omega}_i\times{}^{(i)}I_i^*\cdot{}^{(i)}\bar{\omega}_i\right)\right]+\\+E_A^{(1)}\left[\bar{\theta}(t);\dot{\bar{\theta}}^4(t)\right]\end{array}\right\}; \quad (100)$$

$$\left\{\begin{array}{c}E_A^{(1)}\left[\bar{\theta}(t);\dot{\bar{\theta}}^4(t)\right]=\\\Delta_M^2\cdot\sum_{i=1}^{n}\left\{\frac{1}{2}\cdot{}^{(i)}\bar{\omega}_i^T\cdot\left[{}^{(i)}\bar{\omega}_i^T\cdot{}^{(i)}I_i^*\cdot{}^{(i)}\bar{\omega}_i^T\right]\cdot{}^{(i)}\bar{\omega}_i^T\right\}\end{array}\right\}. \quad (101)$$

Considering the notions from papers [9] – [11], the two components (translational and rotation) of acceleration energy of first order show as:

$$\left\{\begin{array}{l}E_A^{(1)TR}=\frac{1}{2}\cdot\sum_{i=1}^{n}M_i\cdot\left\{\sum_{j=1}^{k^*=n}\sum_{p=1}^{k^*=n}\left[\frac{\partial^2\overset{(m)}{\bar{r}_{C_i}}}{\partial\overset{(m)}{q_j}\cdot\partial\overset{(m)}{q_p}}\cdot\ddot{q}_j\cdot\ddot{q}_p\right.\right.\\\quad+\frac{1}{(m+1)^2}\cdot\frac{\partial^2\overset{(m+1)}{\bar{r}_{C_i}}}{\partial\overset{(m)}{q_j}\cdot\partial\overset{(m)}{q_p}}\cdot\dot{q}_j\cdot\dot{q}_p+\\\left.\left.\frac{1}{m+1}\cdot\frac{\partial\overset{(m)}{\bar{r}_{C_i}}}{\partial\overset{(m)}{q_j}}\cdot\frac{\partial\overset{(m+1)}{\bar{r}_{C_i}}}{\partial\overset{(m)}{q_p}}\cdot\ddot{q}_j\cdot\dot{q}_p\right]\right\}=\frac{1}{2}\cdot\sum_{i=1}^{n}M_i\cdot a_{C_i}^2\end{array}\right\} \quad (102)$$

$$\left\{ E_A^{(1)ROT\varepsilon} = \frac{1}{2}\cdot\sum_{i=1}^{n}\bar{\varepsilon}_i^T\cdot I_i^*\cdot\bar{\varepsilon}_i \right\} = \qquad (103)$$

$$\left\{ \frac{1}{2}\cdot\sum_{i=1}^{n}\sum_{j=1}^{k^*=n}\left[\frac{\partial\overset{(m)}{\bar{\psi}_i}}{\partial\overset{(m)}{q_j}}\cdot\Delta_j\cdot\ddot{q}_j + \frac{1}{m+1}\cdot\frac{\partial\overset{(m+1)}{\bar{\psi}_i}}{\partial\overset{(m)}{q_j}}\cdot\Delta_j\cdot\dot{q}_j\right]^T\cdot I_i^*\cdot\bar{\varepsilon}_i \right\}$$

$$I_i^*\cdot\bar{\varepsilon} = I_i^*\cdot\sum_{j=1}^{k^*}\left[\frac{\partial\overset{(m)}{\bar{\psi}_i}}{\partial\overset{(m)}{q_j}}\cdot\Delta_j\cdot\ddot{q}_j + \frac{1}{m+1}\cdot\frac{\partial\overset{(m+1)}{\bar{\psi}_i}}{\partial\overset{(m)}{q_j}}\cdot\Delta_j\cdot\dot{q}_j\right];$$

$$\left\{ E_A^{(1)ROT\omega\varepsilon} = \sum_{i=1}^{n}\bar{\varepsilon}_i^T\cdot\left(\bar{\omega}_i\times I_i^*\cdot\bar{\omega}_i\right) \right\} = \qquad (104)$$

