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# FORMULATIONS ABOUT ELASTODYNAMICS IN ROBOTICS

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***Abstract:*** *The main objective of this paper consists in the establishment of the generalized elastokinematics and elastodynamics equations for robot structures with flexible links. For kinematics and differential matrices in the case of the robot structures with rigid and elastic links will be applied, among of these, the matrix exponentials, in accordance with the algorithms developed by the author. For dynamical study of robot structures with rigid links, the author of paper will develop kinetic energy with important formulations. According to the author researches, in this paper will be also presented new expressions for acceleration energies of higher order, corresponding to suddenly movements. For the study of elastodynamics, author will establish with new formulations the kinetic energy, acceleration energy of first and second order, corresponding in exclusivity to elastic structure. So, the expressions for generalized inertia forces typical to suddenly motions will be determined, when the robot structures are dominated of elastic links.*

***Key words:*** *elasto-kinematics, elasto-dynamics, advanced mechanics, robotics.*

## 1. INTRODUCTION

This paper is divided in the two main parts. First part will be devoted to establishment the kinematics and dynamics equations for any serial robot with rigid structure. The second part will devoted to determination elastokinematics and elastodynamics equations for any robot with elastic and flexible structure. In the view of this according to [5] – [7], transfer equations of any kinematic chain [*(R)-rotation* and *(T)-prismatic* joints, Fig.1 and Fig.2], are defined by means of the *locating transformations*. So, the mechanical robot structure *(MRS)* is initially represented in the configuration: $\overline{\theta}^{(0)} = [q_i = 0; i = 1 \rightarrow n]^T$ .

In the kinematics of rigid and elastic structure will be defined the linear and angular velocities and accelerations using classical transformations and matrix exponentials. In the dynamics of robot with rigid structure, the author of paper will develop, as well as new expressions for acceleration energies of higher order, according to suddenly movements. In the elastodynamics the author will establish generalized inertia forces, based on new formulations about the kinetic energy and acceleration energy of first and second order, corresponding in exclusivity to elastic structure and suddenly movements.

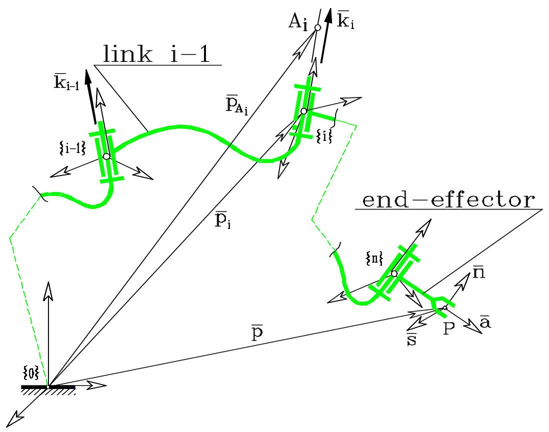


**Fig. 1** Geometrical Parameters of *MRS*

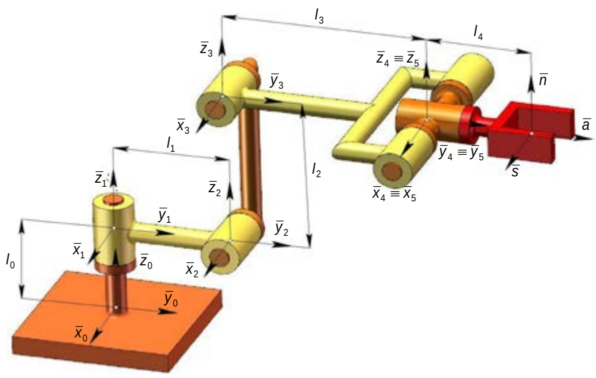

**Fig.2** Mechanical Structure of Robot (MRS)

## 2. KINEMATICS OF RIGID STRUCTURE

The kinematical and dynamical study from this paper [5], [6], [7] is oriented on mechanical structure with opened kinematical chain, where the kinetic ensembles $i=1\to n$ are physically linked by driving joints of fifth order. (Example mechanical structure of robot, see Fig.1).

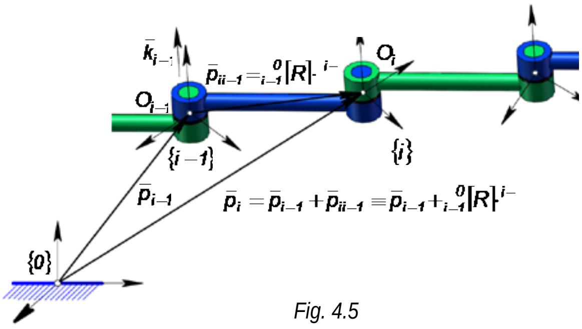


**Fig.3** Sequence of Kinetic Ensembles

This is characterized by $(n\ d.o.f.)$, according to:

$$\bar\theta \neq \bar\theta^{(0)};\quad \bar\theta(t)=[q_i(t);\quad i=1\to n]^T, \tag{1}$$

where $q_i(t)$ is the generalized coordinate from every driving axis. But, considering the current and sudden motions the generalized variables of higher order are developed as follows:

$$\left\{\begin{array}{l}\left\{\bar\theta(t);\ \dot{\bar\theta}(t);\ \ddot{\bar\theta}(t);\cdots;\overset{(m)}{\bar\theta}(t)\right\}=\\ =\left\{\begin{array}{c}q_i(t);\ \dot q_i(t);\ \ddot q_i(t);\cdots;\overset{(m)}{q_i}(t)\\ i=1\to n,\ m\ge 1\end{array}\right\}\end{array}\right\}, \tag{2}$$

and $(m)$ represents the time deriving order. The main objective of this section consists in the establishment of the absolute angular and linear velocities and accelerations for every kinetic ensemble from MRS. Unlike the classical approaches [3] – [5], [7] in the following a few formulations based on the time derivatives of locating matrices will be developed [7], [20].

So, in the Figure 3 a sequence of two kinetic ensembles belonging to MRS is subjected to kinematical study. According to [5] – [7], the locating matrices, for the above sequence are:

$$\left\{\begin{array}{l}{}^0_i[T](t)={}^0_i[T](t)\cdot{}^{i-1}_i[T](t)=\\ =\begin{bmatrix}{}^0_i[R](t) & \bar p_i(t)\\ 0\ 0\ 0 & 1\end{bmatrix}=\\ =\begin{bmatrix}{}^0_{i-1}[R](t)\cdot{}^{i-1}_i[R] & \bar p_{i-1}+\bar p_{ii-1}\\ 0\quad 0\quad 0 & 1\end{bmatrix}\end{array}\right\}, \tag{3}$$

The matrix components from (3) are defined as:

$$ {}^{i-1}_i[R]=R_{ii-1}\cdot R(\bar k_i;q_i(t)\cdot\Delta_i), \tag{4}$$

$$ {}^{i-1}\bar p_{ii-1}={}^{i-1}\bar p^{(0)}_{ii-1}+(1-\Delta_i)\cdot q_i(t)\cdot{}^{i-1}\bar k_i, \tag{5}$$

$$ {}^0_i[R](t)={}^0_{i-1}[R](t)\cdot{}^{i-1}_i[R], \tag{6}$$

$$\bar p_i(t)=\bar p_{i-1}(t)+\bar p_{ii-1}(t)=\bar p_{i-1}(t)+{}^0_{i-1}[R]\cdot{}^{i-1}\bar p_{ii-1}, \tag{7}$$

$$\Delta_i=\{[1,\ i=R];[0,\ i=T]\}. \tag{8}$$

The symbol (9) shows the type of driving joint. On the matrix (3) is applied first time derivative:

$$ {}^0_i[\dot T][q_j(t);\dot q_j(t);j=1\to i]=\begin{bmatrix}{}^0_i[\dot R](t) & \dot{\bar p}_i(t)\\ 0\,0\,0 & 0\end{bmatrix}, \tag{9}$$

$$\left\{\begin{array}{l}{}^0_i[\dot T]=\begin{bmatrix}{}^0_{i-1}[\dot R]\cdot{}^{i-1}_i[R] & \dot{\bar p}_{i-1}+{}^0_{i-1}[\dot R]\cdot{}^{i-1}\bar p_{ii-1}\\ 0\,0\,0 & 1\end{bmatrix}+\\ \begin{bmatrix}\Delta_i\cdot{}^0_{i-1}[R]\cdot{}^{i-1}_i[\dot R] & (1-\Delta_i)\cdot\dot q_i(t)\cdot{}^0_{i-1}[R]\cdot{}^{i-1}\bar k_i+\bar p_{i-1}\\ 0\,0\,0 & 1\end{bmatrix}\end{array}\right\} \tag{10}$$

According to [7], matrix (14) is identical with:

$$\left\{\begin{array}{l}{}^0_i[\dot T][q_j(t);\dot q_j(t);j=1\to i]=\\ =\begin{bmatrix}(\dot{\bar\psi}_i\times) & \dot{\bar p}_i-\dot{\bar\psi}_i\times\bar p_i\\ 0\ 0\ 0 & 0\end{bmatrix}\cdot{}^0_i[T](t)\end{array}\right\}, \tag{11}$$

and $\bar\psi_i$ is orientation vector from $\{i\}$ versus $\{0\}$.

Considering the time derivative property (15), on the matrix (10) a few transformations are:

$$\left\{\begin{array}{l}{}^0_i[\dot T](t)\cdot{}^0_i[T]^{-1}(t)=\\ =\begin{bmatrix}{}^0_i[\dot R]\cdot{}^0_i[R]^T & \dot{\bar p}_i(t)-{}^0_i[\dot R]\cdot{}^0_i[R]^T\cdot\bar p_i(t)\\ 0\,0\,0 & 0\end{bmatrix}\\ =\begin{bmatrix}\{{}^0\bar\omega_i\times\} & \dot{\bar p}_i(t)-\{{}^0\bar\omega_i\times\}\cdot\bar p_i(t)\\ 0\,0\,0 & 0\end{bmatrix}\end{array}\right\} \tag{12}$$

$$\{{}^0\bar\omega_i\times\}={}^0_i[\dot R]\cdot{}^0_i[R]^T,\ \{{}^i\bar\omega_i\times\}={}^0_i[R]^T\cdot{}^0_i[\dot R]; \tag{13}$$

where properties (13) are according to [7] – [8]. The expression (12) is written again as follows:

$$\left\{\begin{array}{l}{}^0_i[\dot T](t)\cdot{}^0_i[T]^{-1}(t)=\\ ={}^0_{i-1}[\dot T](t)\cdot{}^{i-1}_i[T](t)\cdot{}^0_i[T]^{-1}(t)+\\ +{}^0_{i-1}[T](t)\cdot{}^{i-1}_i[\dot T](t)\cdot{}^0_i[T]^{-1}(t)\end{array}\right\}. \tag{14}$$

The first matrix term from (14) becomes thus:

$$\left\{\begin{array}{l}{}^0_{i-1}[\dot T](t)\cdot{}^{i-1}_i[T](t)\cdot{}^0_i[T]^{-1}(t)={}^0_{i-1}[\dot T](t)\cdot{}^0_{i-1}[T]^{-1}(t)\\ =\begin{bmatrix}\{{}^0\bar\omega_{i-1}\times\} & \dot{\bar p}_{i-1}(t)-\{{}^0\bar\omega_{i-1}\times\}\cdot\bar p_{i-1}(t)\\ 0\,0\,0 & 0\end{bmatrix}\end{array}\right\} \tag{15}$$

where $\{{}^0\bar{\omega}_{i-1}\times\} = {}^{0}_{i-1}\left[\dot{R}\right]\cdot{}^{0}_{i-1}[R]^T$, (see (13)).

The second matrix term from (14) is shown as:

$$\left\{\begin{array}{c} {}^{0}_{i-1}[T](t)\cdot{}^{i-1}_{i}\left[\dot{T}\right](t)\cdot{}^{0}_{i}[T]^{-1}(t)= \\ = {}^{0}_{i-1}[T](t)\cdot\left\{{}^{i-1}_{i}\left[\dot{T}\right](t)\cdot{}^{i-1}_{i}[T]^{-1}(t)\right\}\cdot{}^{0}_{i-1}[T]^{-1}(t) \\ = \begin{bmatrix} \left[dR\left(\Delta_i\cdot\dot{q}_i\right)\right] & d\bar{p}\left[\left(1-\Delta_i\right)\cdot\dot{q}_i\right] \\ 0\ 0\ 0 & 0 \end{bmatrix} \end{array}\right\} \quad (16)$$

where $\left\{\Delta_i\cdot\dot{q}_i(t)\cdot{}^{i-1}\bar{k}_i\times\right\} = \Delta_i\cdot{}^{i-1}_{i}\left[\dot{R}\right]\cdot{}^{i-1}_{i}[R]^T$. (17)

The components from (16) are developed thus:

$$\left[dR\left(\Delta_i\cdot\dot{q}_i\right)\right] = \left\{\Delta_i\cdot\dot{q}_i(t)\cdot{}^0\bar{k}_i\times\right\}, \quad (18)$$

$$\left\{\begin{array}{l} d\bar{p}\left[\left(1-\Delta_i\right)\cdot\dot{q}_i\right] = \left(1-\Delta_i\right)\cdot\dot{q}_i(t)\cdot{}^0\bar{k}_i - \\ -\left\{\Delta_i\cdot\dot{q}_i(t)\cdot{}^0\bar{k}_i\times\right\}\cdot\left\{\bar{p}_{i-1}+{}^{0}_{i-1}[R]\cdot{}^{i-1}\bar{p}_{ii-1}\right\} \end{array}\right\}. \quad (19)$$

