USING THE LINEAR PROGRAMMING TO DETERMINE THE PRODUCTION PLAN OF THE ENTERPRISE

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Abstract: Maximizing the profit is the desirability of any enterprise but this is not an easy task when it is desired to achieve several types of products and given that in the equation enters a series of variables such as: the available resources, the resource consumption for the product unit, the profit on each type of product that can be achieved, etc. Therefore the matter of identifying the optimal variant of the quantity of each type of product to be made so the profits earned by the enterprise to be maximum is searched. This paper deals with this problem by transposing data in the form of a mathematical programming problems.

Key words: mathematical programming, linear programming, production plan.

1. INTRODUCTION

The production process can be defined as “the totality of the conscious actions of employees of an enterprise, realized with the help of different machines, equipment or installations upon the raw materials, materials or other components for the purpose of their transformation into products, works or services with a certain market value”. [11]

For the production process to be carried out in optimal conditions and in order to obtain a good productivity an enterprise needs a solid production plan, this because all the activities of the enterprise start from the production plan. Usually, a plan refers to materials, equipment, human resources, training, capacity and the ways to complete the work in a specific time.

That is why in order to achieve a continuous flow of work the production plan must take into account some key elements before the production starts, such as: equipment acquisition, material ordering, bottlenecks, procurement and training of the labor force. [10]

Considering that when the production plan is realized, it is necessary to take into account a series of restrictions regarding the factors of production both in number and as a yield, a plan that has to aim the maximization of profits, it is necessary to use optimization techniques (linear programming, PERT method, queuing theory, inventory optimization, etc.) and computer techniques (computer programs for optimal programming of production).

Linear programming is successfully used to optimize allocation of resources, to allocate production on different machines under conditions of maximizing profits, to determine the quantities of various goods to be produced, to transport products between working places and from there to the distribution points. [11]

By linear programming, it is intended to achieve aims under conditions of restrictions.

2. THE GENERAL FORM OF THE MATHEMATICAL PROGRAMMING PROBLEM

A problem of mathematical programming (optimization problem with restrictions) can be stated as follows:

Find the values of the variables \( x_1, x_2, \ldots, x_n \) that verify the restrictions given in the form of equality or inequality such as:

\[
\begin{align*}
&g_i(x) \leq 0 (\geq 0) \quad i = 1, 2, \ldots, m \\
&h_k(x) = 0 \quad K = 1, 2, \ldots, p
\end{align*}
\]
where \( x = (x_1, x_2, ..., x_n) \in \mathbb{M} \subseteq \mathbb{R}^n \) and which must achieve the maximum (minimum) of the objective function (scope) represented by function \( f \).

The optimization problem with restrictions can be represented symbolically in the following way:

\[
\begin{align*}
\text{min}(\max) & \quad f(x) \\
g_i(x) & \leq 0 \quad (i = 1, 2, ..., m) \\
h_k(x) & = 0 \quad (k = 1, 2, ..., p)
\end{align*}
\]

where:

\[
\begin{align*}
c = (c_1, c_2, ..., c_n), & \quad x = (x_1, x_2, ..., x_n), \\
g_i(x) & = \sum_{k=1}^{n} a_{ik} \cdot x_k - b_i \quad (i = 1, 2, ..., m).
\end{align*}
\]

The problem can be stated: determine the maximum of function \( f \) on the set of restrictions \( g_i(x) \leq 0 \quad (i = 1, 2, ..., m) \).

The linear programming problem can be written in the following forms:[1]

- Canonical:

\[
\max(\min) \left\{ cx : \sum_{k=1}^{n} a_{ik} \cdot x_k \leq b_i, x \geq 0 \right\}
\]

- Standard:

\[
\max(\min) \left\{ cx : \sum_{k=1}^{n} a_{ik} \cdot x_k = b_i, x \geq 0 \right\}
\]

As examples of linear programming problems can be stated[9], [12]: the problem of the optimal production plan, the diet problem (of mixture), the problem of the optimal use of resources and the transport problem.

4. SOLVING LINEAR PROGRAMMING PROBLEMS USING THE DUALITY THEOREMS

The duality wants that for a primal problem \((P)\) – which is a maximization problem with restrictions having the form: \( \max\{f(x): x \in \Omega\} \), to build a dual problem \((D)\) – which is a minimization problem with restrictions having the form: \( \min\{f^*(u): u \in \Omega^*\} \) and the conditions for the two problems to comply with the following properties:

- for each \( x \in \Omega \) and for each \( u \in \Omega^* \) to be \( f(x) \leq f^*(u) \);
- if the primal problem has an optimal solution, then the dual problem has also an optimal
solution, and the optimal values of the objective functions are equal;

- if the dual problem has an optimal solution, then the primal problem has also an optimal solution, and the optimal values of the objective functions are equal.