$$\left\{ \sum_{i=1}^{n}\sum_{j=1}^{k^*=n}\left[\frac{\partial\overset{(m)}{\bar{\psi}_i}}{\partial\overset{(m)}{q_j}}\cdot\Delta_j\cdot\ddot{q}_j + \frac{1}{m+1}\cdot\frac{\partial\overset{(m+1)}{\bar{\psi}_i}}{\partial\overset{(m)}{q_j}}\cdot\Delta_j\cdot\dot{q}_j\right]^T\cdot E_A^{(1)ROT\omega\omega} \right\};$$

$$E_A^{(1)ROT\omega\omega} = \left(\bar{\omega}_i\times I_i^*\cdot\bar{\omega}_i\right) = \qquad (105)$$

$$\left\{ \sum_{i=1}^{n}\sum_{j=1}^{k^*=n}\sum_{p=1}^{k^*=n}\left[\frac{\partial\overset{(m)}{\bar{\psi}_i}}{\partial\overset{(m)}{q_j}}\right]\times\left[I_i^*\cdot\frac{\partial\overset{(m)}{\bar{\psi}_i}}{\partial\overset{(m)}{q_p}}\right]\cdot\Delta_j\cdot\Delta_p\cdot\dot{q}_j\cdot\dot{q}_p \right\}.$$

According to the author researches, [6] – [11], the sudden motion of MBS, the transient motion phases, as well as mechanical systems subjected to the action of a system of external forces, with a time variation law, are characterized by linear and angular accelerations of higher order. So, the acceleration energy of second order has been also developed. According to other papers of author [7] – [9], its explicit form is shown as:

$$E_A^{(2)}\left[\bar{\theta}(t);\dot{\bar{\theta}}(t);\ddot{\bar{\theta}}(t);\dddot{\bar{\theta}}(t)\right] = \qquad (106)$$

$$\left\{\begin{array}{l} = (-1)^{\Delta_M}\cdot\dfrac{1-\Delta_M}{1+3\cdot\Delta_M}\cdot\sum_{i=1}^{n}\left\{\dfrac{1}{2}\cdot M_i\cdot{}^i\dddot{\bar{v}}_{C_i}^T\cdot{}^i\dddot{\bar{v}}_{C_i}\right\} + \\ +\Delta_M^2\cdot\sum_{i=1}^{n}\left\{\dfrac{1}{2}\cdot{}^i\ddot{\bar{\omega}}_i^T\cdot{}^iI_i^*\cdot{}^i\ddot{\bar{\omega}}_i + 2\cdot{}^i\bar{\omega}_i^T\cdot\left({}^i\ddot{\bar{\omega}}_i\times{}^iI_{pi}^*\cdot{}^i\dot{\bar{\omega}}_i\right) + \right. \\ \left. +{}^i\dot{\bar{\omega}}_i^T\cdot\left({}^i\ddot{\bar{\omega}}_i\times{}^iI_{pi}^*\cdot{}^i\bar{\omega}_i\right) - {}^i\bar{\omega}_i^T\cdot\left({}^i\ddot{\bar{\omega}}_i^T\cdot{}^iI_i^*\cdot{}^i\bar{\omega}_i\right)\cdot{}^i\dot{\bar{\omega}}_i\right\} + \\ +E_A^{(2)}\left[\bar{\theta}(t);\dot{\bar{\theta}}(t);\ddot{\bar{\theta}}^2(t)\right] + E_A^{(2)}\left[\bar{\theta}(t);\dot{\bar{\theta}}^6(t);\ddot{\bar{\theta}}(t)\right] \end{array}\right\};$$

$$\text{where}\quad E_A^{(2)}\left[\bar{\theta}(t);\dot{\bar{\theta}}(t);\ddot{\bar{\theta}}^2(t)\right] = \qquad (107)$$