Taking into account on the one hand (12), and on the other hand (14) with the components (15), as well as (16) – (19) the following matrix and differential identity is obtained below:

$$\left\{\begin{array}{l} \begin{bmatrix} \{{}^0\bar{\omega}_i\times\} & \dot{\bar{p}}_i(t)-\{{}^0\bar{\omega}_i\times\}\cdot\bar{p}_i(t) \\ 0\ 0\ 0 & 0 \end{bmatrix} = \\ = \begin{bmatrix} \{{}^0\bar{\omega}_{i-1}\times\} & \dot{\bar{p}}_{i-1}(t)-\{{}^0\bar{\omega}_{i-1}\times\}\cdot\bar{p}_{i-1}(t) \\ 0\ 0\ 0 & 0 \end{bmatrix} + \\ + \begin{bmatrix} \left[dR\left(\Delta_i\cdot\dot{q}_i\right)\right] & d\bar{p}\left[\left(1-\Delta_i\right)\cdot\dot{q}_i\right] \\ 0\ 0\ 0 & 0 \end{bmatrix} \end{array}\right\}. \quad (20)$$

Identifying the angular (rotation) components from the above matrix identity (20), it obtains:

$$\left\{{}^0\bar{\omega}_i\times\right\} = \left\{{}^0\bar{\omega}_{i-1}\times\right\} + \left\{\Delta_i\cdot\dot{q}_i(t)\cdot{}^0\bar{k}_i\times\right\}. \quad (21)$$

In above identity of skew-symmetric matrices, vector equation of angular velocity is selected:

$${}^0\bar{\omega}_i(t) = {}^0\bar{\omega}_{i-1}(t) + \Delta_i\cdot\dot{q}_i(t)\cdot{}^0\bar{k}_i(t). \quad (22)$$

It represents the equation of definition of the angular rotation velocity vector, corresponding to absolute rotation of the kinetic ensemble from MBS with opened kinematical chain (see Fig.3). The last column from (12) – (20) is changed as:

$$\left\{\begin{array}{c} \dot{\bar{p}}_i(t) = \dot{\bar{p}}_{i-1}(t) + \{{}^0\bar{\omega}_{i-1}\times\}\cdot{}^{0}_{i-1}[R]\cdot{}^{i-1}\bar{p}_{ii-1} + \\ +\left(1-\Delta_i\right)\cdot\dot{q}_i(t)\cdot{}^0\bar{k}_i \end{array}\right\}. \quad (23)$$

Using the definition of the linear velocity for the origin of frames: $\{i\}$ and $\{i-1\}$, (see [5] and [7]), the equation (23) is written below as follows:

$$\left\{\begin{array}{c} {}^0\bar{v}_i = {}^0\bar{v}_{i-1} + \{{}^0\bar{\omega}_{i-1}\times\}\cdot{}^{0}_{i-1}[R]\cdot{}^{i-1}\bar{p}_{ii-1} + \\ +\left(1-\Delta_i\right)\cdot\dot{q}_i(t)\cdot{}^0\bar{k}_i \end{array}\right\}. \quad (24)$$

It represents the equation of definition of the linear velocity vector, corresponding to absolute motion of the origin $O_i\in\{i\}$ belonging to kinetic ensemble from MRS with opened chain (Fig.4).

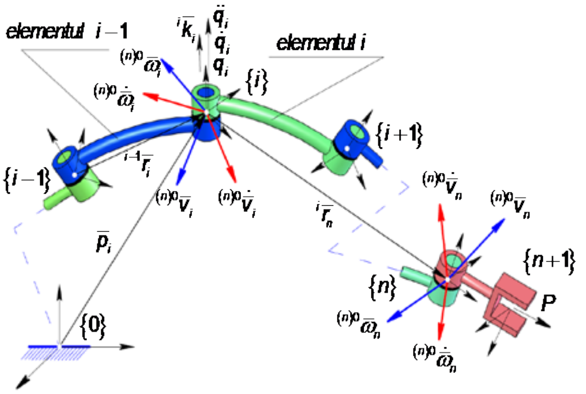


**Fig. 4** Kinematical Parameters for MRS

Applying the absolute time derivatives of first order on (22) and (24), and performing a few differential transformations, the equations of definition for angular and linear accelerations vectors are obtained: ${}^0\dot{\bar{\omega}}_i$ and respectively ${}^0\dot{\bar{v}}_i$. But, especially in the dynamics equations the above kinematical parameters are required by the components with respect to own frame $\{i\}$. The angular and linear velocities and accelerations, corresponding to every kinetic ensemble (Fig.3) are below presented by means of the definition equations with respect to frame $\{i\}$ and $\{0\}$, thus:

$${}^{(0)i}\bar{\omega}_i = {}^{(0)i}_{i-1}[R]\cdot{}^{i-1}\bar{\omega}_{i-1} + \Delta_i\cdot\dot{q}_i\cdot{}^{(0)i}_{i}[R]\cdot{}^{i}\bar{k}_i; \quad (25)$$

$$\left\{\begin{array}{c} {}^{(0)i}\bar{v}_i = {}^{(0)i}_{i-1}[R]\cdot\left\{{}^{i-1}\bar{v}_{i-1} + {}^{i-1}\bar{\omega}_{i-1}\times{}^{i-1}\bar{p}_{ii-1}\right\} + \\ +\left(1-\Delta_i\right)\cdot\dot{q}_i\cdot{}^{(0)i}_{i}[R]\cdot{}^{i}\bar{k}_i \end{array}\right\}; \quad (26)$$

$$\left\{\begin{array}{c} {}^{(0)i}\dot{\bar{\omega}}_i = {}^{(0)i}_{i-1}[R]\cdot{}^{i-1}\dot{\bar{\omega}}_{i-1} + \\ +\Delta_i\cdot\left\{{}^{(0)i}_{i-1}[R]\cdot{}^{i-1}\bar{\omega}_{i-1}\times\dot{q}_i\cdot{}^{(0)i}\bar{k}_i + \ddot{q}_i\cdot{}^{(0)i}\bar{k}_i\right\} \end{array}\right\}; \quad (27)$$

$$\left\{\begin{array}{c} {}^{(0)i}\dot{\bar{v}}_i = {}^{(0)i}_{i-1}[R]\cdot\left[{}^{i-1}\dot{\bar{v}}_{i-1} + {}^{i-1}\dot{\bar{\omega}}_{i-1}\times{}^{i-1}\bar{p}_{ii-1} + \right. \\ \left. + {}^{i-1}\bar{\omega}_{i-1}\times{}^{i-1}\bar{\omega}_{i-1}\times{}^{i-1}\bar{p}_{ii-1}\right] + \\ +\left(1-\Delta_i\right)\cdot{}^{(0)i}_{i}[R]\cdot\left(2\cdot{}^{i}\bar{\omega}_i\times\dot{q}_i\cdot{}^{i}\bar{k}_i + \ddot{q}_i\cdot{}^{i}\bar{k}_i\right) \end{array}\right\}. \quad (28)$$

They are function in exclusivity of parameters included in $[0;i]$ kinematical interval [7]. So, they are applied by outward iterations $i=1\rightarrow n$. When $(i=1)$, within of the equations (25) – (28) the kinematical parameters of the fixed basis from MBS are substituted, according to next:

$$\left\{{}^0\bar{\omega}_0=\bar{0},\ {}^0\dot{\bar{\omega}}_0=\bar{0},\ {}^0\bar{v}_0=0,\ {}^0\dot{\bar{v}}_0=0\right\}. \quad (29)$$

When $(i=n)$, the kinematical parameters of the last kinetic ensemble from MRS are obtained. They are operational velocities and accelerations:

$$\left\{\begin{matrix} {}^{(n)0}\underset{(6\times1)}{\dot{X}}\left[\bar{\theta}(t);\dot{\bar{\theta}}(t)\right]= \\ \left\{{}^{(n)0}\bar{v}_n^T\left[\bar{\theta}(t);\dot{\bar{\theta}}(t)\right] \quad {}^{(n)0}\bar{\omega}_n^T\left[\bar{\theta}(t);\dot{\bar{\theta}}(t)\right]\right\}^T \end{matrix}\right\}; \quad (30)$$

$${}^{(n)0}\underset{(6\times1)}{\ddot{X}}\left[\bar{\theta}(t);\dot{\bar{\theta}}(t);\ddot{\bar{\theta}}(t)\right]=; \quad (31)$$

$$\left\{{}^{(n)0}\dot{\bar{v}}_n^T\left[\bar{\theta}(t);\dot{\bar{\theta}}(t);\ddot{\bar{\theta}}(t)\right] \quad {}^{(n)0}\dot{\bar{\omega}}_n^T\left[\bar{\theta}(t);\dot{\bar{\theta}}(t);\ddot{\bar{\theta}}(t)\right]\right\}^T.$$

The above expressions (30) and (31) represent linear and angular velocities and accelerations corresponding to motion of the last kinetic ensemble of the mechanical structure of the robot relative to absolute Cartesian frame, [5] – [7].

Using algorithm of the matrix exponentials of kinematics (*MEK Algorithm*) from [7], [8] and [20], [21] *the kinematics equations* can be also defined by means of matrix exponentials [22]. So, for every $i=1\rightarrow n$ the next expressions are:

$${}^{0}\bar{\omega}_i=\sum_{j=1}^{i}\left\{\prod_{k=1}^{j-1}exp\left\{\left\{\bar{k}_k^{(0)}\times\right\}q_k\cdot\Delta_k\right\}\right\}\cdot\bar{k}_j^{(0)}\cdot\dot{q}_j\cdot\Delta_j \quad (32)$$

$${}^{i}\bar{\omega}_i=\left\{\left\{R_{i0}^{(0)}\right\}^{-1}\cdot\prod_{j=i}^{1}exp\left\{-\left\{\bar{k}_j^{(0)}\times\right\}q_j\cdot\Delta_j\right\}\right\}\cdot{}^{0}\bar{\omega}_i;$$

$$\left\{\begin{matrix} {}^{0}\dot{\bar{\omega}}_i=\sum_{j=1}^{i}\left\{\prod_{k=1}^{j-1}exp\left\{\left\{\bar{k}_k^{(0)}\times\right\}q_k\cdot\Delta_k\right\}\right\}\cdot\bar{k}_j^{(0)}\cdot\ddot{q}_j\cdot\Delta_j+ \\ +\sum_{j=1}^{i}\sum_{k=0}^{j-1}\left\{\prod_{m=0}^{k}exp\left\{\left\{\bar{k}_m^{(0)}\times\right\}q_m\cdot\Delta_m\right\}\cdot\bar{k}_k^{(0)}\right\}^T\cdot\Omega_{jk}^*,\ and \end{matrix}\right\}$$

$$\Omega_{jk}^*=\left\{\prod_{k=1}^{j-1}exp\left\{\left\{\bar{k}_k^{(0)}\times\right\}q_k\cdot\Delta_k\right\}\right\}\cdot\Delta_j\cdot\Delta_k\cdot\bar{k}_j^{(0)}\cdot\dot{q}_j\cdot\dot{q}_k$$

$$\left\{\begin{matrix} {}^{0}\bar{v}_i=\sum_{j=1}^{i}\left\{\prod_{k=0}^{j-1}exp\left\{\left\{\bar{k}_k^{(0)}\times\right\}q_k\cdot\Delta_k\right\}\cdot\bar{v}_j^{(0)}+\right. \\ +\prod_{l=j}^{i}exp\left\{\left\{\bar{k}_l^{(0)}\times\right\}q_l\cdot\Delta_l\right\}\cdot\bar{p}_i^{(0)}+ \\ \left.+\Delta_j\cdot\prod_{k=0}^{j-1}exp\left\{\left\{\bar{k}_k^{(0)}\times\right\}q_k\cdot\Delta_k\right\}\cdot C_j^*\right\}\cdot\dot{q}_j \end{matrix}\right\}; \quad (33)$$

$$C_j^*=\left\{\bar{k}_j^{(0)}\times\right\}\cdot\left\{\sum_{k=j}^{i}\left\{\prod_{m=j-1}^{k-1}exp\left\{\left\{\bar{k}_m^{(0)}\times\right\}q_m\cdot\delta_m\right\}\cdot\bar{b}_k\right\}\right\};$$

$$\left\{\begin{matrix} \bar{b}_k=\left\{I_3\cdot q_k+\left\{\bar{k}_k^{(0)}\times\right\}\left[1-c\left(q_k\cdot\Delta_k\right)\right]+\right. \\ \left.+\bar{k}_k^{(0)}\cdot\bar{k}_k^{(0)T}\cdot\left[q_k-s\left(q_k\cdot\Delta_k\right)\right]\right\}\cdot\bar{v}_k^{(0)} \end{matrix}\right\}$$

$$\left\{\begin{matrix} {}^{0}\dot{\bar{v}}_i=\sum_{j=1}^{i}\left\{ME\left(V_{j1}\right)\cdot ME\left(V_{j2}\right)\cdot ME\left(V_{j3}\right)\cdot M_{jv}\right\}\cdot\ddot{q}_j+ \\ +\sum_{j=1}^{i}\left\{\frac{d}{dt}\left\{ME\left(V_{j1}\right)\cdot ME\left(V_{j2}\right)\cdot ME\left(V_{j3}\right)\cdot M_{jv}\right\}\right\}\cdot\dot{q}_j \end{matrix}\right\} \quad (34)$$

$$where\ \underset{(3x3)}{ME}\left(V_{i1}\right)=\prod_{j=0}^{i-1}exp\left\{\left\{\bar{k}_j^{(0)}\times\right\}q_j\cdot\Delta_j\right\}; \quad (35)$$

$$\underset{(3\times6)}{ME}\left(V_{i2}\right)=I_3\quad\Delta_i\cdot\left\{\bar{k}_i^{(0)}\times\right\}; \quad (36)$$

$$\underset{[9+3\cdot(n-i)]\times1}{M_{iv}}=\left[\bar{v}_i^{(0)T}\quad\left[\bar{b}_k;\ k=i\rightarrow3\right]^T\quad\bar{p}_n^{(0)T}\right]^T; \quad (37)$$

$$\underset{\{6x[9+3\cdot(n-i)]\}}{ME\left(V_{i3}\right)}=\begin{bmatrix} I_3 & [0] & [0] \\ [0] & ME\left(V_{i322}\right) & ME\left(V_{i323}\right) \end{bmatrix}; \quad (38)$$

$$ME\left(V_{i322}\right)=\left\{\begin{matrix} \left[\begin{matrix}\prod_{m=i-1}^{k-1}exp\left\{\left\{\bar{k}_m^{(0)}\times\right\}q_m\cdot\delta_m\cdot\Delta_m\right\} \\ where\quad k=i\rightarrow n\end{matrix}\right] \\ \delta_m=\left\{\{0;\ m=i-1\};\ \{1;\ m\geq i\}\right\} \end{matrix}\right\},$$

$$and\ ME\left(V_{i323}\right)=\prod_{k=i}^{n}exp\left\{\left\{\bar{k}_k^{(0)}\times\right\}q_k\cdot\Delta_k\right\}. \quad (39)$$

