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STUDY OF AN ELASTIC BEAM, IN CENTRIFUGAL FIELD, USING FINITE ELEMENT METHOD

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Abstract: *The finite element method (FEM) has been applied mainly to the study of the behavior of different elastic bodies, whether static or dynamic. The case of rigid motion of bodies analyzed with MEF, involving new terms due to the effects of inertia and relative motions, has not yet been incorporated into the classic MEF software. Papers to analyze this behavior arose in the 70's for the plane beam and later for a three-dimensional general motion. The present paper aims to develop models previously studied by other researchers and study the influences that the geometric parameters of the bar can have on the dynamic response. The case study is that of a moving bar in a rotation around a fixed axis. The shape functions used are of degree five.*

Key words: *Finite Element Method, beam, eigenvalues, eigenmodes, rotation, centrifugal*

1. INTRODUCTION

The study of elastic elements with a rigid general motion using MEF was begun in the 1970s. The development of the domain was made gradually, first for a single beam using third degree interpolation functions, then for a plane mechanism. The method was developed using degree five interpolation polynomials. From the plane motion to the three-dimensional motion of a beam was used, using the third degree polynomials, then the five-degree polynomials. The finite element used in all these cases was the one-dimensional finite element[2],[3],[5],[22]. The theoretical model for which internal energy was calculated was the Bernoulli model, then went to the Rayleigh model and finally to the Timoshenko model [29]-[34]. The method used to write motion equations was Lagrange's equations and Hamilton's equations method. From one-dimensional finite elements, the researchers passed to the two- and three-dimensional finite elements[23]-[26]. However, the results that contributed to the theoretical development of the field did not lead to the development of the finite element software in this direction, by incorporating the obtained results, due to the difficulties of modeling the respective mechanical systems[1],[4],[6]-[13],[20],[21]. The problems that can solved by this method also involve a previous analysis to determine the velocity and acceleration field, so the use of multibody systems (MBS) models.

In this paper we propose the development of a finite beam model using the five degree shape functions to study the behavior of a beam in a centrifugal field.

2. MODEL AND SHAPE FUNCTIONS

For the present study, a one-dimensional finite element is used, related to an Oxy mobile (local) reference system. The rotation is related to a fixed (global) O'XYZ reference system, around the O'Z axis.

The boundary conditions for the axial displacement *u*, if the linear shape functions are chosen, lead to the equations:

$$u(x) = u_1\left(1-\frac{x}{l}\right)+\frac{u_2 x}{l} = N_1 u_1 + N_2 u_2 .$$

Using the notation $\zeta = \frac{x}{l}$ the shape functions are:

$$N_1 = 1-\frac{x}{l} = 1-\zeta \ ; \ N_2 = \frac{x}{l} = \zeta \ . \qquad (1)$$

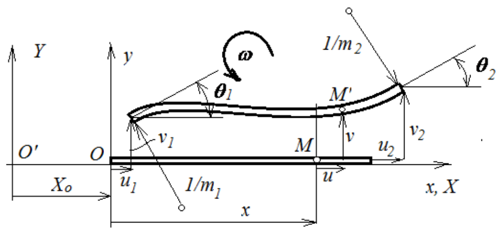

Fig. 1. Model of an one-dimensional finite element

Using the notation $\zeta=\dfrac{x}{l}$ the shape functions are: $N_1=1-\dfrac{x}{l}=1-\zeta$; $N_2=\dfrac{x}{l}=\zeta$. (1)

For the transverse motion, a polynomial of degree 5 is chosen. It is considered, in this case, the displacement of the ends, the rotation of the cross sections and the curvatures at the ends. Writing the end conditions, it will be obtained:

$$v(\zeta)=v_1N_3+v_2N_4+\theta_1N_5+\theta_2N_6+m_1N_7+m_2N_8 \tag{2}$$

with:

$$N_3=1-10\zeta^3+15\zeta^4-6\zeta^5;$$
$$N_4=10\zeta^3-15\zeta^4+6\zeta^5 ;$$
$$N_5=l\left(\zeta-6\zeta^3+8\zeta^4-3\zeta^5\right) ;$$
$$N_6=l\left(-4\zeta^3+7\zeta^4-3\zeta^5\right) ;$$
$$N_7=\frac{l^2}{2}\left(\zeta^2-3\zeta^3+3\zeta^4-\zeta^5\right) ;$$
$$N_8=\frac{l^2}{2}\left(\zeta^3-2\zeta^4+\zeta^5\right).$$

