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FINITE ELEMENT USED IN THE DYNAMIC ANALYSIS OF A MECHANICAL PLANE MBS WITH A PLANAR "RIGID MOTION"

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Abstract: *The paper aims to study a finite element in the case of plane motion of a plane mechanical system. The problem of using two-dimensional finite elements in the dynamic analysis of membranes has been little studied in the literature, which is mainly due to the formalism involved, which requires a special calculation effort. The evolution equations are established and the matrix coefficients are calculated. An example of calculating the eigenvalues for a rectangular shell is presented, which uses the formalism developed in the paper.*

Key words: *Multibody System (MBS), Finite Element Method (FEM), eigenfrequencies, eigenvectors, shell, two-dimensional*

1. INTRODUCTION

The last decades have been characterized by the development of multi-body systems (MBS) with deformable elements mostly in industrial applications. The first researches in the field were made for one-dimensional finite elements using third and fifth degree polynomial shape functions. Then the interest moved on to more complex, two-dimensional or three-dimensional finite elements.

The most common method to obtain the dynamic response of such a system is represented by Lagrange's equations [8-13]. Determining the evolution equations of a single element is the most important step in an analysis of an MBS system. The other procedures that follow in such an analysis are the classic ones, well known from the finite element commercial software. After all these steps, the evolution equations for this problem are obtained. The final shape of the matrix coefficients will ultimately depend on the interpolation functions chosen, for each individual case, for the finite element used. In the present work we aim to determine the evolution equations of an element, used in the study of plane mechanisms with elastic elements[6],[7],[14],[15].

2. EQUATION OF MOTIONS

Consider the plate used for stress analysis in plane elasticity. Let us consider a single finite element referred to the local Oxy coordinate system, in solidarity with the finite element. This reference system is mobile and participates in the parallel plane motion of the plate. It is noted with $\bar{v}\,o(\dot{X}_o, \dot{Y}_o)$ the speed and $\bar{a}\,o(\ddot{X}_o, \ddot{Y}_o)$ acceleration of the origin of the mobile reference system in relation to the fixed reference system O'XY. The mobile reference system will have angular velocity $\omega = \dot{\theta}$ and angular acceleration $\varepsilon = \ddot{\theta}$ (the angle θ defines the angular position of the local coordinate frame system with respect to the fixed reference system)[1],[2],[16]. The orthonormal rotation matrix:

$$[R]=\begin{bmatrix} c_\theta & -s_\theta \\ s_\theta & c_\theta \end{bmatrix} \tag{1}$$

where is denoted $c_\theta = \cos\theta$, $s_\theta = \sin\theta$, makes the transformation from the local to the global reference frame. A vector $\bar{v}(v_x, v_y)$ expressed in the mobile reference system becomes, in the global reference system [3],[4]:

$$\begin{Bmatrix} v_X \\ v_Y \end{Bmatrix} = \begin{bmatrix} c_\theta & -s_\theta \\ s_\theta & c_\theta \end{bmatrix} \begin{Bmatrix} v_x \\ v_y \end{Bmatrix}. \tag{2}$$

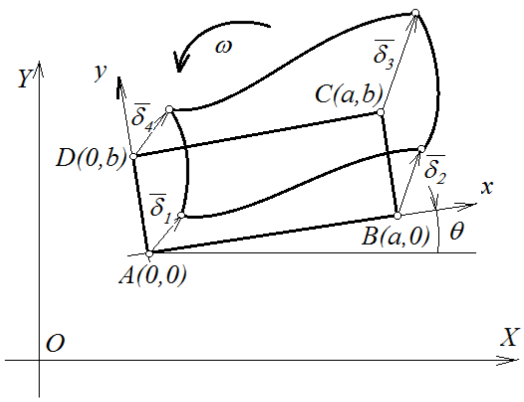

Fig.1. A rectangular finite element

$\{r_M\}_G$ is the position vector of point M:

$$\{r_M\}_G = \{r_O\}_G + \{r\}_G = \{r_O\}_G + [R]\{r\}_L \tag{3}$$

where the index G expresses a size in the fix reference frame and the index L shows that the size is written in the mobile reference frame. If M becomes M', undergoing a small displacement $\{f\}_L$, one can write:

$$\{r_{M'}\}_G = \{r_O\}_G + [R](\{r\}_L + \{f\}_L) = \\ = \begin{Bmatrix} X_0 \\ Y_0 \end{Bmatrix} + [R]\left(\begin{Bmatrix} x \\ y \end{Bmatrix} + \begin{Bmatrix} u \\ v \end{Bmatrix}\right) \tag{4}$$