***Remarks**: The matrix exponentials (ME) enjoy important advantages due to their compact form, easy geometric visualization and especially they avoid the frames typical to every kinetic link. As a result the matrix exponentials will stay at the basis of defining the dynamic control functions for whatever mechanical robot structure, regardless of its building complexity.*

## 3. DYNAMICS OF RIGID STRUCTURE

For understanding mechanical significances of the energies of higher order, at beginning *the kinetic energy* is defined, according to [1] – [32]. First of all is taken in study the rigid body in general motion (Fig.5). The starting equation of the kinetic energy is below written as follows:

$$\left\{\begin{matrix} E_C=\frac{1}{2}\cdot\int v_M^2\cdot dm=\frac{1}{2}\cdot\int\bar{v}_M^T\cdot\bar{v}_M\cdot dm= \\ =\frac{1}{2}\cdot\int Trace\left[\bar{v}_M\cdot\bar{v}_M^T\right]\cdot dm \end{matrix}\right\}. \quad (40)$$

$$E_C=\frac{1}{2}\cdot\int\left(\bar{v}_0+\bar{\omega}\times\bar{\rho}_M\right)^T\cdot\left(\bar{v}_0+\bar{\omega}\times\bar{\rho}_M\right)\cdot dm. \quad (41)$$

Equation of kinetic energy for general motion is:

$$E_C=\frac{1}{2}\cdot M\cdot v_0^2+M\cdot\bar{v}_0^T\cdot\left(\bar{\omega}\times\bar{\rho}_C\right)+\frac{1}{2}\cdot\bar{\omega}^T\cdot I_S'\cdot\bar{\omega}. \quad (42)$$

When $O\equiv C,\ \bar{\rho}_C=0,\ and\ I_S'\equiv I_S^*$, (42) becomes:

$$E_C=\frac{1}{2}\cdot M\cdot v_C^2+\frac{1}{2}\cdot\bar{\omega}^T\cdot I_S^*\cdot\bar{\omega}. \quad (43)$$

This is known as König's theorem of the kinetic energy under the explicit form, as well devoted to general motion. In the case of the systems (see Fig.6), above theorem is modified [19] as:

$$\left\{\begin{array}{l}(-1)^{\Delta_M}\cdot\dfrac{1-\Delta_M}{1+3\cdot\Delta_M}\cdot\left\{\dfrac{1}{2}\cdot M_i\cdot{}^i\bar{v}_{C_i}^T\cdot{}^i\bar{v}_{C_i}\right\}+\\ +\Delta_M^2\cdot\dfrac{1}{2}\cdot{}^i\bar{\omega}_i^T\cdot{}^iI_i^*\cdot{}^i\bar{\omega}_i=E_C^i\left[\bar{\theta}(t);\dot{\bar{\theta}}(t)\right]\end{array}\right\}.(44)$$

To this, the operator is added with significance:

$$\Delta_M=\{(-1;general\ motion);(0;translation);(1;rotation)\}$$

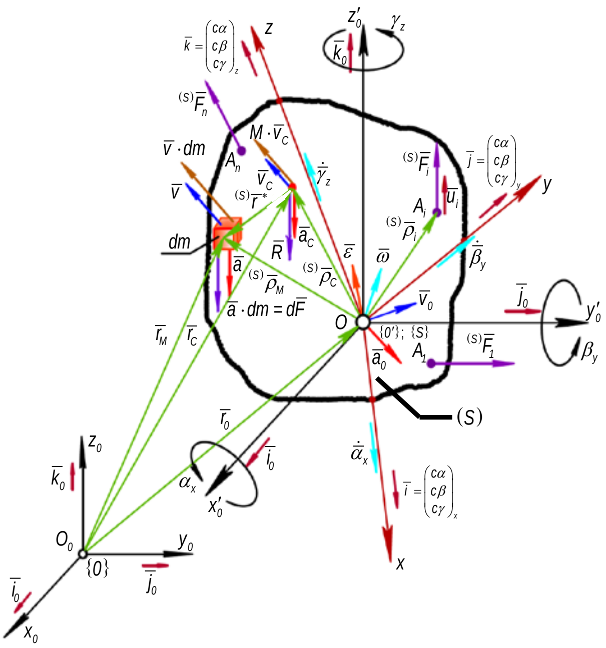


**Fig.5** Free Rigid Body in Cartesian Frame

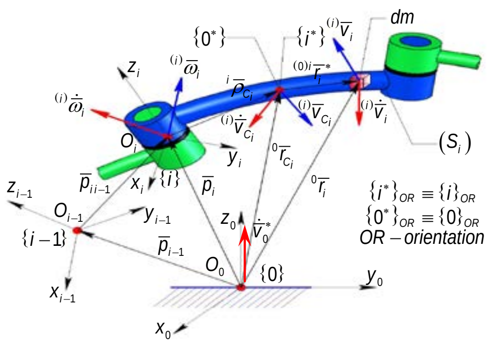


**Fig. 6** Kinetic Ensemble from Robot (MRS)

Considering the notions from others papers of the author [4] – [9], the total kinetic energy of MRS is written by means of the components as follows:

$$\left\{\begin{array}{c}E_C\left[\bar{\theta}(t);\dot{\bar{\theta}}(t)\right]=\\ =\sum_{i=1}^{n}E_C^{iTR}\left[\bar{\theta}(t);\dot{\bar{\theta}}(t)\right]+\sum_{i=1}^{n}E_C^{iROT}\left[\bar{\theta}(t);\dot{\bar{\theta}}(t)\right]\end{array}\right\};(45)$$

The translational and rotation components are changed due to substitution of linear and angular velocities, in accordance with [19]. They are:

$$\left\{\begin{array}{c}\sum_{i=1}^{n}E_C^{iTR}\left[\bar{\theta}(t);\dot{\bar{\theta}}(t)\right]=\\ (-1)^{\Delta_M}\cdot\dfrac{1-\Delta_M}{1+3\cdot\Delta_M}\cdot\dfrac{1}{2}\cdot\sum_{i=1}^{n}M_i\cdot\sum_{j=1}^{k^*=n}\dfrac{1}{m+1}\cdot\dfrac{\partial\,\overset{(m+1)}{\bar{r}_{C_i}}}{\partial\overset{(m)}{q_j}}\cdot\dot{q}_j\end{array}\right\}(46)$$

$$\left\{\begin{array}{c}\sum_{i=1}^{n}E_C^{iROT}\left[\bar{\theta}(t);\dot{\bar{\theta}}(t)\right]=\\ \dfrac{\Delta_M^2}{2}\cdot\sum_{i=1}^{n}\left[\sum_{j=1}^{k^*=n}\dfrac{\partial\overset{(m)}{\bar{\psi}_i}}{\partial\overset{(m)}{q_j}}\cdot\Delta_j\cdot\dot{q}_j\right]\cdot{}^iI_i^*\cdot\left[\sum_{p=1}^{k^*=n}\dfrac{\partial\overset{(m)}{\bar{\psi}_i}}{\partial\overset{(m)}{q_p}}\cdot\Delta_p\cdot\dot{q}_p\right]\end{array}\right\}(47)$$

The expressions (44) and (47) include the inertia tensor axial and centrifugal, relative to $\{i^*\}$:

$${}^iI_i^*=\int\left({}^i\bar{r}_i^*\times\right)\cdot\left({}^i\bar{r}_i^{*T}\times\right)\cdot dm=\begin{bmatrix}{}^iI_x^* & -{}^iI_{xy}^* & -{}^iI_{xz}^*\\ -{}^iI_{yx}^* & {}^iI_y^* & -{}^iI_{yz}^*\\ -{}^iI_{zx}^* & -{}^iI_{zy}^* & {}^iI_z^*\end{bmatrix}.(48)$$

The "*advanced notions*" are found in the analytical dynamics [1]. They are focused on the motion energies, whose central functions are named the accelerations of higher order. They are developed in any sudden and transitory motion of the mechanical systems. The author developed new mathematical formulations on the expressions for acceleration energies of first, second, third and fourth order [7] and [9] – [20]. In this section they will be presented, only in explicit form, typical to the rigid structure.

Considering papers [9] – [20], in following *the acceleration energies of order* $(p\geq 1)$ will be defined. The starting equation shows as follows:

$$\overset{(k-1)}{E}{}_A^{(p)}\left[\bar{\theta}(t);\dot{\bar{\theta}}(t);\cdots;\overset{(p+k)}{\bar{\theta}}(t)\right]=\qquad(49)$$

$$\left\{\begin{array}{c}\dfrac{1}{2}\cdot\sum_{i=1}^{n}Trace\left\{\overset{(p+k)}{{}_i^0[R]}\cdot\left[\int{}^i\bar{r}_i^*\cdot{}^i\bar{r}_i^{*T}\cdot dm+{}^i\bar{r}_{C_i}\cdot{}^i\bar{r}_{C_i}^T\cdot\int dm\right]\cdot\overset{(p+k)}{{}_i^0[R]^T}\right\}\\ +\dfrac{1}{2}\cdot\sum_{i=1}^{n}Trace\left\{\dfrac{d^{k-1}}{dt^{k-1}}\left[\overset{(p+1)}{\bar{p}_i}\cdot\overset{(p+1)}{\bar{p}_i^T}\right]\right\}\cdot\int dm=\\ \dfrac{1}{2}\cdot\sum_{i=1}^{n}Trace\dfrac{d^{k-1}}{dt^{k-1}}\left\{\overset{(p+1)}{{}_i^0[R]}\cdot\left[{}^iI_{pi}^*+M_i\cdot{}^i\bar{r}_{C_i}\cdot{}^i\bar{r}_{C_i}^T\right]\cdot\overset{(p+1)}{{}_i^0[R]^T}\right\}\\ +\dfrac{1}{2}\cdot\sum_{i=1}^{n}Trace\left\{\dfrac{d^{k-1}}{dt^{k-1}}\left[\overset{(p+1)}{\bar{p}_i}\cdot\overset{(p+1)}{\bar{p}_i^T}\right]\right\}\cdot M_i\end{array}\right.$$

$$\left\{\begin{array}{l} where\ p\geq 1,\ \ k\geq 1,\ \{p;\,k\}=\{1;2;3;4;5;.....\} \\ and\ \ \overset{(0)}{E_A^{(p)}}\left[\bar{\theta}(t);\dot{\bar{\theta}}(t);\cdots;\overset{(p+k)}{\bar{\theta}}(t)\right]= \\ =E_A^{(p)}\left[\bar{\theta}(t);\dot{\bar{\theta}}(t);\cdots;\overset{(p+1)}{\bar{\theta}}(t)\right] \end{array}\right\}. \quad (50)$$

The expression (49) includes the inertia tensor planar and centrifugal, relative to the frame $\{i^*\}$:

$$^{i}I_{pi}^{*}=\int {}^{i}\bar{r}_i^{*}\cdot{}^{i}\bar{r}_i^{*T}\cdot dm=\begin{bmatrix} ^{i}I_{xx}^{*} & ^{i}I_{xy}^{*} & ^{i}I_{xz}^{*} \\ ^{i}I_{yx}^{*} & ^{i}I_{yy}^{*} & ^{i}I_{yz}^{*} \\ ^{i}I_{zx}^{*} & ^{i}I_{zy}^{*} & ^{i}I_{zz}^{*} \end{bmatrix}. \quad (51)$$

The acceleration energies will be also defined for rigid body and multibody systems.

In the view of this input parameters of advanced kinematics and mass properties become [3] – [7]. According to papers [9] – [20], the author was established the acceleration energy in the generalized form, corresponding to rigid body founded in the general motion. This was named *acceleration energy of first order*, as follows:

$$\left\{\begin{array}{l} E_A^{(1)}=\dfrac{1}{2}\cdot\int a_M^2\cdot dm=\dfrac{1}{2}\cdot\int\dot{\bar{v}}_M^T\cdot\dot{\bar{v}}_M\cdot dm= \\ \quad =\dfrac{1}{2}\cdot\int Trace\left[\dot{\bar{v}}_M\cdot\dot{\bar{v}}_M^T\right]\cdot dm \end{array}\right\}; \quad (52)$$

$$where\ \ \bar{a}_M=\dot{\bar{v}}_M=\left(\bar{a}_0+\varepsilon\times\bar{\rho}_M+\bar{\omega}\times\bar{\omega}\times\bar{\rho}_M\right);$$

$$\left\{\begin{array}{l} E_A^{(1)}=\dfrac{1}{2}\cdot M\cdot\bar{a}_0^T\cdot\bar{a}_0+M\cdot\bar{a}_0^T\cdot\left(\bar{\varepsilon}\times\bar{\rho}_C\right)+ \\ +M\cdot\bar{a}_0^T\cdot\left(\bar{\omega}\times\bar{\omega}\times\bar{\rho}_C\right)+\dfrac{1}{2}\cdot\bar{\varepsilon}^T\cdot I_S'\cdot\bar{\varepsilon}+ \\ +\bar{\varepsilon}^T\cdot\left(\bar{\omega}\times I_S'\cdot\bar{\omega}\right)+\dfrac{1}{2}\cdot\bar{\omega}^T\cdot\left[\bar{\omega}^T\cdot I_S'\cdot\bar{\omega}\right]\cdot\bar{\omega} \end{array}\right\}; \quad (53)$$