The primal problem (P):
\[
\text{Max}\{cx: Ax \leq b, x \geq 0\}
\]
(can also be written as: \(\text{min}\{-cx: -Ax \geq -b, x \geq 0\}\)), which has the dual (D) of form:
\[
\text{min}\{b^Tu: A^Tu \geq c^T, u \geq 0\}
\]
(can also be written as: \(\text{max}\{-b^Tu: -A^Tu \leq -c, u \geq 0\}\)).

For the two problems we have the following notations:
- \(c = (c_1, c_2, \ldots, c_n)\),
- \(x = (x_1, x_2, \ldots, x_n)^T\),
- \(A = (a_{ik})\) a matrix of type \((m \times n)\),
- \(b = (b_1, b_2, \ldots, b_n)^T\),
- \(f(x) = cx\), an objective function for (P);
- \(f'(u) = b^Tu\), an objective function for (D);
- \(\Omega = \{x \in R^n: Ax \leq b, x \geq 0\}\), the set of admissible solutions for (P);
- \(\Omega^* = \{u \in R^m: A^Tu \geq c^T, u \geq 0\}\), the set of admissible solutions for (D);

The solving of a linear program of the form (P) is done with the simplex algorithm, which determines whether the program solution is optimal, and otherwise it shows the optimal infinite situation or actually builds a better solution than the current one, the simplex method being a systematic research process of linear program solutions.

A special quality of the simplex algorithm is that it produces not only the optimal solution to the problem to which it applies but also the optimal solution to the dual problem.

Simplex tables are used to manually solve small-scale linear programs, while for large-scale linear programs, computer programs are used to eliminate the "rough" calculations inherent in solving this algorithm.

Further, the simplex table attached to the two problems is provided:

<table>
<thead>
<tr>
<th>Simplex table</th>
<th>(v_1 = )</th>
<th>(v_2 = )</th>
<th>(v_s = )</th>
<th>(v_m = )</th>
<th>(f^* = )</th>
</tr>
</thead>
<tbody>
<tr>
<td>(-x_1)</td>
<td>(-x_3)</td>
<td>(-x_s)</td>
<td>(-x_n)</td>
<td>(1)</td>
<td></td>
</tr>
<tr>
<td>(u_1, y_1 = )</td>
<td>(a_{11})</td>
<td>(a_{12})</td>
<td>(\ldots)</td>
<td>(a_{1s})</td>
<td>(a_{1n})</td>
</tr>
<tr>
<td>(u_2, y_2 = )</td>
<td>(a_{21})</td>
<td>(a_{22})</td>
<td>(\ldots)</td>
<td>(a_{2s})</td>
<td>(a_{2n})</td>
</tr>
<tr>
<td>(\ldots)</td>
<td>(\ldots)</td>
<td>(\ldots)</td>
<td>(\ldots)</td>
<td>(\ldots)</td>
<td>(\ldots)</td>
</tr>
<tr>
<td>(u_r, y_r = )</td>
<td>(a_{r1})</td>
<td>(a_{r2})</td>
<td>(\ldots)</td>
<td>(a_{rs})</td>
<td>(a_{rn})</td>
</tr>
<tr>
<td>(\ldots)</td>
<td>(\ldots)</td>
<td>(\ldots)</td>
<td>(\ldots)</td>
<td>(\ldots)</td>
<td>(\ldots)</td>
</tr>
<tr>
<td>(u_m, y_m = )</td>
<td>(a_{m1})</td>
<td>(a_{m2})</td>
<td>(\ldots)</td>
<td>(a_{ms})</td>
<td>(a_{mn})</td>
</tr>
<tr>
<td>(1, f = )</td>
<td>(-c_1)</td>
<td>(-c_2)</td>
<td>(\ldots)</td>
<td>(-c_s)</td>
<td>(-c_n)</td>
</tr>
</tbody>
</table>

In order to build the dual of a linear programming problem, the following should be taken into account:[1]

- a) the variables number of the dual is equal to the constraints number of the primal problem;
b) the constraints number of the dual is equal to the variables number of the primal problem;
c) the matrix of the dual is the transpose of the primal matrix;
d) to each equality of the form \(\sum_{j=1}^{n} a_{ij} \cdot x_j \leq b_i\) with \(i \in \{1, 2, \ldots, m_1\}\) from the primal problem corresponds in the dual the variable \(u_i \geq 0, i = 1, 2, \ldots, m\);
e) to each equality of the form \(\sum_{j=1}^{n} a_{ij} \cdot x_j \leq b_i\) with \(i \in \{m_1+1, \ldots, m\}\) from the primal corresponds the variable \(u_i, i = m_1+1, \ldots, m\), unbound, in the dual;
f) to each primal variable \(x_i \geq 0\), \(i \in \{1,2,...,n\}\) corresponds in the dual an equation of form \(\sum_{j=1}^{m} a_{ij} u_i = c_j\).

