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THE TWO STAGES CIRCLE FITTING METHOD

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Abstract: *The problem of circle fitting (and also of other conics, especially of ellipse) to a number of given points in plane is one of the great importance in computer graphics, image processing, quality control, metrology and other fields of science and engineering. Even if it is necessary to fit an entire circle or just an arc, the same methods are used. The solving of linear and nonlinear algebraic equations, the extensive use of well-known method of least-squares (for the first time discovered by Legendre [10]), the error analysis, are the main mathematical tools used in these procedures for perform the accurate fit of the points with a circle, if this is the demand. The first part of the paper summarizes the existing methods of algebraic and geometric fit, using a circle: the methods of Kåsa [8], Pratt [13] and Taubin [17], having in common the use of least-squares method. With these methods, the coordinates of the centre of the circle and its radius result simultaneously, after solving the system of nonlinear equations. The authors present a new original method for solving the same problem: the coordinates of the centre of the circle are obtained first as a consequence of approximate solving of an over-determined system of linear equations and, in the second stage, is determined the value of the radius by minimizing the objective function. The paper also contains two examples of fitting (with a complete circle and with an arc) solved with the new-imaged procedure.*
Key words: *Least-square method, circle fitting, algebraic and geometric fitting, Newton iterative method, nonlinear equations.*

1. INTRODUCTION

In different fields of science and engineering (physics, biology, quality control, metrology, image processing, computer vision and pattern recognition) it is necessary to fit some geometric curves or surfaces to the given points located in plane or space.

More than a century ago Pearson [12] solved the problem of fitting lines and planes to points situated in plane or space. Many other scientists have studied the problems regarding the fitting of different curves (especially conics) to points placed in plane and of some surfaces that have to fit to points in space.

Because the fitting of circles or circular arcs to points in plane is necessary to be performed in many field of science and engineering such as: computer vision, pattern recognition, metrology, quality control a.o., this issue has been given great attention especially in the last decades [1]-[4], [6], [9], [11], [15], [16], [19]. The existing algorithms solved the problem of minimizing the mean square distance from the circle to the given points in plane, using the least-squares method.

In the scientific literature two possible methods are presented, both determining approximate results: the algebraic circle fitting and the geometric circle fitting.

2. THEORETICAL BACKGROUND

The following problem has to be solved: there are **m** pints placed in the Oxy plane, having known coordinates $\{x_1, y_1, x_2, y_2, \ldots, x_m, y_m\}$, disposed near a circle or circle arc. The coordinates of the centre of the circle and its radius are initially unknown and the goal is to find their values by respecting the requirement that the circle to perform the best fit to all the points. Also one must specify what the best fit is, which are the mathematical expressions that have to minimalize.

As known, the analytical equation of the circle is: $(x-x_c)^2+(y-y_c)^2-r^2=0$.

The error function (the objective function) may have one of the following forms:

$$E=\begin{cases}\sum_{i=1}^{m}\left|(x_i-x_C)^2+(y_i-y_C)^2-r^2\right| & \text{(a)}\\ \sum_{i=1}^{m}\left[(x_i-x_C)^2+(y_i-y_C)^2-r^2\right]^2 & \text{(b)}\\ \sum_{i=1}^{m}\left[\sqrt{(x_i-x_C)^2+(y_i-y_C)^2}-r\right]^2 & \text{(c)}\end{cases} \quad (1)$$

The cases (a) and (b) correspond to so called algebraic fitting, when the sum of deviations (or squares) of the circle equation is considered. In this case the current coordinates are replaced by the points coordinates.

2.1 The algebraic fitting of the circle

Especially the case (b) is considered:

$$E_{alg}=\sum_{i=1}^{m}\left[\underbrace{x_C^2+y_C^2-r^2}_{u}+x_i^2+y_i^2-2x_ix_C-2y_iy_C\right],$$

$u=x_C^2+y_C^2-r^2$ (2)

We have to find the values of the unknowns x_C, y_C and r that minimize the objective function. After calculating partial derivatives of the expression E_{alg} with respect to the unknowns x_C, y_C and r, and setting them to zero:

$$\frac{\partial E_{alg}}{\partial x_C}=0,\ \frac{\partial E_{alg}}{\partial y_C}=0,\ \frac{\partial E_{alg}}{\partial u}=0$$

three simultaneous linear equations are obtained, resulting x_C, y_C, and $r=\sqrt{x_C^2+y_C^2-u}$, the situation is given in relation (3).