When $O\equiv C,\ \ \bar{\rho}_C=0,\ and\ I_S'\equiv I_S^*$, (98) becomes:

$$\left\{\begin{array}{l} E_A^{(1)}=\dfrac{1}{2}\cdot M\cdot\bar{a}_C^T\cdot\bar{a}_C+\dfrac{1}{2}\cdot\bar{\varepsilon}^T\cdot I_S^*\cdot\bar{\varepsilon}+ \\ +\bar{\varepsilon}^T\cdot\left(\bar{\omega}\times I_S^*\cdot\bar{\omega}\right)+\dfrac{1}{2}\cdot\bar{\omega}^T\cdot\left[\bar{\omega}^T\cdot I_S^*\cdot\bar{\omega}\right]\cdot\bar{\omega} \end{array}\right\}; \quad (54)$$

$$\left\{\begin{array}{c} E_A^{(1)}\left[\bar{\theta}(t);\dot{\bar{\theta}}(t);\ddot{\bar{\theta}}(t)\right]= \\ (-1)^{\Delta_M}\cdot\dfrac{1-\Delta_M}{1+3\cdot\Delta_M}\displaystyle\sum_{i=1}^{n}\left[\dfrac{1}{2}\cdot M_i\cdot{}^{(i)}\dot{\bar{v}}_{C_i}^T\cdot{}^{(i)}\dot{\bar{v}}_{C_i}\right] \\ +\Delta_M^2\cdot\displaystyle\sum_{i=1}^{n}\dfrac{1}{2}\cdot{}^{(i)}\dot{\bar{\omega}}_i^T\cdot{}^{(i)}I_i^*\cdot{}^{(i)}\dot{\bar{\omega}}_i+ \\ +\Delta_M^2\cdot\displaystyle\sum_{i=1}^{n}\left[{}^{(i)}\dot{\bar{\omega}}_i^T\cdot\left({}^{(i)}\bar{\omega}_i\times{}^{(i)}I_i^*\cdot{}^{(i)}\bar{\omega}_i\right)\right]+ \\ +E_A^{(1)}\left[\bar{\theta}(t);\dot{\bar{\theta}}^4(t)\right] \end{array}\right\}; \quad (55)$$

$$\left\{\begin{array}{c} where\quad E_A^{(1)}\left[\bar{\theta}(t);\dot{\bar{\theta}}^4(t)\right]= \\ \Delta_M^2\cdot\displaystyle\sum_{i=1}^{n}\left\{\dfrac{1}{2}\cdot{}^{(i)}\bar{\omega}_i^T\cdot\left[{}^{(i)}\bar{\omega}_i^T\cdot{}^{(i)}I_i^*\cdot{}^{(i)}\bar{\omega}_i^T\right]\cdot{}^{(i)}\bar{\omega}_i^T\right\} \end{array}\right\}. \quad (56)$$

According to [7] – [20], in the case of multibody systems (MRS), the definition equation of the *acceleration energy of first order* is (55) / (56). Considering the notions from papers [7] – [19], the two components (translational and rotation) of acceleration energy of first order show thus:

$$\left\{\begin{array}{l} E_A^{(1)TR}=\dfrac{1}{2}\cdot\displaystyle\sum_{i=1}^{n}M_i\cdot\left\{\sum_{j=1}^{k^*=n}\sum_{p=1}^{k^*=n}\left[\dfrac{\partial^2\overset{(m)}{\bar{r}_{C_i}}}{\partial\overset{(m)}{q_j}\cdot\partial\overset{(m)}{q_p}}\cdot\ddot{q}_j\cdot\ddot{q}_p\right.\right. \\ \quad +\dfrac{1}{(m+1)^2}\cdot\dfrac{\partial^2\overset{(m+1)}{\bar{r}_{C_i}}}{\partial\overset{(m)}{q_j}\cdot\partial\overset{(m)}{q_p}}\cdot\dot{q}_j\cdot\dot{q}_p+ \\ \left.\left.\dfrac{1}{m+1}\cdot\dfrac{\partial\overset{(m)}{\bar{r}_{C_i}}}{\partial\overset{(m)}{q_j}}\cdot\dfrac{\partial\overset{(m+1)}{\bar{r}_{C_i}}}{\partial\overset{(m)}{q_p}}\cdot\ddot{q}_j\cdot\dot{q}_p\right]\right\}=\dfrac{1}{2}\cdot\displaystyle\sum_{i=1}^{n}M_i\cdot a_{C_i}^2 \end{array}\right\} \quad (57)$$

$$\left\{E_A^{(1)ROT\varepsilon}=\dfrac{1}{2}\cdot\sum_{i=1}^{n}\bar{\varepsilon}_i^T\cdot I_i^*\cdot\bar{\varepsilon}_i\right\}= \quad (58)$$

$$\left\{\dfrac{1}{2}\cdot\sum_{i=1}^{n}\sum_{j=1}^{k^*=n}\left[\dfrac{\partial\overset{(m)}{\bar{\psi}_i}}{\partial\overset{(m)}{q_j}}\cdot\Delta_j\cdot\ddot{q}_j+\dfrac{1}{m+1}\cdot\dfrac{\partial\overset{(m+1)}{\bar{\psi}_i}}{\partial\overset{(m)}{q_j}}\cdot\Delta_j\cdot\dot{q}_j\right]^T\cdot I_i^*\cdot\bar{\varepsilon}_i\right\}$$

$$I_i^*\cdot\bar{\varepsilon}=I_i^*\cdot\sum_{j=1}^{k^*}\left[\dfrac{\partial\overset{(m)}{\bar{\psi}_i}}{\partial\overset{(m)}{q_j}}\cdot\Delta_j\cdot\ddot{q}_j+\dfrac{1}{m+1}\cdot\dfrac{\partial\overset{(m+1)}{\bar{\psi}_i}}{\partial\overset{(m)}{q_j}}\cdot\Delta_j\cdot\dot{q}_j\right];$$

$$\left\{E_A^{(1)ROT\omega\varepsilon}=\sum_{i=1}^{n}\bar{\varepsilon}_i^T\cdot\left(\bar{\omega}_i\times I_i^*\cdot\bar{\omega}_i\right)\right\}= \quad (59)$$

$$\left\{\sum_{i=1}^{n}\sum_{j=1}^{k^*=n}\left[\dfrac{\partial\overset{(m)}{\bar{\psi}_i}}{\partial\overset{(m)}{q_j}}\cdot\Delta_j\cdot\ddot{q}_j+\dfrac{1}{m+1}\cdot\dfrac{\partial\overset{(m+1)}{\bar{\psi}_i}}{\partial\overset{(m)}{q_j}}\cdot\Delta_j\cdot\dot{q}_j\right]^T\cdot E_A^{(1)ROT\omega\omega}\right\};$$

$$E_A^{(1)ROT\omega\omega}=\left(\bar{\omega}_i\times I_i^*\cdot\bar{\omega}_i\right)= \quad (60)$$

$$\left\{\sum_{i=1}^{n}\sum_{j=1}^{k^*=n}\sum_{p=1}^{k^*=n}\left[\dfrac{\partial\overset{(m)}{\bar{\psi}_i}}{\partial\overset{(m)}{q_j}}\right]\times\left[I_i^*\cdot\dfrac{\partial\overset{(m)}{\bar{\psi}_i}}{\partial\overset{(m)}{q_p}}\right]\cdot\Delta_j\cdot\Delta_p\cdot\dot{q}_j\cdot\dot{q}_p\right\}.$$

According to the author researches, [9] – [20], the sudden motion of MRS, transient motion phases, as well as mechanical systems subjected to the action of a system of external forces, with time variation law, are dominated by linear and angular accelerations of higher order. As a result acceleration energies of higher order are defined.

# 4. EXPONENTIALS AT FLEXIBLE ROBOT

This section is devoted to define the generalized elastodynamics equations, when the robot links are dominated of flexibility properties. At first, a few kinematic transformations are described. In the aria of the small deflections, and considering the aspects from Fig.7, the time functions for the angular and linear deformations of the link $(i)$ are written, according to [2], [12], [21] – [28] and [32] as:

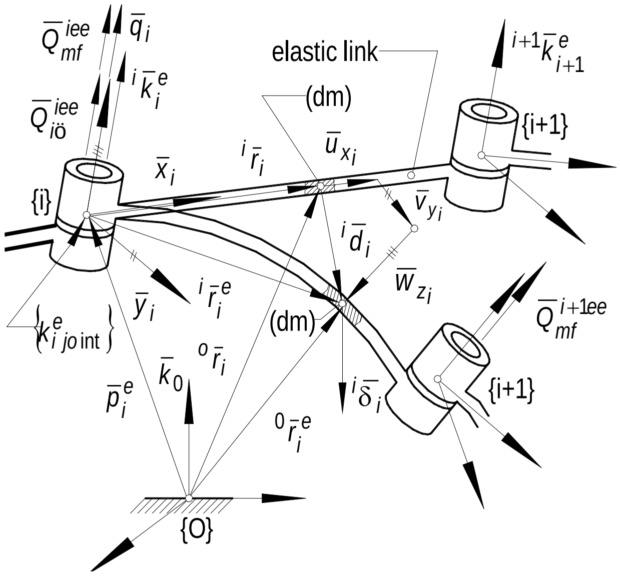


Fig. 7 Elastic Link from Robot Structure

$$ {}^{i}\bar{\delta}_{i}=\begin{pmatrix}\delta_{xi}\\ \delta_{yi}\\ \delta_{zi}\end{pmatrix}=\left\{\sum_{j=1}^{m_i} q_{ij}(t)\cdot {}^{i}\bar{\delta}_{ij}\right\}=\sum_{j=1}^{m_i} q_{ij}(t)\cdot\begin{pmatrix}\delta_{xij}\\ \delta_{yij}\\ \delta_{zij}\end{pmatrix} \quad (61) $$

$$ {}^{i}\bar{d}_{i}=\begin{pmatrix}u_{xi}\\ v_{yi}\\ w_{zi}\end{pmatrix}=\left\{\sum_{j=1}^{m_i} q_{ij}(t)\cdot {}^{i}\bar{d}_{ij}\right\}=\sum_{j=1}^{m_i} q_{ij}(t)\cdot\begin{pmatrix}u_{ij}\\ v_{ij}\\ w_{ij}\end{pmatrix} \quad (62) $$

where $q_{ij}(t)$ are time amplitude of proper modes, and they are completing variables $q_i(t)$. The position vector of the elementary mass $dm$ is:

$$ {}^{i}\bar{r}_{i}^{e}={}^{i}\bar{r}_{i}+{}^{i}\bar{d}_{i}\,;\ {}^{0}\bar{r}_{i}^{e}=\bar{p}_{i}^{e}+R_{i0}^{e}\cdot\left({}^{i}\bar{r}_{i}+{}^{i}\bar{d}_{i}\right). \quad (63) $$

The symbol $(e)$ highlights elasticity of kinetic link. After a few transformations, the locating matrix between adjoining elastic links shows as:

$$ T_{i\,i-1}^{e}=T_{ii-1}\cdot\left\{\begin{bmatrix}I_3 & {}^{i}\bar{r}_{i}\\ 0\,0\,0 & 1\end{bmatrix}+\begin{bmatrix}\left\{{}^{i}\bar{\delta}_{i}\times\right\} & {}^{i}\bar{d}_{i}\\ 0\,0\,0 & 0\end{bmatrix}\right\}; \quad (64) $$

The locating matrix $T_{i\,i-1}$ is answerable to rigid link, while for the small deformations of link is corresponding to matrix operator $\Delta T_{ij}^{e}$, written as:

$$ \Delta T_{ij}^{e}=\left\{\begin{array}{l}\begin{bmatrix}I_3 & {}^{i}\bar{r}_{i}\\ 0\,0\,0 & 1\end{bmatrix}+\begin{bmatrix}\left\{{}^{i}\bar{\delta}_{i}\times\right\} & {}^{i}\bar{d}_{i}\\ 0\,0\,0 & 0\end{bmatrix}=\\ \begin{bmatrix}I_3 & {}^{i}\bar{r}_{i}\\ 0\,0\,0 & 1\end{bmatrix}+\sum_{j=1}^{m_i} q_{ij}\begin{bmatrix}\left\{{}^{i}\bar{\delta}_{ij}\times\right\} & {}^{i}\bar{d}_{ij}\\ 0\,0\,0 & 0\end{bmatrix}\end{array}\right\}; \quad (65) $$

$$ \left\{\begin{array}{l}T_{i\,i-1}^{e}=T_{ii-1}\cdot\Delta T_{ij}^{e}=\\ =T_{i\,i-1}^{ee}+T_{i\,i-1}\cdot\left\{\sum_{j=1}^{m_i} q_{ij}\cdot\left[\begin{bmatrix}\left\{\bar{\delta}_{ij}\times\right\} & \bar{d}_{ij}\\ 0\,0\,0 & 0\end{bmatrix}\right]\right\}=\\ =T_{i\,i-1}^{ee}+T_{i\,i-1}\cdot\left\{\sum_{j=1}^{m_i} q_{ij}\cdot\Delta T_{ij}\right\}\end{array}\right\}; \quad (66) $$

$$ \left\{\begin{array}{l}T_{i0}^{e}=\prod_{j=1}^{i-1}T_{j\,j-1}^{e}\cdot T_{i\,i-1}=\\ \prod_{j=1}^{i-1}\left\{T_{j\,j-1}^{ee}+T_{j\,j-1}\cdot\left\{\sum_{k=1}^{m_j} q_{jk}\cdot\left[\begin{bmatrix}\left\{\bar{\delta}_{jk}\times\right\} & \bar{d}_{jk}\\ 0\,0\,0 & 0\end{bmatrix}\right]\right\}\right\}\cdot T_{i\,i-1}\end{array}\right\} \quad (67) $$