It results:

$$\theta(\zeta)=\frac{\partial v}{\partial x}=\frac{\partial v}{l\partial\zeta}=$$
$$=v_1N_3'+v_2N_4'+\theta_1N_5'+\theta_2N_6'+m_1N_7'+m_2N_8' ; \tag{3}$$

$$m(\zeta)=\frac{\partial^2 v}{\partial x^2}=\frac{\partial^2 v}{l^2\partial\zeta^2}=$$
$$=v_1N_3''+v_2N_4''+\theta_1N_5''+\theta_2N_6''+m_1N_7''+m_2N_8'' , \tag{4}$$

where:

$$N_i'=\frac{\partial N_i}{\partial x}=\frac{\partial N_i}{l\,\partial\zeta} ; \; N_i''=\frac{\partial^2 N}{\partial x^2}=\frac{\partial^2 N}{l^2\,\partial^2\zeta} \tag{5}$$

$$\begin{Bmatrix} u(\zeta)\\ v(\zeta)\\ \theta(\zeta)\\ m(\zeta)\end{Bmatrix}=\begin{bmatrix} N_1 & 0 & 0 & 0 & N_2 & 0 & 0 & 0\\ 0 & N_3' & N_5' & N_7' & 0 & N_4' & N_6' & N_8'\\ 0 & N_3'' & N_5'' & N_7'' & 0 & N_4'' & N_6'' & N_8''\\ 0 & N_3 & N_5 & N_7 & 0 & N_4 & N_6 & N_8\end{bmatrix}\cdot\begin{Bmatrix} u_1\\ v_1\\ \theta_1\\ m_1\\ u_2\\ v_2\\ \theta_2\\ m_2\end{Bmatrix}=$$

$$=\begin{bmatrix} N_{(1)}\\ N_{(2)}\\ N'_{(2)}\\ N''_{(2)}\end{bmatrix}\begin{Bmatrix}\delta_1\\ \delta_2\end{Bmatrix}, \tag{6}$$

where the notations were used:

$$\{\delta_1\}=\begin{Bmatrix} u_1\\ v_1\\ \theta_1\\ m_1\end{Bmatrix} \; ; \; \{\delta_2\}=\begin{Bmatrix} u_2\\ v_2\\ \theta_2\\ m_2\end{Bmatrix} \tag{7}$$

3. LAGRANGIAN OF AN ONE DIMENSIONAL FINITE ELEMENT METHOD AND MOTION EQUATIONS

Lagrange's function for the beam is:

$$L=E_c-E_i+W^c+W^d \tag{8}$$

where E_c is the kinetic energy of the bar, E_i the internal energy, W^c+W^d the work of concentrated and distributed forces.

The position vector of an arbitrary point of the deformed fiber is:

$$\{r_{M'}\}=\{r_0\}+[R]\cdot\begin{Bmatrix} l\zeta\\ 0\end{Bmatrix}+[R]\cdot\begin{Bmatrix} u\\ v\end{Bmatrix}=$$
$$=\{r_0\}+[R]\cdot\begin{Bmatrix} l\zeta\\ 0\end{Bmatrix}+[R]\cdot[N]\cdot\{\delta\}. \tag{9}$$

The velocity of this point is:

$$\{\dot r_{M'}\}=\{\dot r_0\}+[\dot R]\cdot\begin{Bmatrix} l\zeta\\ 0\end{Bmatrix}+[\dot R]\cdot[N]\cdot\{\delta\}+[R]\cdot[N]\cdot\{\dot\delta\} \tag{10}$$

where the matrix $[R]$ is:

$$[R]=\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}; \; [\dot R]=\omega\begin{bmatrix}-\sin\theta & -\cos\theta\\ \cos\theta & -\sin\theta\end{bmatrix} \tag{12}$$

where $\dot{\theta} = \omega$ (the angular velocity of the rigid motion of the beam).