To solve a primal problem \((P)\) of the form \(\max \{cx: Ax \leq b, x \geq 0\}\) three stages must be passed through:

I. Elimination of independent variables \(x_1, x_2, ..., x_n\), is no longer necessary because these variables are subject to non-negativity restrictions, meaning \(x_i \geq 0\).

II. Determination of an admissible solution.

At this stage, the choice of the pivot element is sought, so that after a finite number of steps a basic solution is obtained or is found out that the system is incompatible.

Be the dual of a linear programming problem written in the form of:

\[
\begin{align*}
\text{Table 2.} \\
\text{The dual of a linear programming problem} \\
& v_1 = \begin{bmatrix} u_1 & u_2 & \ldots & u_i & \ldots & u_n & 1 \end{bmatrix} \\
& v_2 = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1i} & \ldots & a_{1n} \end{bmatrix} \\
& \ldots \\
& v_r = \begin{bmatrix} a_{ri} & a_{r2} & \ldots & a_{ri} & \ldots & a_{rn} \end{bmatrix} \\
& \ldots \\
& v_m = \begin{bmatrix} a_{mi} & a_{m2} & \ldots & a_{mi} & \ldots & a_{mn} \end{bmatrix} \\
& f^* = \begin{bmatrix} q_1 & q_2 & \ldots & q_i & \ldots & q_n & 0 \end{bmatrix} \\
& \text{The pivot elements are chosen according to the minimal ratio rule, which is built between the last line and the pivot line:} \\
& \frac{q_i}{a_{is}} = \min_{1 \leq i \leq m} \left\{ \frac{q_i}{a_{is}} > 0, a_{is} \neq 0 \right\}. \quad (6)
\end{align*}
\]

where: \(s = \text{Gauss-Jordan number of steps; so in order to find the basic admissible solution, the last line from Table 2 is analyzed.}\)

If the line of objective function \((f^*)\) is positive, this step is not necessary, and even under these conditions the rule of the minimum ratio will continue to be used.

III. Improvement of the admissible solution and finding the optimal solution.

This stage will start when at the end of the second stage, the following situation will be encountered in the resulting table: \(d_1 \geq 0, d_2 \geq 0, \ldots, d_m \geq 0\). To find the optimal solution, the last column will be analyzed. If all the values on this column are positive, then the problem has the optimal solution. To determine this, proceed as follows: the elements \(u_i\) from above the table will be zero, and for the elements \(u_i\) that are to the left of the table the values are read from the table, from the column "1".

After choosing the pivot element, marked by framing the pivot element into a square, a Gauss-Jordan step is performed according to the rules:

1) the pivot element is replaced by 1;
2) the other elements in the pivot column remain unchanged;
3) the other elements in the pivot line change the sign;
4) the elements of the other lines and columns are calculated with the formula:
\[
b_{ik} = \begin{bmatrix} a_{ik} & a_{ik} \\ a_{ik} & a_{ik} \end{bmatrix} \\
\text{(7)}
\]

The main diagonal must be considered to be always the diagonal with the pivot element \((a_{rs})\).

5) all the elements of the new table are divided with the pivot element.

5. CASE STUDY

Knowing that an enterprise produces 4 types of products \(P_1, P_2, P_3, P_4\), for which 5 resources \(R_1, R_2, R_3, R_4\) and \(R_5\) are used, it is desired to determine a production plan so the benefit obtained by the enterprise to be maximum. The data of the available quantities of the 5 resources, resource consumption for the product unit and the benefit to be achieved by selling a quantity of each product type are given in the following table:

\[
\begin{align*}
\text{Table 3.} \\
\text{Products} & \text{Resources} & \text{Data of the problem} \\
\hline
\text{Product} & \text{P}_1 & \text{P}_2 & \text{P}_3 & \text{P}_4 & \text{Available} \\
\hline
\text{R}_1 & 1 & 2 & 4 & 3 & 12 \\
\text{R}_2 & 3 & 5 & 1 & 0 & 30 \\
\text{R}_3 & 0 & 1 & 2 & 4 & 9 \\
\text{R}_4 & 1 & 2 & 1 & 3 & 10 \\
\text{R}_5 & 1 & 1 & 4 & & 8 \\
\text{Benefit} & 2 & 3 & 4 & 1 & \\
\end{align*}
\]