The above *kinematic transformations* (66)/(67) can be also obtained by means of *the matrix exponentials*. According to [12], [21] and [32], the exponentials are applied for elastic links as:

$$ T_{i\,i-1}^{e}=T_{i\,i-1}^{ee(0)}\cdot exp\left\{U_i\cdot q_i\right\}+T_{i\,i-1}^{(0)}\cdot\prod_{j=1}^{m_i} exp\left\{\Delta T_{ij}\cdot q_{ij}\right\}; \quad (68) $$

$$ \left\{\begin{array}{l}exp\left\{\Delta T_{ij}\cdot q_{ij}\right\}=exp\left\{\begin{bmatrix}\left\{\bar{\delta}_{ij}\times\right\} & \bar{d}_{ij}\\ 0\,0\,0 & 0\end{bmatrix}\cdot q_{ij}\right\}\\ =exp\left\{\begin{bmatrix}exp\left\{\bar{\delta}_{ij}\times\right\}q_{ij} & \bar{b}_{ij}\\ 0\,0\,0 & 1\end{bmatrix}\right\}\end{array}\right\}; \quad (69) $$

$$ exp\left\{\bar{\delta}_{ij}\times\right\}q_{ij}=exp\left\{\begin{bmatrix}0 & -\delta_{zij} & \delta_{yij}\\ \delta_{zij} & 0 & -\delta_{xij}\\ -\delta_{yij} & \delta_{xij} & 0\end{bmatrix}\cdot q_{ij}\right\}=; \quad (70) $$

$$ =exp\left\{U_{ix}\cdot\delta_{xij}\cdot q_{ij}\right\}\cdot exp\left\{U_{iy}\cdot\delta_{yij}\cdot q_{ij}\right\}\cdot exp\left\{U_{iz}\cdot\delta_{zij}\cdot q_{ij}\right\}; $$

$$ \left\{\begin{array}{l}exp\left\{U_{iu}\cdot\delta_{uij}\cdot q_{ij}\right\}\\ where\ u=\{x;y;z\}\end{array}\right\}=\bar{u}_{i}^{(0)}\cdot\bar{u}_{i}^{(0)T}\cdot\left[1-c\left(\delta_{xij}\cdot q_{ij}\right)\right]+ $$

$$ +I_3\cdot c\left(\delta_{xij}\cdot q_{ij}\right)+\left\{\bar{u}_{i}^{(0)}\times\right\}s\left(\delta_{uij}\cdot q_{ij}\right); \quad (71) $$

$$ \left\{\begin{array}{l}T_{i0}^{e}=\prod_{j=1}^{i-1}\left\{T_{j\,j-1}^{ee(0)}\cdot exp\left\{U_j\cdot q_j\right\}\right\}\cdot T_{i\,i-1}^{(0)}\cdot exp\left\{U_i\cdot q_i\right\}\\ +\prod_{j=1}^{i-1}\left\{T_{j\,j-1}^{(0)}\cdot\prod_{k=1}^{m_j}\left\{q_{jk}\cdot\Delta T_{jk}\right\}\right\}\cdot T_{i\,i-1}^{(0)}\cdot exp\left\{U_i\cdot q_i\right\}\end{array}\right\} \quad (72) $$

On the locating matrix (72), which it expresses the locating of the frame $\{i\}$ with respect to fixed basis, the time derivatives of first and second order are applied. The expressions are:

$$\left\{\begin{array}{c}\dot{T}^{e}_{i0}=\begin{bmatrix}\dot{R}^{e}_{i0} & \dot{\bar{p}}^{e}_{i}\\ 0\,0\,0 & 0\end{bmatrix}=\\ \sum_{k=1}^{i}T^{e}_{k0}\cdot U_k\cdot T^{e}_{ik}\cdot\dot{q}_k+\sum_{k=1}^{i-1}\sum_{l=1}^{m_k}T_{k\,k-1}\varDelta T^{e}_{kl}\cdot T^{e}_{il}\cdot\dot{q}_{kl}\end{array}\right\}; \quad (73)$$

$$\ddot{T}^{e}_{i0}=\begin{bmatrix}\ddot{R}^{e}_{i0} & \ddot{\bar{p}}^{e}_{i}\\ 0\,0\,0 & 0\end{bmatrix}=\ddot{T}^{e}_{i0A}+\ddot{T}^{e}_{i0B}+\ddot{T}^{e}_{i0C}; \quad (74)$$

$$\ddot{T}^{e}_{i0A}=\sum_{k=1}^{i}T^{e}_{k0}\cdot U_k\cdot T^{e}_{ik}\cdot\ddot{q}_k+\sum_{k=1}^{i-1}\sum_{l=1}^{m_i}T_{k\,k-1}\cdot\varDelta T^{e}_{kl}\cdot T^{e}_{il}\cdot\ddot{q}_{kl} \quad (75)$$

$$\ddot{T}^{e}_{i0B}=\left\{\begin{array}{c}\sum_{k=1}^{i}\sum_{m=1}^{k}T^{e}_{m0}\cdot U_m\cdot T^{e}_{km}\cdot U_k\cdot T^{e}_{ik}\cdot\dot{q}_m\cdot\dot{q}_k+\\ +\sum_{k=1}^{i}\sum_{l=k}^{i}T^{e}_{k0}\cdot U_k\cdot T^{e}_{lk}\cdot U_l\cdot T^{e}_{il}\cdot\dot{q}_l\cdot\dot{q}_k+\\ +\sum_{k=1}^{i-1}\sum_{l=1}^{m_i}\dot{T}_{kk-1}\cdot\varDelta T^{e}_{kl}\cdot T^{e}_{il}\cdot\dot{q}_k\cdot\dot{q}_{kl}\end{array}\right\}; \quad (76)$$

$$\ddot{T}^{e}_{i0C}=\sum_{k=1}^{i-1}\sum_{l=1}^{m_i}\sum_{m=l}^{i}T_{k\,k-1}\cdot\varDelta T^{e}_{kl}\cdot T^{e}_{mk}\cdot U_m\cdot T^{e}_{il}\cdot\dot{q}_{kl}\cdot\dot{q}_m \quad (77)$$

In these expressions are substituted the matrices defined by means of the exponentials above shown. Considering (73) and (74), the angular rotation velocity and acceleration are defined as:

$$vect\{\,{}^{0}\bar{\omega}^{e}_{i}\times\}=vect\{\dot{R}^{e}_{i0}\cdot R^{eT}_{i0}\}=\left[\,{}^{0}\omega^{e}_{ix}\quad {}^{0}\omega^{e}_{iy}\quad {}^{0}\omega^{e}_{iz}\right]^{T};$$

$${}^{0}\dot{\bar{\omega}}^{e}_{i}=vect\left\{\ddot{R}^{e}_{i0}\cdot\left\{R^{e}_{i0}\right\}^{T}+\dot{R}^{e}_{i0}\cdot\left\{\dot{R}^{e}_{i0}\right\}^{T}\right\}. \quad (78)$$

The linear velocity and acceleration of the elementary mass $dm$ are defined by means of the time derivative applied on the position vector (63). As a result, next expressions are:

$${}^{0}\bar{r}^{e}_{i}=\bar{p}^{e}_{i}+R^{e}_{i0}\cdot\left({}^{i}\bar{r}_{i}+{}^{i}\bar{d}_{i}\right)+R^{e}_{i0}\cdot{}^{i}\bar{d}_{i};$$

$$\left\{\begin{array}{l}{}^{\dot{0}}\bar{r}^{e}_{i}=\dot{\bar{p}}^{e}_{i}+\dot{R}^{e}_{i0}\cdot\left({}^{\bar{\imath}}r_i+{}^{\bar{\imath}}d_i\right)+R^{e}_{i0}\cdot{}^{\dot{\bar{\imath}}}d_i=\\ =\dot{\bar{p}}^{e}_{i}+{}^{0}\bar{\omega}^{e}_{i}\times R^{e}_{i0}\cdot\left({}^{\bar{\imath}}r_i+{}^{\bar{\imath}}d_i\right)+R^{e}_{i0}\cdot{}^{\dot{\bar{\imath}}}d_i\end{array}\right\}; \quad (79)$$

$$\left\{\begin{array}{c}{}^{\ddot{0}}\bar{r}^{e}_{i}=\ddot{\bar{p}}^{e}_{i}+2\cdot{}^{0}\bar{\omega}^{e}_{i}\times R^{e}_{i0}\cdot{}^{\dot{\bar{\imath}}}d_i+R^{e}_{i0}\cdot{}^{\ddot{\bar{\imath}}}d_i+\\ {}^{\dot{0}}\bar{\omega}^{e}_{i}\times R^{e}_{i0}\left({}^{\bar{\imath}}r_i+{}^{\bar{\imath}}d_i\right)+{}^{0}\bar{\omega}^{e}_{i}\times{}^{0}\bar{\omega}^{e}_{i}\times R^{e}_{i0}\cdot\left({}^{\bar{\imath}}r_i+{}^{\bar{\imath}}d_i\right)\end{array}\right\} \quad (80)$$

The column vector of the generalized variables, in the case of the structures with flexible links, is completed with (61) and (62) as below:

$$\bar{\theta}^{e}(t)=\left[\left[\theta^{eT}_{ij}(t)\;\; j=0\to m_i\right]\;\; i=1\to n\right]^{T}; \quad (81)$$

$$\theta^{eT}_{ij}(t)=\left\{\left\{q_i(t)\ if\ j=0\right\};\left\{q_{ij}(t)\ if\ j\geq 1\right\}\right\}; \quad (82)$$

$$\dot{\bar{\theta}}^{e}(t)=\left[\dot{\bar{\theta}}^{eT}_{ij}(t)\left[\dot{\theta}^{eT}_{ij}(t)\quad j=0\to m_i\right]\quad i=1\to n\right]^{T};$$

$$\dot{\theta}^{eT}_{ij}(t)=\left\{\{\dot{q}_i(t) if j=0\};\{\dot{q}_{ij}(t) if j\geq 1\}\right\}; \quad (83)$$

$$\ddot{\bar{\theta}}^{e}(t)=\left[\ddot{\bar{\theta}}^{eT}_{ij}(t)=\left[\ddot{\theta}^{eT}_{ij}(t)\quad j=0\to m_i\right]\quad i=1\to n\right]^{T};$$

$$\ddot{\theta}^{eT}_{ij}(t)=\left\{\{\ddot{q}_i(t) if j=0\};\{\ddot{q}_{ij}(t) if j\geq 1\}\right\}; \quad (84)$$

$$\left\{{}^{0}J^{e}_{i}\begin{pmatrix}q_j(t) & q_{jk}(t)\\ j=1\to i & k=1\to m_j\end{pmatrix}=\begin{bmatrix}{}^{0}J^{e}_{iv}\\ {}^{0}J^{e}_{i\omega}\end{bmatrix}\right\}\in{}^{0}J(\bar{\theta})^{e}. \quad (85)$$

The above expression shows that every column of Jacobian matrix is function of generalized variables. Considering [8], [12] and [21], its expression is defined by means of the classical transformations or matrix exponentials.

## 5. ELASTODYNAMICS EQUATIONS

This section is devoted to establishment the generalized elastodynamics forces: generalized inertia forces, as well generalized active forces answerable to gravity and manipulating load.

On the basis of the *NE-type equations* in this section an elastodynamics expressions are determined, [2] and [12]. Unlike the expressions answerable to structure with rigid links, [7] [10] [20], *the generalized inertia forces* are changed for robot structures with flexible links. First of all, resultant active force in the new elastic restrictions is characterized by the equations:

$$\left\{\begin{array}{c}{}^{i}\bar{F}^{e}_{i}=\left[{}^{i}F^{e}_{ix}\;\;{}^{i}F^{e}_{iy}\;\;{}^{i}F^{e}_{iz}\right]^{T}=\\ =M_i\cdot{}^{i}\dot{\bar{v}}^{e}_{C_i}+\sum_{j=1}^{m_i}\left({}^{i}\dot{\bar{\omega}}^{e}_{i}\times q_{ij}+{}^{i}\bar{\omega}^{e}_{i}\times{}^{i}\bar{\omega}^{e}_{i}\times q_{ij}+\right.\\ \left.+2\cdot{}^{i}\bar{\omega}^{e}_{i}\times\dot{q}_{ij}+R^{e}_{i0}\cdot\ddot{q}_{ij}\right)\cdot\int_{link}{}^{i}\bar{d}_{ij}\,dm\end{array}\right\}; \quad (86)$$

$$where\ M_i\cdot{}^{0}\dot{\bar{v}}^{e}_{C_i}=\int_{link}\ddot{\bar{p}}^{e}_{i}\cdot dm=M_i\cdot\left(\ddot{\bar{p}}^{e}_{i}+\ddot{R}^{e}_{i0}\cdot{}^{i}\bar{r}^{e}_{C_i}\right). \quad (87)$$

Above expression is corresponding to the rigid ensemble. Performing a few transformations, the resultant moment of the active forces becomes:

$$\left\{\begin{array}{c}{}^{i}\bar{N}^{e}_{i}={}^{i}\bar{N}^{e}_{i\omega}+\sum_{j=1}^{m_i}\ddot{q}_{ij}\cdot{}^{i}I^{e}_{i\,rd}+\sum_{j=1}^{m_i}\sum_{k=1}^{m_i}q_{ij}\cdot\ddot{q}_{ik}\cdot{}^{i}I^{e}_{ik\,dd}+\\ +\left\{{}^{i}I^{e}_{i}+2\sum_{j=1}^{m_i}q_{ij}\cdot{}^{i}I^{e}_{i\,rd}+\sum_{j=1}^{m_i}\sum_{k=1}^{m_i}q_{ij}\cdot q_{ik}\cdot{}^{i}I^{e}_{ik\,dd}\right\}{}^{i}\dot{\bar{\omega}}^{e}_{i}\end{array}\right\} \quad (88)$$

$$\left\{\begin{array}{c}{}^{i}\bar{N}^{e}_{i\omega}=2\left\{\int_{link}\left\{{}^{i}\bar{d}_{i}\times\right\}\left\{{}^{i}\dot{\bar{d}}_{i}\times\right\}^{T}dm\right\}{}^{i}\bar{\omega}^{e}_{i}+\\ +{}^{i}\bar{\omega}^{e}_{i}\times{}^{i}I_{ir\dot{d}\dot{d}}\cdot{}^{i}\bar{\omega}^{e}_{i}\end{array}\right\}. \quad (89)$$