$$\begin{bmatrix} 2\sum_{i=1}^{m}x_i^2 & 2\sum_{i=1}^{m}x_iy_i & -\sum_{i=1}^{m}x_i\\ 2\sum_{i=1}^{m}x_iy_i & 2\sum_{i=1}^{m}y_i^2 & -\sum_{i=1}^{m}y_i\\ 2\sum_{i=1}^{m}x_i & 2\sum_{i=1}^{m}y_i & -m\end{bmatrix}\begin{bmatrix}x_C\\ y_C\\ u\end{bmatrix}=\begin{bmatrix}\sum_{i=1}^{m}(x_i^3+x_iy_i^2)\\ \sum_{i=1}^{m}(x_i^2y_i+y_i^3)\\ \sum_{i=1}^{m}(x_i^2+y_i^2)\end{bmatrix} \quad (3)$$

This method was introduced by Kåsa [8].

In spite of the fact that algebraic fit is easy to use (the elements of 3x3 matrix and of 3x1 column are determined and a system of three equation is solved) and gives a rapid result, this method has serious drawbacks in accuracy, the determined circle having an unintended geometric feature, especially when the points are situated near the circle arc [1], [2], [4].

To improve this procedure, Pratt [13] and Taubin [17] have modified the objective function and introduced some constraints. Practical experience shows that these two fits, proposed by Pratt and Taubin, are more stable and accurate than Kåsa fit, and they perform nearly equally well.

2.2 The geometric fitting of the circle

In the case of geometric fitting, the error function contains the sum of geometric distances between the given points and the geometric feature (in this studied case - the circle). Considering the third expression, noted (c), as objective function:

$$E_{geom}=\sum_{i=1}^{m}\left[\sqrt{(x_i-x_C)^2+(y_i-y_C)^2}-r\right]^2 \quad (4)$$

and setting to zero the computed partial derivatives:

$$\frac{\partial E_{geom}}{\partial x_C}=f_1(\underbrace{x_C,y_C,r}_{unknowns},x_1,y_1,x_2,y_2,\ldots,x_m,y_m)=0$$

$$\frac{\partial E_{geom}}{\partial y_C} = f_2(\underbrace{x_C, y_C, r}_{unknowns}, x_1, y_1, x_2, y_2, \ldots, x_m, y_m) = 0$$

$$\frac{\partial E_{geom}}{\partial r} = f_3(\underbrace{x_C, y_C, r}_{unknowns}, x_1, y_1, x_2, y_2, \ldots, x_m, y_m) = 0$$

one obtains a nonlinear system of equations:

$$F(X)=0 \quad X_0 = X_{alg} \tag{5}$$

The proper method for solving this system is the Newton's method [14], [18], [20].

The performance of this iterative algorithm heavily depends on the choice of the initial guess.

The initial values of unknown, used in the first iteration, will be considered the results obtained after solving the algebraic fitting problem [5], [7]:

$$x_{C\ geom\ init} = x_{C\ alg}, \quad y_{C\ geom\ init} = y_{C\ alg}, \quad r_{geom\ init} = r_{alg}$$

The basic formula of the Newton's method is the following:

$$X_{k+1} = X_k - \left[J(X_k)\right]^{-1} F(X_k) \tag{6}$$

where $J(X_k)$ is the Jacobi matrix

$$J(X_k) = \begin{bmatrix} \frac{\partial f_1}{\partial x_C} & \frac{\partial f_1}{\partial y_C} & \frac{\partial f_1}{\partial r} \\ \frac{\partial f_2}{\partial x_C} & \frac{\partial f_2}{\partial y_C} & \frac{\partial f_2}{\partial r} \\ \frac{\partial f_3}{\partial x_C} & \frac{\partial f_3}{\partial y_C} & \frac{\partial f_3}{\partial r} \end{bmatrix}$$

Usually two steps are performed: the solving of the linear algebraic system $J(X_k)\Delta_k = -F(X_k)$, followed by the modification of column matrix containing the solutions $X_{k+1} = X_k + \Delta_k$.

In the last decades this method was presented in the literature in several variants.

2.3 The new original procedure

As we notice, according to the previously described methods, the centre circle coordinates and its radius result simultaneously after solving the system of nonlinear equations.

The same problem may be solved in a different way, as described below. In the first stage, the coordinates of circle centre are determined, as a consequence of approximate solving of an over-determined system of linear equations. In the second stage is determined the radius value by minimizing the objective function.