$$\left\{\begin{array}{l} = \Delta_M^2\cdot\sum_{i=1}^{n}\left\{2\cdot{}^i\dot{\bar{\omega}}_i^T\cdot\left({}^i\bar{\omega}_i^T\cdot{}^iI_i^*\cdot{}^i\dot{\bar{\omega}}_i\right)\cdot{}^i\bar{\omega}_i + \right. \\ +2\cdot{}^i\bar{\omega}_i^T\cdot\left[{}^i\dot{\bar{\omega}}_i^T\cdot{}^iI_{pi}^*\cdot{}^i\dot{\bar{\omega}}_i\right]\cdot{}^i\bar{\omega}_i - \\ -5\cdot\left({}^i\dot{\bar{\omega}}_i^T\cdot{}^iI_{pi}^*\right)\cdot\left({}^i\dot{\bar{\omega}}_i^T\cdot{}^i\bar{\omega}_i\right)\cdot{}^i\bar{\omega}_i + \\ +\dfrac{5}{2}\cdot\left({}^i\dot{\bar{\omega}}_i^T\cdot{}^i\bar{\omega}_i\right)\cdot Trace\left({}^iI_{pi}^*\right)\cdot\left({}^i\dot{\bar{\omega}}_i^T\cdot{}^i\bar{\omega}_i\right) \\ \left. +\dfrac{1}{2}\cdot{}^i\dot{\bar{\omega}}_i^T\cdot\left[{}^i\bar{\omega}_i^T\cdot{}^iI_{pi}^*\cdot{}^i\bar{\omega}_i\right]\cdot{}^i\dot{\bar{\omega}}_i\right\} + \end{array}\right\},$$

$$\text{and}\quad E_A^{(2)}\left[\bar{\theta}(t);\dot{\bar{\theta}}^6(t);\ddot{\bar{\theta}}(t)\right] = \qquad (108)$$

$$\left\{\begin{array}{l} = \Delta_M^2\cdot\left\{\sum_{i=1}^{n}{}^i\bar{\omega}_i^T\cdot\left[{}^i\bar{\omega}_i^T\cdot\left({}^i\ddot{\bar{\omega}}_i\times{}^iI_{pi}^*\cdot{}^i\bar{\omega}_i\right)\right]\cdot{}^i\bar{\omega}_i + \right. \\ \left. \dfrac{1}{2}\cdot{}^i\bar{\omega}_i^T\cdot\left[{}^i\bar{\omega}_i^T\cdot\left({}^i\bar{\omega}_i^T\cdot{}^iI_i^*\cdot{}^i\bar{\omega}_i\right)\cdot{}^i\bar{\omega}_i\right]\cdot{}^i\bar{\omega}_i\right\} \end{array}\right\}.$$

The study of advanced dynamics is extended on acceleration energy of third order. According to [6] – [11], author proposes explicit equation of the acceleration energy of third order thus:

$$E_A^{(3)}\left[\bar{\theta}(t);\dot{\bar{\theta}}(t);\ddot{\bar{\theta}}(t);\dddot{\bar{\theta}}(t);\ddddot{\bar{\theta}}(t)\right] = \qquad (109)$$