According to [12] and [32], the mass elements (88) and (89) are pseudoinertial matrices, thus:

$$\sum_{j=1}^{m_i}q_{ij}\cdot{}^{i}I^{e}_{i\,rd}=\int_{link}\left\{{}^{i}\bar{r}_{i}\times\right\}\left\{{}^{i}\bar{d}_{i}\times\right\}^{T}dm;$$

$$\sum_{j=1}^{m_i} q_{ij}\cdot {}^iI^e_{i\,rd} = \sum_{j=1}^{m_i} q_{ij}\int_{link}\left\{ {}^i\overline{r}_i \times\right\}\left\{ {}^i\overline{d}_{ij}\times\right\}^T dm\,; \quad (90)$$

$$\left\{\begin{array}{c} \sum_{j=1}^{m_i}\ddot{q}_{ij}\cdot {}^iI^e_{i\,rd} = \int_{link}\left\{ {}^i\overline{r}_i\times\right\}\left\{ {}^i\ddot{\overline{d}}_i\times\right\}^T dm \\ = \sum_{j=1}^{m_i}\ddot{q}_{ij}\int_{link}\left\{ {}^i\overline{r}_i\times\right\}\left\{ {}^i\overline{d}_{ij}\times\right\}^T dm \end{array}\right\}; \quad (91)$$

$$\left\{\begin{array}{c} \sum_{j=1}^{m_i}\sum_{k=1}^{m_i} q_{ij}\cdot q_{ik}\cdot {}^iI^e_{ik\,dd} = \int_{link}\left\{ {}^i\overline{d}_i\times\right\}\left\{ {}^i\overline{d}_i\times\right\}^T dm \\ \sum_{j=1}^{m_i}\sum_{k=1}^{m_i} q_{ij}\cdot q_{ik}\int_{link}\left\{ {}^i\overline{d}_{ij}\times\right\}\left\{ {}^i\overline{d}_{ik}\times\right\}^T dm \end{array}\right\}; \quad (92)$$

$$\left\{\begin{array}{c} \sum_{j=1}^{m_i}\sum_{k=1}^{m_i} q_{ij}\cdot \dot{q}_{ik}\cdot {}^iI^e_{ik\,dd} = \int_{link}\left\{ {}^i\overline{d}_i\times\right\}\left\{ {}^i\dot{\overline{d}}_i\times\right\}^T dm \\ \sum_{j=1}^{m_i}\sum_{k=1}^{m_i} q_{ij}\cdot \dot{q}_{ik}\int_{link}\left\{ {}^i\overline{d}_{ij}\times\right\}\left\{ {}^i\overline{d}_{ik}\times\right\}^T dm \end{array}\right\};$$

$$\left\{\begin{array}{c} \sum_{j=1}^{m_i}\sum_{k=1}^{m_i} q_{ij}\cdot \ddot{q}_{ik}\cdot {}^iI^e_{ik\,dd} = \int_{link}\left\{ {}^i\overline{d}_i\times\right\}\left\{ {}^i\ddot{\overline{d}}_i\times\right\}^T dm \\ \sum_{j=1}^{m_i}\sum_{k=1}^{m_i} q_{ij}\cdot \ddot{q}_{ik}\int_{link}\left\{ {}^i\overline{d}_{ij}\times\right\}\left\{ {}^i\overline{d}_{ik}\times\right\}^T dm \end{array}\right\}; \quad (93)$$

$$\left\{\begin{array}{c} {}^iI_{ird\dot{d}\ddot{d}} = \int_{link}\left\{ {}^i\overline{r}_i\times\right\}\left\{ {}^i\overline{r}_i\times\right\}^T\cdot dm + \\ +2\left\{\int_{link}\left\{ {}^i\overline{r}_i\times\right\}\left\{ {}^i\overline{d}_i\times\right\}^T dm\right\} + \\ +\left\{\int_{link}\left\{ {}^i\overline{d}_i\times\right\}\left\{ {}^i\overline{d}_i\times\right\}^T\cdot dm\right\} + \int_{link}\left\{ {}^i\overline{d}_i\times\right\}\left\{ {}^i\dot{\overline{d}}_i\times\right\}^T dm \\ +\int_{link}\left\{ {}^i\overline{r}_i\times\right\}\left\{ {}^i\ddot{\overline{d}}_i\times\right\}^T dm + \int_{link}\left\{ {}^i\overline{d}_i\times\right\}\left\{ {}^i\ddot{\overline{d}}_i\times\right\}^T dm \end{array}\right\}$$

$${}^iI^e_i = \int_{link}\left\{ {}^i\overline{r}_i\times\right\}\left\{ {}^i\overline{r}_i\times\right\}^T\cdot dm\,. \quad (94)$$

$$\left\{\begin{array}{c} {}^i\overline{N}^e_{i\,\omega} = 2\cdot\sum_{j=1}^{m_i}\sum_{k=1}^{m_i} q_{ij}\cdot\dot{q}_{ik}\cdot {}^iI^e_{ik\,dd}\cdot {}^i\overline{\omega}^e_i + \\ {}^i\overline{\omega}^e_i\times\left\{ {}^iI^e_i + 2\sum_{j=1}^{m_i} q_{ij}\cdot {}^iI^e_{i\,rd} + \sum_{j=1}^{m_i}\sum_{k=1}^{m_i} q_{ij}\cdot q_{ik}\cdot {}^iI^e_{ik\,dd} + \right. \\ +\sum_{j=1}^{m_i}\sum_{k=1}^{m_i} q_{ij}\cdot\dot{q}_{ik}\cdot {}^iI^e_{ik\,dd} + \sum_{j=1}^{m_i}\ddot{q}_{ij}\cdot {}^iI^e_{i\,rd} \\ \left. +\sum_{j=1}^{m_i}\sum_{k=1}^{m_i} q_{ij}\cdot\ddot{q}_{ik}\cdot {}^iI^e_{ik\,dd}\right\}\cdot {}^i\overline{\omega}^e_i \end{array}\right\} (95)$$

Among of these, the (94) is the inertial tensor axial and centrifugal typical to the rigid link with respect to moving frame $\{i\}$. The others are *the inertia matrices*, in accordance with [6].

Applying the transfer matrices, according to [12], *the generalized inertia forces* show as:

$$Q^{i\,ee}_{i\ddot{o}} = {}^0J^e_i\cdot {}^0\ddot{o}^{-\,*ee}_{X\,i} = {}^0J^e_i\cdot\left[ {}^0\overline{F}^{*eeT}_{X\,i} \quad {}^0\overline{F}^{*eeT}_{X\,i}\right]^T; \quad (96)$$

$$\text{where}\quad {}^0\overline{F}^{*ee}_{X\,i} = \sum_{j=i}^{n} R^e_{j0}\cdot {}^j\overline{F}^e_j\,;$$

$$\text{and}\quad {}^0\overline{N}^{*ee}_{X\,i} = \sum_{j=i}^{n}\left\{\,\overline{p}^e_{jn}\times R^e_{j0}\cdot {}^j\overline{F}^e_j + R^e_{j0}\cdot {}^j\overline{N}^e_j\right\}.$$

In the following, the generalized inertia forces will be also determined by means of the *LE-type equations*. At first, *the kinetic energy* answerable to a flexible link is established [12]:

$$\left\{\begin{array}{c} \dot{\overline{r}}^{ee}_i = \dot{T}^e_{i0}\cdot {}^i\overline{r}_i + \dot{T}^e_{i0}\cdot {}^i\overline{d}_i + T^e_{i0}\cdot {}^i\dot{\overline{d}}_i \\ \dot{\overline{r}}^{eeT}_i = {}^i\overline{r}^T_i\cdot\dot{T}^{eT}_{i0} + {}^i\overline{d}^T_i\cdot\dot{T}^{eT}_{i0} + {}^i\dot{\overline{d}}^T_i\cdot T^{eT}_{i0} \end{array}\right\}; \quad (97)$$

$$E^{iee}_C = \frac{1}{2}\cdot\int_{link} Trace\left[\dot{\overline{r}}^{ee}_i\cdot\dot{\overline{r}}^{eeT}_i\right]\cdot dm =$$

$$\left\{\begin{array}{c} =\frac{1}{2}\cdot Tr\left[\dot{T}^e_{i0}\cdot {}^iI^e_{psi}\cdot\dot{T}^{eT}_{i0}\right] + \\ +\frac{1}{2}\cdot\sum_{j=1}^{m_i}\sum_{k=1}^{m_i} Tr\left[\dot{T}^e_{i0}\cdot {}^iI^e_{ps\,jk\,dd}\cdot\dot{T}^{eT}_{i0}\right]\cdot q_{ij}\cdot q_{ik} + \\ +\frac{1}{2}\cdot\sum_{j=1}^{m_i}\sum_{k=1}^{m_i} Tr\left[T^e_{i0}\cdot {}^iI^e_{ps\,jk\,dd}\cdot T^{eT}_{i0}\right]\cdot\dot{q}_{ij}\cdot\dot{q}_{ik} + \\ +\sum_{j=1}^{m_i} Tr\left[\dot{T}^e_{i0}\cdot {}^iI^e_{ps\,i\,rd}\cdot\dot{T}^{eT}_{i0}\right]\cdot q_{ij} + \\ +\sum_{j=1}^{m_i} Tr\left[\dot{T}^e_{i0}\cdot {}^iI^e_{ps\,i\,rd}\cdot T^{eT}_{i0}\right]\cdot\dot{q}_{ij} + \\ +\sum_{j=1}^{m_i}\sum_{k=1}^{m_i} Tr\left[\dot{T}^e_{i0}\cdot {}^iI^e_{ps\,jk\,dd}\cdot T^{eT}_{i0}\right]\cdot q_{ij}\cdot\dot{q}_{ik} \end{array}\right\}. \quad (98)$$

The column vector of generalized velocities is defined, from (82), by time derivative as below:

$$\left\{\begin{array}{c} \dot{\overline{\theta}}^e(t) = \left[\dot{\theta}^{eT}_{ij}(t)\ \text{where}\ j=0\to m_i;\ i=1\to n\right]^T \\ \dot{\theta}^{eT}_{ij}(t) = \left\{\left\{\dot{q}_i(t)\ \text{if}\ j=0\right\};\left\{\dot{q}_{ij}(t)\ \text{if}\ j\geq 1\right\}\right\} \end{array}\right\}.$$

Considering that mechanical structure of robot is characterized through ($n$) flexible links, *the kinetic energy* in the matrix form, which it expresses the elastodynamics behavior of the robot structure, is shown in the following as:

$$\left\{\begin{array}{c} E^{ee}_C\left(\overline{\theta}^e;\dot{\overline{\theta}}^e\right) = \sum_{i=1}^{n} E^{iee}_C\left(\overline{\theta}^{eT}_{jk};\dot{\overline{\theta}}^{eT}_{jk};\ j=1\to i\right) \\ =\frac{1}{2}\sum_{i=1}^{n}\sum_{j=0}^{m_i}\sum_{k=1}^{n}\sum_{l=0}^{m_k} M^{ee}_{ijkl}\left(\overline{\theta}^e\right)\cdot\dot{\overline{\theta}}^e_{ij}\cdot\dot{\overline{\theta}}^e_{kl} \end{array}\right\}; \quad (99)$$

$$\left\{\begin{array}{c} E^{ee}_C\left(\overline{\theta}^e;\dot{\overline{\theta}}^e\right) = \frac{1}{2}\cdot\dot{\overline{\theta}}^{eT}\cdot M^{ee}\left(\overline{\theta}^e\right)\cdot\dot{\overline{\theta}}^e = \\ \frac{1}{2}\cdot\left[\dot{\overline{\theta}}^{eT}_{ij}\ \ i=1\to n\right]^T\cdot M^{ee}\left(\overline{\theta}^e\right)\cdot\left[\dot{\overline{\theta}}^{eT}_{ij}\ \ i=1\to n\right] \end{array}\right\}.$$

Above, $M^{ee}\left(\bar\theta^{e}\right)$ is the inertia matrix, positive and symmetric, typical to the kinetic energy.

In keeping with the papers [1] and [12], the generalized inertia forces can be likewise determined by means of the generalization of the Appell's equations. In the view of this, the acceleration energy of first order for flexible link from mechanical robot structure (MRS) is established with elastodynamics expressions:

$$\left\{\begin{aligned} \ddot{\bar r}_i^{ee} &= \ddot T_{i0}^{e} \cdot {}^{i}\bar r_i + \ddot T_{i0}^{e} \cdot {}^{i}\bar d_i + 2\cdot \dot T_{i0}^{e} \cdot {}^{i}\dot{\bar d}_i + T_{i0}^{e} \cdot {}^{i}\ddot{\bar d}_i \\ \ddot{\bar r}_i^{eeT} &= {}^{i}\bar r_i^{T} \cdot \ddot T_{i0}^{eT} + {}^{i}\bar d_i^{T} \cdot \ddot T_{i0}^{eT} + 2\cdot {}^{i}\dot{\bar d}_i^{T} \cdot \dot T_{i0}^{eT} + {}^{i}\ddot{\bar d}_i^{T} \cdot T_{i0}^{eT} \end{aligned}\right\};$$

$$E_A^{1iee} = \frac{1}{2}\cdot \int_{link} Trace\left[\ddot{\bar r}_i^{ee} \cdot \ddot{\bar r}_i^{eeT}\right] \cdot dm = \qquad (100)$$

$$\left\{\begin{aligned} &= \frac{1}{2}\cdot Tr\left[\ddot T_{i0}^{e} \cdot {}^{i}I_{psi}^{e} \cdot \ddot T_{i0}^{eT}\right] + \\ &+ \frac{1}{2}\cdot \sum_{j=1}^{m_i}\sum_{k=1}^{m_i} Tr\left[\ddot T_{i0}^{e} \cdot {}^{i}I_{ps\,jk\,dd}^{e} \cdot \ddot T_{i0}^{eT}\right]\cdot q_{ij}\cdot q_{ik} + \\ &+ \frac{1}{2}\cdot \sum_{j=1}^{m_i}\sum_{k=1}^{m_i} Tr\left[T_{i0}^{e} \cdot {}^{i}I_{ps\,jk\,dd}^{e} \cdot T_{i0}^{eT}\right]\cdot \ddot q_{ij}\cdot \ddot q_{ik} + \\ &+ 2\sum_{j=1}^{m_i}\sum_{k=1}^{m_i} Tr\left[\dot T_{i0}^{e} \cdot {}^{i}I_{ps\,jk\,dd}^{e} \cdot \dot T_{i0}^{eT}\right]\cdot \dot q_{ij}\cdot \dot q_{ik} + \\ &+ 2\sum_{j=1}^{m_i} Tr\left[\ddot T_{i0}^{e} \cdot {}^{i}I_{ps\,i\,rd}^{e} \cdot \dot T_{i0}^{eT}\right]\cdot \dot q_{ij} \\ &+ 2\sum_{j=1}^{m_i}\sum_{k=1}^{m_i} Tr\left[\ddot T_{i0}^{e} \cdot {}^{i}I_{ps\,jk\,dd}^{e} \cdot \dot T_{i0}^{eT}\right]\cdot q_{ij}\cdot \dot q_{ik} + \\ &+ 2\sum_{j=1}^{m_i}\sum_{k=1}^{m_i} Tr\left[\dot T_{i0}^{e} \cdot {}^{i}I_{ps\,jk\,dd}^{e} \cdot T_{i0}^{eT}\right]\cdot \dot q_{ij}\cdot \ddot q_{ik} + \\ &+ \sum_{j=1}^{m_i} Tr\left[\ddot T_{i0}^{e} \cdot {}^{i}I_{ps\,i\,rd}^{e} \cdot \ddot T_{i0}^{eT}\right]\cdot q_{ij} + \sum_{j=1}^{m_i} Tr\left[\ddot T_{i0}^{e} \cdot {}^{i}I_{ps\,i\,rd}^{e} \cdot T_{i0}^{eT}\right]\cdot \ddot q_{ij} + \\ &+ \sum_{j=1}^{m_i}\sum_{k=1}^{m_i} Tr\left[\ddot T_{i0}^{e} \cdot {}^{i}I_{ps\,jk\,dd}^{e} \cdot T_{i0}^{eT}\right]\cdot q_{ij}\cdot \ddot q_{ik} \end{aligned}\right\}$$