Let's look at figure 1. Considering two different given points, D_k and D_ℓ, we notice that the circle passing through that points has the centre on the straight perpendicular line on the middle point of the segment D_kD_ℓ.

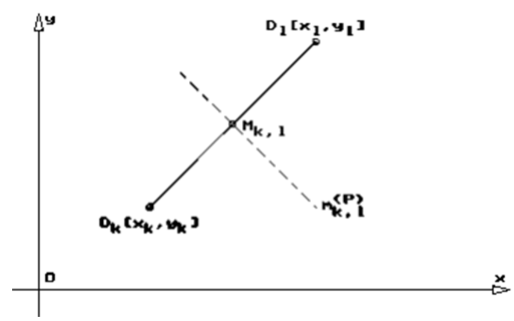

Fig. 1. The new proposed procedure

The slopes of the segment D_kD_ℓ and of the perpendicular line are as follows:

$$m_{k,\lambda} = \frac{y_\lambda - y_k}{x_\lambda - x_k}, \quad m_{k,\lambda}^{(P)} = -\frac{1}{m_{k,\lambda}} = \frac{x_k - x_\lambda}{y_\lambda - y_k}$$

and we can write the equation of the straight line containing the circle centre:

$$\begin{gathered} y - y_M = m_{k,\lambda}^{(P)}(x - x_M), \\ y - \frac{y_k + y_\lambda}{2} = \frac{x_k - x_\lambda}{y_\lambda - y_k}\left(x - \frac{x_k + x_\lambda}{2}\right) \\ 2(x_k - x_\lambda)x + 2(y_k - y_\lambda)y = y_k^2 - y_\lambda^2 + x_k^2 - x_\lambda^2, \\ k, \lambda \in \{1,2,3,.\,,n\},\ k \neq \lambda \end{gathered} \tag{7}$$

Considering all possible combinations of two given points D_k and D_ℓ we obtain $m = \frac{1}{2}n(n-1)$ linear equations of perpendicular lines

$$a_{i,1}x+a_{i,2}y=b_i,\quad i=\overline{1,m} \tag{8}$$

The coefficients of unknown x and y and the right terms of equations have the expressions:

$$\begin{aligned}&a_{i,1}=2(x_k-x_\lambda),\quad a_{i,2}=2(y_k-y_\lambda),\\&b_i=y_k^2-y_\lambda^2+x_k^2-x_\lambda^2,\\&i=1,2,3,\dots,m=\tfrac{1}{2}n(n-1)\end{aligned} \tag{9}$$

An over-determined system of linear algebraic equations is obtained:

$$\begin{aligned}&A_{m,n}X_n=B_m,\quad m>n=2\\&x_C=x_1,\ y_C=x_2\end{aligned} \tag{10}$$

Systems of this type are approximately solved using the well-known method of least squares, according to the following algorithm.

Considering **m** equations with **s** unknowns (m>s):

$$\begin{bmatrix}a_{1,1}&a_{1,2}&a_{1,3}&\dots&a_{1,s}\\a_{2,1}&a_{2,2}&a_{2,3}&\dots&a_{2,s}\\a_{3,1}&a_{3,2}&a_{3,3}&\dots&a_{3,s}\\\dots&\dots&\dots&\dots&\dots\\\dots&\dots&\dots&\dots&\dots\\\dots&\dots&\dots&\dots&\dots\\a_{m,1}&a_{m,2}&a_{m,3}&\dots&a_{m,s}\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\\\dots\\x_s\end{bmatrix}=\begin{bmatrix}b_1\\b_2\\b_3\\\dots\\\dots\\\dots\\b_m\end{bmatrix},\quad m>s \tag{11}$$

the solutions of this system minimize the objective function value:

$$S(x_1,x_2,x_{3,\dots},x_s)=\sum_{i=1}^{m}\left(b_i-\sum_{j=1}^{s}a_{i,j}x_j\right)^2$$

By setting to zero the partial derivatives:

$$\begin{aligned}&\frac{\partial S}{\partial x_1}=0,\\&\dots,\\&\frac{\partial S}{\partial x_j}=-2\sum_{i=1}^{m}\left(b_i-\sum_{k=1}^{s}a_{i,k}x_k\right)a_{i,j}=0,\\&\dots,\\&\frac{\partial S}{\partial x_s}=0\end{aligned}$$

a system of **s** linear equations is obtained:

$$\sum_{k=1}^{s}\left(\sum_{i=1}^{m}a_{i,j}a_{i,k}\right)x_k=\sum_{i=1}^{m}a_{i,j}b_i,\quad j=\overline{1,s} \tag{12}$$