The kinetic energy for a beam element *dm* is:

$$E_c = \frac{1}{2}\int_0^l \rho\left(A\{\dot{r}_{M'}\}^T \cdot \{\dot{r}_{M'}\} + I_z\omega^{*2}\right)\ ld\zeta \tag{13}$$

where: $$\omega^* = \omega + \dot{\theta} = \omega + \left[N_{(2)}\right]' \cdot \{\dot{\delta}\}. \tag{14}$$

If the calculus are made for kinetic energy, it obtains:

$$E_c = \frac{1}{2}\int_0^l \rho A(\{\dot{r}_0\}^T\{\dot{r}_0\} + 2\{\dot{r}_0\}^T[\dot{R}]\begin{Bmatrix} l\zeta \\ 0 \end{Bmatrix} + 2\{\dot{r}_0\}^T[\dot{R}][N]\{\delta\} +$$
$$+2\{\dot{r}_0\}^T[R][N]\{\dot{\delta}\} + [x \quad 0]\left[\dot{R}\right]^T[\dot{R}]\begin{Bmatrix} l\zeta \\ 0 \end{Bmatrix} +$$
$$+2[l\zeta \quad 0]\left[\dot{R}\right]^T[\dot{R}][N]\{\delta\} + 2[l\zeta \quad 0]\left[\dot{R}\right]^T[R][N]\{\dot{\delta}\} +$$
$$+\{\delta\}^T[N]^T\left[\dot{R}\right]^T[\dot{R}][N]\{\delta\} + 2\{\delta\}^T[N]^T\left[\dot{R}\right]^T[R][N]\{\dot{\delta}\} +$$
$$+\{\dot{\delta}\}^T[N]^T[R]^T[R][N]\{\dot{\delta}\})ld\zeta +$$
$$+\frac{1}{2}\int_0^l \rho I_z(\omega^2 + 2\omega\left[N_{(2)}\right]'\{\dot{\delta}\} + \{\dot{\delta}\}^T\left[N_{(2)}\right]'^T\left[N_{(2)}\right]'\{\dot{\delta}\})ld\zeta \tag{15}$$

If Lagrange equations are written [14]-[19], [27],[28]:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\delta}} - \frac{\partial L}{\partial \delta} = 0, \tag{17}$$

the motion equations are obtained as:

$$([m_e] + l[m_r])\{\ddot{\delta}\} + 2\omega([m_{21}] - [m_{12}])\{\dot{\delta}\} +$$
$$\left[[k_i]+[k_a]+[k_e^G]+\varepsilon([m_{21}]-[m_{12}])-\omega^2([m_{11}]+[m_{22}])\right]\{\delta\}==$$
$$-\omega^2 l\{m_{1x}\}^T + \{\dot{r}_0\}^T[R][m_{oe}^i]^T\omega +$$
$$+\varepsilon(l\{m_{ee}^i\}^T + l\{m_{2x}\}^T) + \{\ddot{r}_0\}^T[R][m_{oe}^i]^T). \tag{18}$$

4. INFLUENCE OF GEOMETRIC PARAMETERS ON EIGEN-FREQUENCIES

Using the previously obtained equations (18), a study was performed to see how the eigenvalues of the elastic beam is affected by the geometrical parameters. This is practically useful as it is found that for some sets of parameters, the system becomes unstable. The lengths of the element and its diameter have varied (the circular section from the beam is considered to reduce the number of parameters defining the bar geometry), the concentrated mass that can be applied to the end of the bar and the angular speed of the beam. The results are presented in the following figures (Fig.2-10)