and the objective function is: \(2x_1 + 3x_2 + 4x_3 + x_4 \rightarrow \max\), function which aims to maximize the benefit obtained by the enterprise as a result of the products' realization.
The objective function is written together with the constraints system and so the primal problem (P) is obtained:

\[ 2x_1 + 3x_2 + 4x_3 + x_4 \rightarrow \text{max} \quad (8) \]

\[
\begin{aligned}
&x_1 + 2x_2 + 4x_3 + 3x_4 \leq 12 \\
&3x_1 + 5x_2 + x_3 \leq 30 \\
&x_2 + 2x_3 + 4x_4 \leq 9 \\
&x_1 + 2x_2 + x_3 + 3x_4 \leq 10 \\
&x_1 + x_2 + 4x_3 + x_4 \leq 8 \\
&x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, x_4 \geq 0
\end{aligned} \quad (9) \]

In the above system, \( x_1, \ldots, x_4 \) are the quantities that need to be produced from the products \( P_1, \ldots, P_4 \) in conformity with the 5 restrictions that occur due to the available resources for each product. In addition, it must be taken into account that these quantities can only have positive values, which is why to the system will be added the conditions of non-negativity \( x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, x_4 \geq 0 \).

The dual problem (D) is associated to the primal problem:

\[ 12u_1 + 30u_2 + 9u_3 + 10u_4 + 8u_5 \rightarrow \text{min} \quad (10) \]

\[
\begin{aligned}
&u_1 + 3u_2 + u_4 + u_5 - 2 \geq 0 \\
&2u_1 + 5u_2 + u_3 + 2u_4 + u_5 - 3 \geq 0 \\
&4u_1 + u_2 + 2u_3 + u_4 + 4u_5 - 4 \geq 0 \\
&3u_1 + 4u_3 + 3u_4 + u_5 - 1 \geq 0 \\
u_1 \geq 0, u_2 \geq 0, u_3 \geq 0, u_4 \geq 0, u_5 \geq 0
\end{aligned} \quad (11) \]

The optimal solution to the dual problem is:

\[ u_1=0, u_2=0, u_3=0, u_4=1, u_5=1. \]

\[ \min f^* = 18 = f^*(0, 0, 0, 1, 1) \]

The optimal solution to the primal problem is:

\[ x_1=6, x_2=2, x_3=0, x_4=0. \]

\[ \max f = 18 = f(6, 2, 0, 0) \]

6. CONCLUSIONS

The maximum benefit obtained by the enterprise is 18 monetary units, and in order to achieve this benefit 8 products: 6 of \( P_1 \), 2 of \( P_2 \) and 0 of \( P_3 \) and \( P_4 \) have to be realized.

In order to justify the solution obtained by solving the problem, reference should be made to the economic interpretation of duality. According to this, the enterprise aims to
minimize the total amount of the consumed resources. Therefore, analyzing Table 3 from the statement of the problem, it can be found the following:

- for product P\textsubscript{1}, 6 units of resources are consumed with a benefit of 2 units;
- for product P\textsubscript{2}, 11 units of resources are consumed with a benefit of 3 units;
- for product P\textsubscript{3}, 12 units of resources are consumed with a benefit of 4 units;
- for product P\textsubscript{4}, 11 units of resources are consumed with a benefit of 1 unit.

In the final result are found:

- product P\textsubscript{1} with a minimum resource consumption of 6;
- product P\textsubscript{2} which has the same resource consumption of 11, as well as the product P\textsubscript{4}, but has a superior benefit.

7. REFERENCES


Utilizarea programării liniare pentru determinarea planului de producție al întreprinderii

Rezumat: Maximizarea profitului este dezideratul oricărei întreprinderi dar acest lucru nu este o sarcină ușoară atunci când se dorește realizarea mai multor tipuri de produse și având în vedere că în ecuație intră o serie de variabile precum: resursele disponibile, consumurile de resurse pe unitatea de produs, profitul pe fiecare tip de produs ce se poate obține, etc. De aceea, se pune problema identificării variantei optime a cantității din fiecare tip de produs care trebuie realizată astfel încât profitul obținut de companie să fie maxim. Lucrarea de față tratează această problemă prin transpunerea datelor sub forma unei probleme de programare matematică.

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