The system may be written under the following matrix form:

$$\begin{bmatrix}c_{1,1}&c_{1,2}&..&c_{1,s}\\c_{2,1}&c_{2,2}&..&c_{2,s}\\\dots&\dots&..&\dots\\c_{s,1}&c_{s,2}&\dots&c_{s,s}\end{bmatrix}\begin{bmatrix}x_1\\x_2\\..\\x_s\end{bmatrix}=\begin{bmatrix}d_1\\d_2\\..\\d_s\end{bmatrix} \tag{13}$$

with the following notations:

$$c_{j,k}=\sum_{i=1}^{m}a_{i,j}a_{i,k},\quad d_j=\sum_{i=1}^{m}a_{i,j}b_i,\quad j=\overline{1,s}$$

If the number of unknowns is only two, $n=2$, $x_C=x_1$, $y_C=x_2$, the system (13) becomes:

$$\begin{bmatrix}\sum_{i=1}^{m}a_{i,1}^2&\sum_{i=1}^{m}a_{i,1}a_{i,2}\\\sum_{i=1}^{m}a_{i,1}a_{i,2}&\sum_{i=1}^{m}a_{i,2}^2\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}\sum_{i=1}^{m}a_{i,1}b_i\\\sum_{i=1}^{m}a_{i,2}b_i\end{bmatrix} \tag{14}$$

and the values of $x_1=x_C$, $x_2=y_C$ are obtained. The first stage of the proposed procedure is accomplished.

Now, considering the **m** given points and the determined centre of the fitting circle we may compute the values r_{min} and r_{max}, representing the minimal and maximal distances between the centre C and the given points. Of course the values $r=r_{min}$ or $r=r_{max}$ do not yield the minimal values for the objective function.

$$E(r)=\sum_{i=1}^{m}\left[\sqrt{(x_i-x_C)^2+(y_i-y_C)^2}-r\right]^2 \tag{15}$$

Such values will be obtained searching for the minimal objective function value, considering the values for r in the interval $[r_{min}; r_{max}]$ with an adequate step.

This new original method was implemented in C and some numerical results are shown below.

3. NUMERICAL EXPERIMENTS

3.1. First example

In figure 2 are presented five points with known coordinates and the perpendicular lines on the segments linking the given points.

Table 1

First example - Coordinates of the five points

i	0	1	2	3	4
x_i	212	149	33	47	145
y_i	80	168	143	27	-20

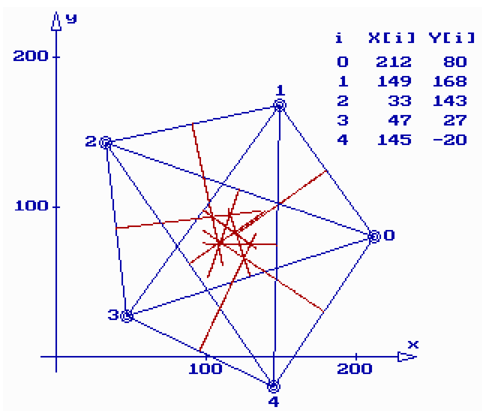

Fig. 2. First example - The five given points, the ten segments and perpendiculars

Using the relations (9) we can write the following over-determined system:

$$\begin{bmatrix} 126 & -176 \\ 358 & -126 \\ 330 & 106 \\ 134 & 200 \\ 232 & 50 \\ 204 & 282 \\ 8 & 376 \\ -28 & 232 \\ -224 & 326 \\ -196 & 94 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 919 \\ 29806 \\ 48406 \\ 29919 \\ 28887 \\ 47487 \\ 29000 \\ 18600 \\ 113 \\ -18487 \end{bmatrix}$$

and according to (12), system (14) has the following particular form:

$$\begin{cases} 4.5578 \times 10^5 x_C - 3.1312 \times 10^4 y_C = 5.0468 \times 10^7 \\ -3.1312 \times 10^4 x_C + 4.9042 \times 10^5 y_C = 3.5551 \times 10^7 \end{cases}$$

witch can also be written in a simplified way:

$$\begin{cases} 45.578 x_C - 3.1312 y_C = 5046.8 \\ -3.1312 x_C + 49.042 y_C = 3555.1 \end{cases}$$

having the solutions:

$$x_C = 116.219658, \quad y_C = 79.911566$$

In figure 3 the circle centre is placed among the multiple intersections of the perpendicular lines.