$$\left\{\begin{array}{l} = (-1)^{\Delta_M}\cdot\dfrac{1-\Delta_M}{1+3\cdot\Delta_M}\cdot\sum_{i=1}^{n}\left\{\dfrac{1}{2}\cdot M_i\cdot{}^i\ddddot{\bar{v}}_{C_i}^T\cdot{}^i\ddddot{\bar{v}}_{C_i}\right\} + \\ +\Delta_M^2\cdot\sum_{i=1}^{n}\left\{\dfrac{1}{2}\cdot{}^i\dddot{\bar{\omega}}_i^T\cdot{}^iI_i^*\cdot{}^i\dddot{\bar{\omega}}_i + 3\cdot\bar{\omega}_i^T\cdot\left(\dddot{\bar{\omega}}_i\times I_{pi}^*\cdot\ddot{\bar{\omega}}_i\right) + \right. \\ +3\cdot\dot{\bar{\omega}}_i^T\cdot\left(\dddot{\bar{\omega}}_i\times I_{pi}^*\cdot\dot{\bar{\omega}}_i\right) + 3\cdot\ddot{\bar{\omega}}_i^T\cdot\left(\dddot{\bar{\omega}}_i\times I_{pi}^*\cdot\bar{\omega}_i\right) + \\ +2\cdot\left(\bar{\omega}_i\times\dddot{\bar{\omega}}_i\right)^T\cdot I_{pi}^*\cdot\left(\dot{\bar{\omega}}_i\times\bar{\omega}_i\right) - \\ -5\cdot\dot{\bar{\omega}}_i^T\cdot\left[\bar{\omega}_i^T\cdot I_i^*\cdot\dddot{\bar{\omega}}_i\right]\cdot\bar{\omega}_i - \bar{\omega}_i^T\cdot\left[\dot{\bar{\omega}}_i^T\cdot I_i^*\cdot\dddot{\bar{\omega}}_i\right]\cdot\bar{\omega}_i + \\ \left. +\bar{\omega}_i^T\cdot\left[\bar{\omega}_i^T\cdot I_{pi}^*\cdot\left(\dddot{\bar{\omega}}_i\times\bar{\omega}_i\right)\right]\cdot\bar{\omega}_i\right\} + \end{array}\right\}$$

The component $E_A^{(3)}\left[\bar{\theta}(t);\dot{\bar{\theta}}(t);\ddot{\bar{\theta}}(t);\dddot{\bar{\theta}}^2(t)\right]$ is not included in the definition equation (109).

4. CONCLUSIONS

By means of the researches of the author, in the first two sections of this paper formulations concerning the classical notions and theorems from dynamics were presented. So, momentum and theorem of momentum, also named motion theorem of the mass center, angular momentum and theorem of the angular momentum, as well as the kinetic energy and theorem of the kinetic energy were defined in the explicit form.

In the third section, the study was extended on the energies of higher order. Using the researches of the author, form other papers, first of all the acceleration energy of first order for rigid body in the general motion was demonstrated. In the following, expressions of definition in explicit form have been presented for the acceleration energy of first, second and third order. They are corresponding to the current and sudden motions of rigid body and multi body systems. These formulations also contain the absolute time derivatives of higher order of the advanced notions, according to differential equations of higher order, typically to analytical dynamics.

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Energii de Ordin Superior în Dinamica Avansată a Sistemelor

Studiul dinamic al mişcărilor curente şi rapide ale corpului rigid şi sistemelor mecanice multicorp (MBS), spre exemplu structurile mecanice de roboţi seriali şi în conformitate cu principiile diferenţiale specifice dinamicii analitice a sistemelor, se bazează, printre altele, pe noţiunile avansate, cum sunt: impulsul şi momentul cinetic, energia cinetică, energiile de acceleraţii de diferite ordine şi derivatele absolute în raport cu timpul a acestora de ordin superior. Noţiunile avansate sunt dezvoltate în conexiune directă cu variabilele generalizate, de asemenea, denumite parametrii independenţi corespunzători sistemelor mecanice olonome. Dar, expresiile de definiţie ale noţiunilor avansate conţin pe de o parte parametrii cinematici şi transformările lor diferenţiale corespunzătoare mişcării absolute, iar pe de altă parte proprietăţile maselor. Cu ajutorul, în special, cercetărilor autorului în această lucrare se vor prezenta reformulări şi formulări noi cu privire la noţiunile şi teoremele din dinamica clasică. În continuare,pe baza aceloraşi cercetări ale autorului, studiul se va extinde asupra energiilor de ordin superior. Astfel se vor prezenta expresiile de definiţie în formă explicită pentru energia de acceleraţii de ordin întâi, doi şi trei, corespunzătoare mişcărilor curente şi rapide ale corpului rigid şi sistemelor multicorp. Aceste formulări vor conţine derivatele absolute în raport cu timpul de ordin superior ale noţiunilor avansate, conform cu ecuaţiile diferenţiale de ordin superior, specifice dinamicii analitice a sistemelor.

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