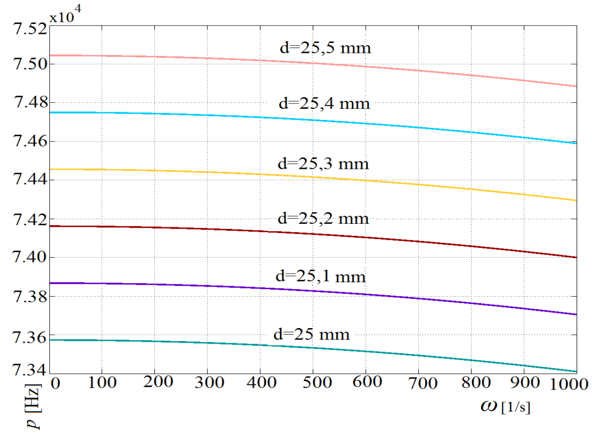

Fig.3. Beam with L=0.7m in a centrifugal field

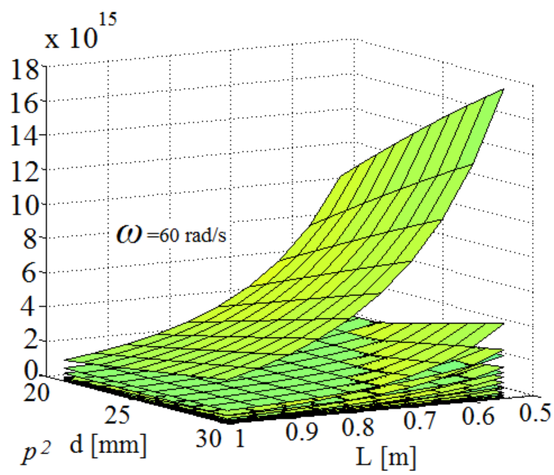

Fig.4. First ten eigenfrequencies, L variable, d variable

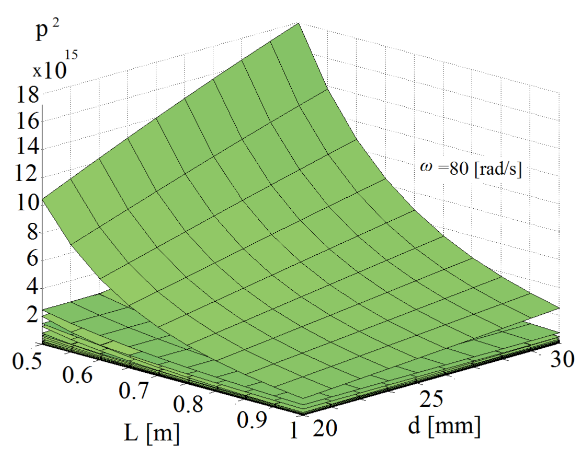

Fig.5. First ten eigenfrequencies, L variable, d variable

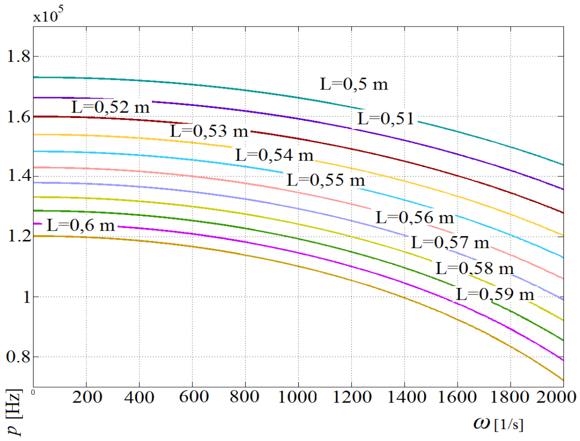

Fig.6. Beam with d=0.03 m, L variable

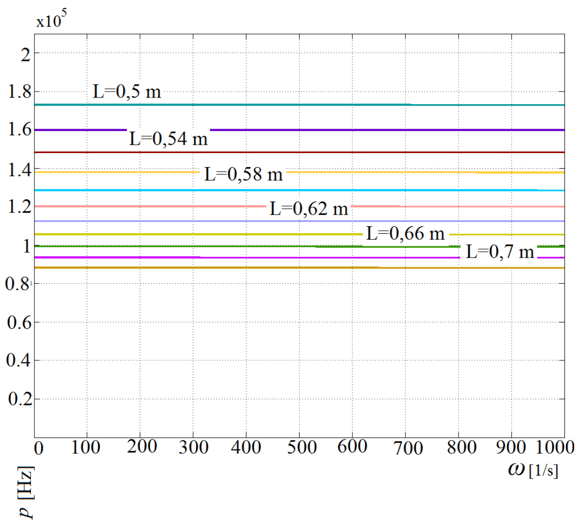

Fig.7. Beam with d=0.02 m, L variable

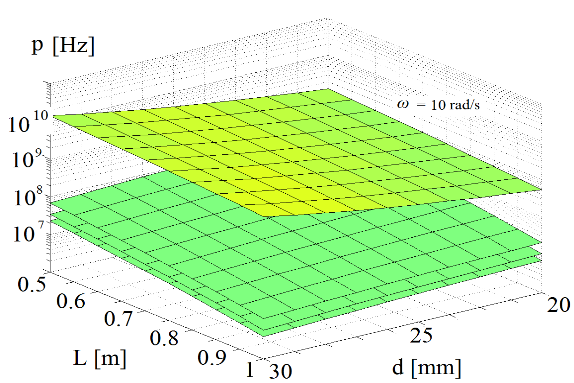

Fig.8. The first eigenfrequency, L variable, d variable, endmass variable. Logaritmic scale.

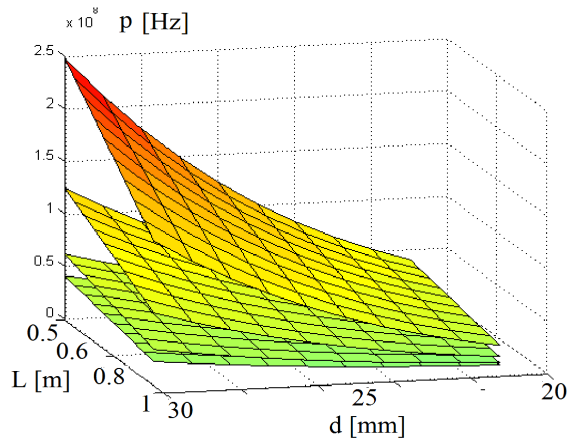

Fig.10. The first eigenfrequency, L variable, d variable, end mass variable. Linear scale.

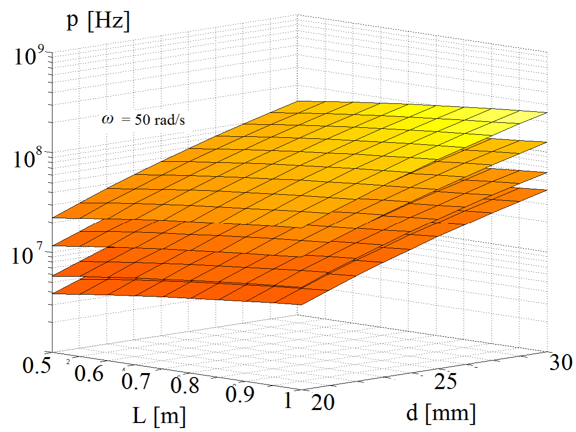

Fig.10. The first eigenfrequency, L variable, d variable, endmass variable. Logaritmic scale.

5. CONCLUSION

In this paper, the motion equations for a beam in a rotary motion were established, being considered fifth-degree shape functions. After determining these equations, it was determined how different parameters defining the properties of the bar can influence the beam eigenfrequencies. There is a problem in the sense that, in a certain range of values, the beam may enter into an unstable response. It determines the eigenfrequencies variation of the beam if the diameter varies, its length changes or the concentrated mass at the end of the bar is varied. All these parameters can influence their eigenfrequencies and, for some values, instability phenomena can be reached.

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Studiul unei bare elastice într-un câmp centrifugal, utilizând metoda elementelor finite

Rezumat. *Metoda elementelor finite a fost aplicată, cu precădere, la studiul comportării diferitelor corpuri elastic, în caz static sau dynamic. Cazul mişcărilor de rigid ale corpurilor analizate cu MEF, care implică termeni noi, datoraţi efectelor inerţiei şi a mişcărilor relative nu a fost încă incorporate în softul clasic de MEF. Lucrări care să analizeze această comportare au apărut în anii 70, pentru bara cu mişcare plană şi, mai apoi, pentru corpuri cu mişcarea general, tridimensională. Lucrarea de faţă îşi propune dezvoltarea unor modele studiate anterior de alţi cercetători şi studiul influenţelor pe care parametrii geometrici ai barei le poate avea asupra răspunsului dinamic al barei. Cazul studiat este acela al unei bare în mişcare de rotaţie cu axă fixă. Funcţiile de interpolare utilizate sunt cele de gradul cinci.*

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