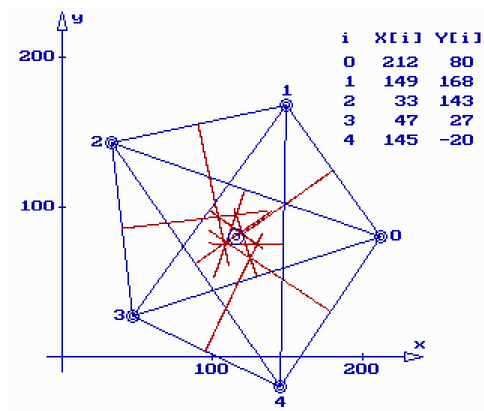

Fig. 3. First example - The circle centre is determined

In figure 4 are shown the two circles having radii equal to r_{min} and r_{max}. Figure 5 shows the shape of the objective function with minimal value for searched circle radius (r=97.076).

Finally, in figure 6 one may see the circle having the centre in the point determined in first stage of the procedure and the radius determined in the second stage.

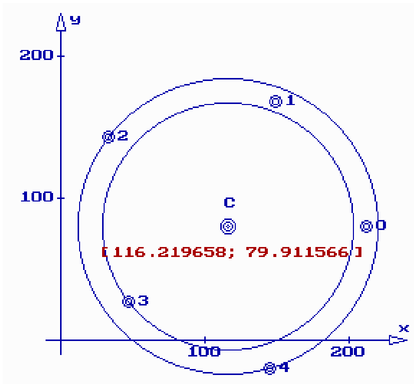

Fig. 4. First example - The circles with minimal and maximal radii

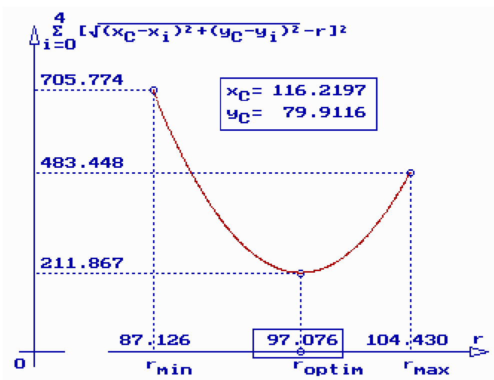

Fig. 5. First example - The shape of the objective function

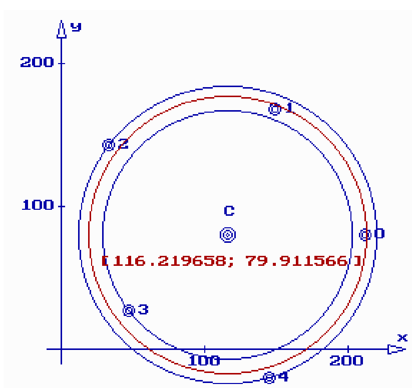

Fig. 6. First example **-** The fitting circle with radius r=97.076

3.2. Second example

There is presented another numerical example solved by using the same method, the difference consisting in the fact that the fitting is performed to an arc of circle.

Five plan points are considered, with known coordinates, given in Table 2.

Table 2

Second example - Coordinates of the five points

i	0	1	2	3	4
x_i	102	51	12	21	62
y_i	175	151	102	47	-6

The over-determined system of linear algebraic equations is:

$$\begin{bmatrix} 102 & 48 \\ 180 & 146 \\ 162 & 256 \\ 80 & 362 \\ 78 & 98 \\ 60 & 208 \\ -22 & 314 \\ -18 & 110 \\ -100 & 216 \\ -82 & 106 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 15627 \\ 30481 \\ 38379 \\ 37149 \\ 14854 \\ 22752 \\ 21522 \\ 7898 \\ 6668 \\ -1230 \end{bmatrix}$$

and according to (12) results the system (14):

$$\begin{cases} 1.0266\text{x}10^5\, x_C + 8.2552\text{x}10^4\, y_C = 1.7612\text{x}10^7 \\ 8.2552\text{x}10^4\, x_C + 4.4166\text{x}10^5\, y_C = 4.3598\text{x}10^7 \end{cases}$$

with final form:

$$\begin{cases} 10.266\, x_C + 8.2552\, y_C = 1761.2 \\ 8.2552\, x_C + 44.166\, y_C = 4359.8 \end{cases}$$

and the solutions:

$$x_C = 108.477323\,, \quad y_C = 78.438803\,.$$

In figures 7, 8 and 9 are shown the results of the first and second stage of the previous presented procedure for solving the fitting problem and also the objective function.