The column vector of generalized accelerations is defined by time derivative (84) rewritten as:

$$\left\{\begin{aligned} &\ddot{\bar\theta}^{e}(t) = \left[\ddot{\bar\theta}_{ij}^{eT}(t) \ \text{where} \ j = 0 \to m_i;\ i = 1 \to n\right]^{T} \\ &\ddot\theta_{ij}^{eT}(t) = \left\{\left\{\ddot q_i(t) \ \text{if} \ j = 0\right\};\left\{\ddot q_{ij}(t) \ \text{if} \ j \ge 1\right\}\right\} \end{aligned}\right\}.$$

Considering (84) and (100), for the whole *MRS*, supposing that all $(n)$ kinetic links are flexible *the acceleration energy* of first order in the new matrix expression is shown in the two variants:

$$\left\{\begin{aligned} &E_A^{1ee}\left(\bar\theta^{e};\dot{\bar\theta}^{e};\ddot{\bar\theta}^{e}\right) = \\ &= \sum_{i=1}^{n} E_A^{1iee}\left(\bar\theta_{jk}^{eT};\dot{\bar\theta}_{jk}^{eT};\ddot{\bar\theta}_{jk}^{eT};\ j = 1 \to i\right) \end{aligned}\right\}; \qquad (101)$$

$$\left\{\begin{aligned} &E_A^{1ee}\left(\bar\theta^{e};\dot{\bar\theta}^{e};\ddot{\bar\theta}^{e}\right) = \\ &\frac{1}{2}\cdot \ddot{\bar\theta}^{eT} \cdot M^{ee}\left(\bar\theta^{e}\right)\cdot \ddot{\bar\theta}^{e} + V^{ee}\left(\bar\theta^{e};\dot{\bar\theta}^{e}\right)\cdot \ddot{\bar\theta}^{e} \\ &+ \frac{1}{2}\cdot \dot{\bar\theta}^{eT} \cdot D^{ee}\left(\bar\theta^{e};\dot{\bar\theta}^{e}\right)\cdot \dot{\bar\theta}^{e} \end{aligned}\right\}; \qquad (102)$$

$$\left\{\begin{aligned} &E_A^{1ee}\left(\bar\theta^{e};\dot{\bar\theta}^{e};\ddot{\bar\theta}^{e}\right) = \\ &+ \frac{1}{2}\cdot \sum_{i=1}^{n}\sum_{j=0}^{m_i}\sum_{k=1}^{n}\sum_{l=0}^{m_k} M_{ijkl}^{ee}\left(\bar\theta^{e}\right)\cdot \ddot\theta_{ij}^{e} \cdot \ddot\theta_{kl}^{e} + \\ &+ \left[\frac{1}{2}\cdot \dot{\bar\theta}^{eT} \cdot D^{ee}\left(\bar\theta^{e};\dot{\bar\theta}^{e}\right)\cdot \dot{\bar\theta}^{e}\right] + \\ &+ \sum_{i=1}^{n}\sum_{j=0}^{m_i}\sum_{k=1}^{n}\sum_{l=0}^{m_k}\sum_{p=1}^{n}\sum_{r=0}^{m_p} V_{ijklpr}^{ee}\left(\bar\theta^{e}\right)\cdot \ddot\theta_{ij}^{e} \cdot \dot\theta_{kl}^{e} \cdot \dot\theta_{pr}^{e} \end{aligned}\right\}; \qquad (103)$$

$$\left\{\begin{aligned} &\text{where} \quad \dot{\bar\theta}^{eT} \cdot D^{ee}\left(\bar\theta^{e};\dot{\bar\theta}^{e}\right)\cdot \dot{\bar\theta}^{e} = \\ &\sum_{i=1}^{n}\sum_{j=0}^{m_i}\sum_{k=1}^{n}\sum_{l=0}^{m_k}\sum_{p=1}^{n}\sum_{r=0}^{m_p}\sum_{s=1}^{n}\sum_{u=0}^{m_s} D_{ijklprsu}^{ee}\left(\bar\theta^{e}\right)\cdot \dot\theta_{ij}^{e} \cdot \dot\theta_{kl}^{e} \cdot \dot\theta_{pr}^{e} \cdot \dot\theta_{su}^{e} \end{aligned}\right\}.$$

In the above expressions of the acceleration energy of first order it remarks existence a few elastodynamics matrices: inertia and pseudo inertia matrices, as well as the generalization of matrices of the Coriolis and centrifugal terms.

According to *LE-type equations* on the one hand and on the other hand generalization of *Appell's equations* [1] and [12], *the generalized inertia forces* (96), supposing that all $(n)$ kinetic links of the robot have flexible features, are also defined with matrix and explicit expressions:

$$Q_{iö}^{ee}\left(\bar\theta^{e}\right) = \left\{\frac{d}{dt}\left(\frac{\partial E_C^{ee}}{\partial\dot{\bar\theta}^{e}}\right) - \frac{\partial E_C^{ee}}{\partial\bar\theta^{e}} = \frac{\partial E_A^{1ee}}{\partial\ddot{\bar\theta}^{e}}\right\} = (104)$$

$$= \left\{\frac{d}{dt}\left\{M^{ee}\left(\bar\theta^{e}\right)\cdot \dot{\bar\theta}^{e}\right\} - \frac{1}{2}\cdot \dot{\bar\theta}^{eT} \cdot \frac{\partial}{\partial\bar\theta^{e}}\left\{M^{ee}\left(\bar\theta^{e}\right)\right\}\cdot \dot{\bar\theta}^{e}\right\};$$

$$\left\{\begin{aligned} &Q_{iö}^{kl\,ee} = \frac{d}{dt}\left\{\sum_{i=1}^{n}\sum_{j=0}^{m_i} M_{ijkl}^{ee}\left(\bar\theta^{e}\right)\cdot \dot\theta_{ij}^{e}\right\} - \\ &- \frac{\partial}{\partial\theta_{kl}^{e}}\left\{\frac{1}{2}\sum_{i=1}^{n}\sum_{j=0}^{m_i}\sum_{p=1}^{n}\sum_{r=0}^{m_p} M_{ijpr}^{ee}\left(\bar\theta^{e}\right)\cdot \dot\theta_{ij}^{e} \cdot \dot\theta_{pr}^{e}\right\} = \\ &\sum_{i=1}^{n}\sum_{j=1}^{m_i} M_{ijkl}^{ee} \cdot \ddot\theta_{ij}^{e} + \\ &+ \sum_{i=1}^{n}\sum_{j=0}^{m_i}\sum_{p=1}^{n}\sum_{r=0}^{m_p} \frac{\partial}{\partial\theta_{pr}^{e}}\left\{M_{ijkl}^{ee}\right\}\cdot \dot\theta_{ij}^{e} \cdot \dot\theta_{pr}^{e} - \\ &- \frac{\partial}{\partial\theta_{kl}^{e}}\left\{\frac{1}{2}\sum_{i=1}^{n}\sum_{j=0}^{m_i}\sum_{p=1}^{n}\sum_{r=0}^{m_p} M_{ijpr}^{ee} \cdot \dot\theta_{ij}^{e} \cdot \dot\theta_{pr}^{e}\right\} \end{aligned}\right\}; \ (105)$$

The above two expressions can be also written in accordance with the following development:

$$\left\{\begin{array}{c} Q_{i\ddot{o}}^{kl\,ee}\left(\bar{\theta}_{kl}^{e}\right)=\left\{\frac{d}{dt}\left(\frac{\partial E_{C}^{ee}}{\partial\dot{\theta}_{kl}^{e}}\right)-\frac{\partial E_{C}^{ee}}{\partial\theta_{kl}^{e}}=\frac{\partial E_{A}^{1ee}}{\partial\ddot{\theta}_{kl}^{e}}\right\}= \\ =\frac{d}{dt}\left\{\sum_{i=1}^{n}\sum_{j=1}^{m_i}M_{ijkl}^{ee}\left(\bar{\theta}^{e}\right)\cdot\dot{\theta}_{ij}^{e}\right\}- \\ -\frac{1}{2}\cdot\dot{\bar{\theta}}^{eT}\cdot\frac{\partial}{\partial\theta_{kl}^{e}}\left\{M^{ee}\left(\bar{\theta}^{e}\right)\right\}\cdot\dot{\bar{\theta}}^{e} \end{array}\right\} \quad (106)$$

*Remarks:* In the above matrix and explicit expressions, the generalized inertia forces, when $l=0$, are corresponding to generalized variable $q_k$ from the driving joints with rigid features, and the others are answerable to the generalized variables $q_{kl}$ considering the flexible links of robot have features of generalized deformations.

But the author has developed in many other papers [7],[9], [13] – [20], new formulations on the acceleration energies of higher order, as well as the dynamics equations of higher order corresponding to suddenly movements for any multibody system, as example serial robots. In this section a new expression about *acceleration energy of second order* for elastic structure of the robot will be presented. As a result, in the beginning a few kinematics transformations for elastic structure are defined by means of time derivative of third order, as below follows:

$$\dddot{T}_{i0}^{e}=\begin{bmatrix}\dddot{R}_{i0}^{e} & \dddot{\bar{p}}_{i}^{e}\\ 0\;0\;0 & 0\end{bmatrix}=\dddot{T}_{i0A}^{e}+\dddot{T}_{i0B}^{e}+\dddot{T}_{i0C}^{e}; \quad (107)$$

$$\dddot{T}_{i0A}^{e}=\sum_{k=1}^{i}T_{k0}^{e}\cdot U_k\cdot T_{ik}^{e}\cdot\dddot{q}_k+\sum_{k=1}^{i-1}\sum_{l=1}^{m_i}T_{kk-1}\cdot\Delta T_{kl}^{e}\cdot T_{il}^{e}\cdot\dddot{q}_{kl}; \quad (108)$$

$$\dddot{T}_{i0B}^{e}=\dddot{T}_{i0B1}^{e}+\dddot{T}_{i0B2}^{e}+\dddot{T}_{i0B3}^{e}, \quad (109)$$

$$\left\{\begin{array}{c} \dddot{T}_{i0C}^{e}=\sum_{k=1}^{i-1}\sum_{l=1}^{m_i}\sum_{m=l}^{i}\dot{T}_{kk-1}\cdot\Delta T_{kl}^{e}\cdot T_{mk}^{e}\cdot U_m\cdot T_{il}^{e}\cdot\ddot{q}_{kl}\cdot\dot{q}_m+ \\ +\sum_{k=1}^{i-1}\sum_{l=1}^{m_i}\sum_{m=l}^{i}\dot{T}_{kk-1}\cdot\Delta T_{kl}^{e}\cdot T_{mk}^{e}\cdot U_m\cdot T_{il}^{e}\cdot\dot{q}_k\cdot\dot{q}_{kl}\cdot\dot{q}_m+ \\ +\sum_{k=1}^{i-1}\sum_{l=1}^{m_i}\sum_{m=l}^{i}\sum_{p=k}^{m}T_{kk-1}\cdot\Delta T_{kl}^{e}\cdot T_{pk}^{e}\cdot U_p\cdot T_{mp}^{e}\cdot U_m\cdot T_{il}^{e}\cdot\dot{q}_k\cdot\dot{q}_{kl}\cdot\dot{q}_m \\ +\sum_{k=1}^{i-1}\sum_{l=1}^{m_i}\sum_{m=l}^{i}\sum_{r=l}^{i}T_{kk-1}\cdot\Delta T_{kl}^{e}\cdot T_{mk}^{e}\cdot U_m\cdot T_{rl}^{e}\cdot U_r\cdot T_{ir}^{e}\cdot\dot{q}_r\cdot\dot{q}_{kl}\cdot\dot{q}_m \end{array}\right\};$$

$$\dddot{T}_{i0B2}^{e}=\dddot{T}_{i0B21}^{e}+\dddot{T}_{i0B22}^{e}, \quad (110)$$

$$\left\{\begin{array}{c} \dddot{T}_{i0B21}^{e}=2\cdot\sum_{k=1}^{i}\sum_{l=k}^{i}T_{k0}^{e}\cdot U_k\cdot T_{lk}^{e}\cdot U_l\cdot T_{il}^{e}\cdot\ddot{q}_l\cdot\dot{q}_k+ \\ +\sum_{k=1}^{i}\sum_{l=k}^{i}\sum_{p=1}^{k}T_{p0}^{e}\cdot U_p\cdot T_{kp}^{e}\cdot U_k\cdot T_{lk}^{e}\cdot U_l\cdot T_{il}^{e}\cdot\dot{q}_p\cdot\dot{q}_l\cdot\dot{q}_k+ \end{array}\right\}$$