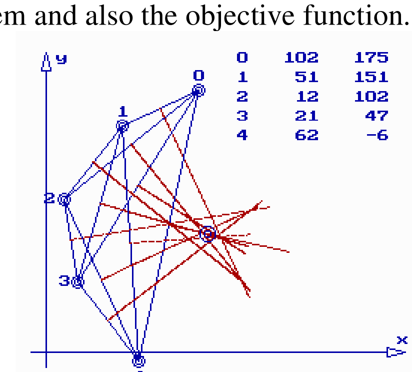

Fig. 7. Second example - The coordinates of circle arc centre are: x_C=108.477, y_C=78.439

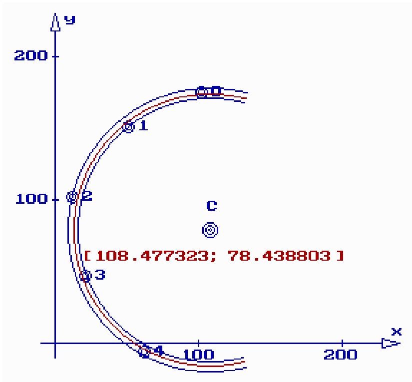

Fig. 8. Second example - The fitting circle arc radius is r=95.603

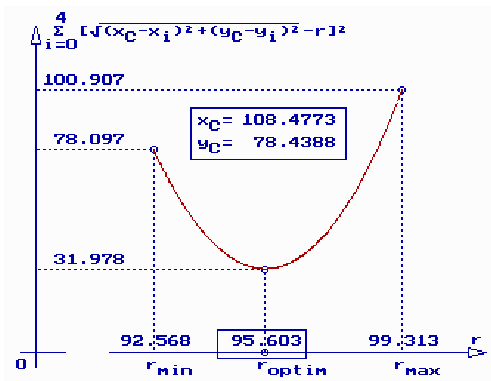

Fig. 9. Second example - The shape of objective function

4. CONCLUSION

This method for performing the circle fitting is based on the least-squared method applied for approximate solving of over determined linear system of equations, resulting the circle centre coordinate values. As it is known using the so called algebraic circle fitting the position of circle centre are determined with errors, sometimes too big.

Considering the obtained position of the centre, the radius of the circle may be determined very accurate, in the second stage of the procedure that was presented in this paper.

The simplicity and correctness of this method is obvious as well as the convenient C code programming.

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METODA DE APROXIMARE A CERCULUI CU DOUĂ ETAPE

Rezumat: Problema aproximării cercului (și a altor conice, mai ales a elipsei) la un anumit număr de puncte date, situate în plan, este una de importanța deosebită în grafica computerizată, prelucrarea imaginilor, controlul calității, metrologie și alte domenii ale științei și ingineriei. Chiar dacă este necesar să se aproximeze un cerc întreg sau doar un arc de cerc, metodele utilizate sunt aceleaşi. Rezolvarea ecuațiilor algebrice liniare şi neliniare, utilizarea extensivă a bine cunoscutei metode a celor mai mici pătrate (pentru prima dată descoperită de Legendre [10]), analiza erorilor, sunt principalele instrumente matematice utilizate în aceste proceduri pentru a realiza o cât mai bună aproximare a punctelor date cu un cerc, dacă aceasta este cerinţa. Prima parte a lucrării prezintă pe scurt metodele existente pentru aproximare algebrică şi geometrică, folosind un cerc: metodele lui Kåsa [8], Pratt [13] și Taubin [17], care au în comun utilizarea metodei celor mai mici pătrate. Cu aceste metode, coordonatele centrului cercului și raza lui rezultă simultan, după rezolvarea sistemului de ecuații neliniare. Autorii prezintă o metodă noua, originală, pentru rezolvarea aceleiași probleme: coordonatele centrului cercului sunt obținute mai întâi ca o consecință a rezolvării aproximative a unui sistem supra-determinat de ecuații lineare și, în a doua etapă, se determină valoarea razei prin minimizarea funcției obiectiv. Lucrarea conţine, de asemenea, două exemple de aproximare (cu un cerc complet și cu un arc) rezolvate prin procedura nou-imaginată.

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