$$\left\{\begin{array}{c} \dddot{T}_{i0B22}^{e}=\sum_{k=1}^{i}\sum_{l=k}^{i}\sum_{r=k}^{l}T_{k0}^{e}\cdot U_m\cdot T_{rk}^{e}\cdot U_r\cdot T_{lr}^{e}\cdot U_l\cdot T_{il}^{e}\cdot\dot{q}_r\cdot\dot{q}_l\cdot\dot{q}_k \\ +\sum_{k=1}^{i}\sum_{m=1}^{k}\sum_{s=l}^{i}T_{k0}^{e}\cdot U_k\cdot T_{lk}^{e}\cdot U_l\cdot T_{sl}^{e}\cdot U_s\cdot T_{is}^{e}\cdot\dot{q}_s\cdot\dot{q}_l\cdot\dot{q}_k \end{array}\right\}$$

$$\left\{\begin{array}{c} \dddot{T}_{i0B3}^{e}=\sum_{k=1}^{i-1}\sum_{l=1}^{m_i}\ddot{T}_{kk-1}\cdot\Delta T_{kl}^{e}\cdot T_{il}^{e}\cdot\ddot{q}_k\cdot\dot{q}_{kl}+ \\ +\sum_{k=1}^{i-1}\sum_{l=1}^{m_i}\dot{T}_{kk-1}\cdot\Delta T_{kl}^{e}\cdot T_{il}^{e}\cdot\dot{q}_k\cdot\ddot{q}_{kl}+ \\ +\sum_{k=1}^{i-1}\sum_{l=1}^{m_i}\sum_{p=l}^{i}\dot{T}_{kk-1}\cdot\Delta T_{kl}^{e}\cdot T_{pl}^{e}\cdot U_p\cdot T_{ip}^{e}\cdot\dot{q}_p\cdot\dot{q}_k\cdot\ddot{q}_{kl} \end{array}\right\}$$

$$\left\{\begin{array}{c} \dddot{T}_{i0C}^{e}=\sum_{k=1}^{i-1}\sum_{l=1}^{m_i}\sum_{m=l}^{i}\dot{T}_{kk-1}\cdot\Delta T_{kl}^{e}\cdot T_{mk}^{e}\cdot U_m\cdot T_{il}^{e}\cdot\ddot{q}_{kl}\cdot\dot{q}_m+ \\ +\sum_{k=1}^{i-1}\sum_{l=1}^{m_i}\sum_{m=l}^{i}\dot{T}_{kk-1}\cdot\Delta T_{kl}^{e}\cdot T_{mk}^{e}\cdot U_m\cdot T_{il}^{e}\cdot\dot{q}_k\cdot\dot{q}_{kl}\cdot\dot{q}_m+ \\ +\sum_{k=1}^{i-1}\sum_{l=1}^{m_i}\sum_{m=l}^{i}\sum_{p=k}^{m}T_{kk-1}\cdot\Delta T_{kl}^{e}\cdot T_{pk}^{e}\cdot U_p\cdot T_{mp}^{e}\cdot U_m\cdot T_{il}^{e}\cdot\dot{q}_k\cdot\dot{q}_{kl}\cdot\dot{q}_m \\ +\sum_{k=1}^{i-1}\sum_{l=1}^{m_i}\sum_{m=l}^{i}\sum_{r=l}^{i}T_{kk-1}\cdot\Delta T_{kl}^{e}\cdot T_{mk}^{e}\cdot U_m\cdot T_{rl}^{e}\cdot U_r\cdot T_{ir}^{e}\cdot\dot{q}_r\cdot\dot{q}_{kl}\cdot\dot{q}_m \end{array}\right\}$$

For determine the equation of definition of the acceleration energy of second order, first of all time derivative of third order for position vector in homogeneous coordinates are established as:

$$\left\{\begin{array}{c} \dddot{\bar{r}}_{i}^{ee}=\dddot{T}_{i0}^{e}\cdot{}^{i}\bar{r}_{i}+\dddot{T}_{i0}^{e}\cdot{}^{i}\bar{d}_{i}+ \\ +3\cdot\ddot{T}_{i0}^{e}\cdot{}^{i}\dot{\bar{d}}_{i}+3\cdot\dot{T}_{i0}^{e}\cdot{}^{i}\ddot{\bar{d}}_{i}+T_{i0}^{e}\cdot{}^{i}\dddot{\bar{d}}_{i} \end{array}\right\}; \quad (111)$$

$$\left\{\begin{array}{c} \dddot{\bar{r}}_{i}^{eeT}={}^{i}\bar{r}_{i}^{T}\cdot\dddot{T}_{i0}^{eT}+{}^{i}\bar{d}_{i}^{T}\cdot\dddot{T}_{i0}^{eT}+ \\ +3\cdot{}^{i}\dot{\bar{d}}_{i}^{T}\cdot\ddot{T}_{i0}^{eT}+3\cdot{}^{i}\ddot{\bar{d}}_{i}^{T}\cdot\dot{T}_{i0}^{eT}+{}^{i}\dddot{\bar{d}}_{i}^{T}\cdot T_{i0}^{eT} \end{array}\right\}.$$

According to (49), where $p=1$ *and* $k=1$, as well as the papers of author [14] – [20], the equation for acceleration energy of second order shows as:

$$\left\{\begin{array}{c} E_{A}^{2iee}=\frac{1}{2}\cdot\int_{link}Trace\left[\dddot{\bar{r}}_{i}^{ee}\cdot\dddot{\bar{r}}_{i}^{eeT}\right]\cdot dm= \\ =\frac{1}{2}\cdot Trace\left[\dddot{T}_{i0}^{e}\cdot{}^{i}I_{psi}^{e}\cdot\dddot{T}_{i0}^{eT}\right]+\cdots\cdots \\ +\cdots\frac{1}{2}\cdot Trace\left\{T_{i0}^{e}\cdot\left\{\int_{link}\left[{}^{i}\dddot{\bar{d}}_{i}\cdot{}^{i}\dddot{\bar{d}}_{i}^{T}\right]\cdot dm\right\}\cdot T_{i0}^{eT}\right\} \end{array}\right\}. \quad (112)$$

This expression will develop in another paper. The generalized inertia force of higher order is:

$$\left\{\begin{array}{c} \frac{1}{m}\cdot\frac{d^{k-1}}{dt^{k-1}}\left[\frac{\partial\overset{(m)}{E_{C}^{ee}}}{\partial\overset{(m)}{\theta_{ij}^{e}}}-(m+1)\cdot\frac{\partial E_{C}^{ee}}{\partial\theta_{ij}^{e}}\right]= \\ =\overset{(k-1)}{Q_{i\ddot{o}}^{jee}}\left[\bar{\theta}^{e}(t);\dot{\bar{\theta}}^{e}(t);\cdots;\overset{(m)}{\bar{\theta}^{e}}(t)\right] \end{array}\right\}; \quad (113)$$

$$\overset{(k-1)}{Q}{}^{jee}_{i\ddot{o}}\left[\overline{\theta}^e(t);\dot{\overline{\theta}}^e(t);\cdots;\overset{(m)}{\overline{\theta}^e}(t)\right]= \quad (114)$$

$$\left\{\frac{d^{k-1}}{dt^{k-1}}\left\{\frac{\partial \overset{(m-2)}{E_A^{(1)ee}}}{\partial \overset{(m)}{\theta_{ij}^e}}\right\}=\frac{1}{m+1}\cdot\frac{d^{k-1}}{dt^{k-1}}\frac{\partial}{\partial \overset{(m)}{\theta_{ij}^e}}\left[2\cdot \overset{(m-3)}{E_A^{(2)ee}}+\overset{(m-1)}{E_A^{(1)ee}}\right]\right\}.$$

The above equations are in consonance with the researches of the author from [14] – [20]. In the same papers the author proposed, the generalized differential equations of higher order, in the case of the mechanical systems (MRS), dynamically characterized by sudden and transitory motions. When the mechanical systems are dominated by elastic links, then the equations are changing as:

$$\left\{\begin{array}{l}\dfrac{(k-1)!\cdot m!}{(m+k-1)!}\cdot\dfrac{\partial}{\partial \overset{(m)}{\theta_{ij}^e}}\left\{\left(\sum\limits_{p=1}^{k}\Delta_p\right)\cdot \overset{(m+k)-(2\cdot p+1)}{E_A^{(p)ee}}\right\}\\ =\overset{(k-1)}{Q}{}^{jee}_{i\ddot{o}}\left[\overline{\theta}^e(t);\dot{\overline{\theta}}^e(t);\cdots;\overset{(m)}{\overline{\theta}^e}(t)\right]\end{array}\right\}; \quad (115)$$

$$\left\{\begin{array}{l} where \quad E_A^{(p)ee}=E_A^{(p)ee}\left[\overline{\theta}^e(t);\dot{\overline{\theta}}^e(t);\cdots;\overset{(p+1)}{\overline{\theta}^e}(t)\right]\\ and \quad \sum\limits_{p=1}^{k}\Delta_p=\sum\limits_{p=1}^{k}\left[\dfrac{p\cdot(p+1)}{2}-\delta_p\right]\end{array}\right\}.$$

The necessary conditions in (115) are following:

$$\left\{\begin{array}{l} p=1\rightarrow k\ ;\ \ \delta_p=\{\{0;p=1\};\{1;p>1\}\}\\ and \quad k\geq 1;\quad k=\{1;2;3;4;5;.....\}\\ m\geq(k+1);\quad m=\{2;3;4;5;.....\}\end{array}\right\}. \quad (116)$$

Generalized differential equations (115) contain acceleration energies of order $(p=1\rightarrow 2)$, whose expressions of definition, in explicit and matrix form, are detailed presented in this section.

In the robot with elastic structures, the generalized forces answerable to gravity loads are also developed due to the elastic driving joints and deformations to every flexible link:

$$Q_{ig}^{iee}={}^0J_i^e\cdot{}^0\,\ddot{o}\,\overline{\phantom{x}}^{ee}_{Xi}={}^0J_i^e\cdot\left[{}^0\overline{F}_{Xi}^{eeT}\quad {}^0\overline{F}_{Xi}^{eeT}\right]^T; \quad (117)$$

$$\left\{\begin{array}{l}{}^0\overline{F}_{Xi}^{ee}=\left\{\sum\limits_{j=i}^{n}M_j+\sum\limits_{j=i}^{n}\sum\limits_{k=1}^{m_j}\overline{d}_{jk}^T\right\}\cdot{}^0\overline{g};\ \ {}^0\overline{g}=-g\cdot\overline{k}_g^T\cdot\overline{k}_0\\ {}^0\overline{N}_{Xi}^{ee}=\sum\limits_{j=i}^{n}M_j\left\{\overline{r}_{Cj}^{ee}-\overline{p}_n^{ee}\right\}\times{}^0\overline{g}+\sum\limits_{j=i}^{n}\sum\limits_{k=1}^{m_j}\left\{{}^0\overline{r}_j^{ee}-\overline{p}_n^{ee}\right\}\times\overline{d}_{jk}\end{array}\right\}.$$

Above expressions of generalized gravitational forces are functions of matrix transformations in elasticity conditions of the mechanical structure, of robot, as well as Jacobian matrix, (81) – (85) presented in the fourth section of this paper.

## 6. CONCLUSIONS

This paper was divided in the two essential parts. First part was devoted to establishment the kinematics and dynamics equations for any serial robot with rigid structure. The second part was devoted to determination elastokinematics and elastodynamics equations for serial robot with elastic and flexible structure.

For define the kinematics and differential matrices functions in the case of the robot structures with rigid links, the author have been applied, in accordance with its researches, the classical transformations, as well as matrix exponentials, based on *MEK Algorithm*. By means of the matrix exponentials have been also determined all kinematic parameters. They characterize the equations of direct and control kinematics for any mechanical robot structure, regardless of its constructive complexity. For dynamical study of robot structures with rigid links, the author of paper developed, kinetic energy with important formulations. According to the author researches, the suddenly motion, transient motion phases, as well as mechanical systems subjected to the action of a system of external forces, with time variation law, are dominated by linear and angular accelerations of higher order. In this paper new expressions for acceleration energies of higher order have been presented, according to suddenly movements.

In the second part of this paper an elastic structure of the serial robot was analyzed from view point of elastokinematics, as well as from view point of elastodynamics behavior. As a result, using the properties of the matrix exponentials, the locating matrices and their time derivatives corresponding to small deformations been established. For the study of elastodynamics author established generalized inertia forces, as well generalized active forces answerable to gravity load. So, the author has defined with fundamental new formulations the kinetic energy, acceleration energy of first and second order, corresponding in exclusivity to elastic structure. By means of proper researches, the paper author established the expressions for generalized inertia forces typical to suddenly and transitory motions, when mechanical structure of the robot is dominated of elastic links. This study also included generalized gravitational forces.

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## Formulări asupra elastodinamicii în robotică

***Rezumat:*** *Obiectivul principal al acestei lucrări îl constituie stabilirea ecuațiilor generalizate ale elastocinematicii și elastodinamicii structurilor robotului cu elemente flexibile. Pentru matricele cinematicii și diferențiale. În cazul structurilor de roboți cu elemente rigide și elastice, se vor aplica printre altele exponențiale de matrice în consonanță cu algoritmii dezvoltați de către autor. Pentru studiul dinamic al structurilor de robot cu elemente rigide, în cadrul lucrării vor fi prezentate autorul va dezvolta energia cinetică cu formulări importante. În conformitate cu cercetările autorului, vor fi de asemenea prezentate expresii noi pentru energiile de accelerații de ordin superior corespunzătoare mișcărilor rapide. Pentru studiul elastodinamicii, autorul va stabili, cu formulări noi, energia cinetică și energia de accelerații de ordinul întâi și doi, corespunzătoare, în exclusivitate, structurilor elastice. Astfel, expresiile pentru forțele generalizate de inerție se vor determina conform cu mișcările rapide și pentru robotul dominat de elemente elastice.*

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