The continuous field of displacements is approximated, in the method of the finite elements, by the relation:

$$\{f\}_L = \begin{Bmatrix} u \\ v \end{Bmatrix} = [\Phi(x,y)]\{\delta(t)\}_L \tag{5}$$

where $[\Phi(x,y)]$ is the matrix of the shape function. The vector $\{\delta(t)\}_L$ represents the vector of the independent coordinates, expressed in the mobile coordinate frame. We consider the displacements u and v of some point as completely defined by the nodal displacements. For this triangular finite element, with the nodes at the ends, the shape functions are chosen:

$$\begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{bmatrix} \Phi_1 & 0 & \Phi_2 & 0 & \Phi_3 & 0 & \Phi_4 & 0 \\ 0 & \Phi_1 & 0 & \Phi_2 & 0 & \Phi_3 & 0 & \Phi_4 \end{bmatrix} \begin{Bmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \\ \delta_4 \end{Bmatrix} = \\ = [\Phi]\{\delta\}_L \tag{6}$$

$$\{\delta_i\} = \begin{Bmatrix} u_i \\ v_i \end{Bmatrix}, \; i = 1,2 \tag{7}$$

where it was noted:

$$\xi = \frac{x}{a} \quad ; \quad \eta = \frac{y}{b} \quad ; \tag{8}$$

$$\Phi_1 = (1-\xi)(1-\eta) \quad ; \quad \Phi_2 = \xi(1-\eta) \quad ; \\ \Phi_3 = \xi\eta \quad ; \quad \Phi_4 = \eta(1-\xi) \tag{9}$$

The velocity of point M 'can be expressed by:

$$\{v_{M'}\}_G = \{\dot{r}_{M'}\}_G = \begin{Bmatrix} \dot{X}_0 \\ \dot{Y}_0 \end{Bmatrix} + [\dot{R}]\begin{Bmatrix} x \\ y \end{Bmatrix} + \\ + [\dot{R}][\Phi]\{\delta\}_L + [R][\Phi]\{\dot{\delta}\}_L \tag{10}$$

The expression of kinetic energy for a single finite element is:

$$E_c = \frac{1}{2}\int_V \rho v^2 dV = \frac{1}{2}\int_V \rho \{v_{M'}\}_G^T \{v_{M'}\}_G dV = \\ = \frac{1}{2}\int_V \rho\left([\dot{X}_0 \; \dot{Y}_0]\begin{Bmatrix} \dot{X}_0 \\ \dot{Y}_0 \end{Bmatrix} + [x \; y][\dot{R}]^T[\dot{R}]\begin{Bmatrix} x \\ y \end{Bmatrix} + \right. \\ + \{\delta\}_L^T[\Phi]^T[\dot{R}]^T[\dot{R}][\Phi]\{\delta\}_L +$$

$$\left\{\dot{\delta}\right\}_L^T[\Phi]^T[R]^T[R][\Phi]\left\{\dot{\delta}\right\}_L + +2[\dot{X}_0\ \dot{Y}_0][\dot{R}]\begin{Bmatrix}x\\y\end{Bmatrix} + 2[\dot{X}_0\ \dot{Y}_0][R][\Phi]\left\{\dot{\delta}\right\}_L + +2[\dot{X}_0\ \dot{Y}_0][\dot{R}][\Phi]\{\delta\}_L + 2\{\delta\}_L^T[\Phi]^T[\dot{R}]^T[\dot{R}]\begin{Bmatrix}x\\y\end{Bmatrix} + +2\{\dot{\delta}\}_L^T[\Phi]^T[R]^T[\dot{R}]\begin{Bmatrix}x\\y\end{Bmatrix} + +2\{\delta\}_L^T[\Phi]^T[\dot{R}]^T[R][\Phi]\left\{\dot{\delta}\right\}_L)dV \qquad (11)$$

The concentrated and distributed loads gives us the work:

$$W + W^c = \int_V \{p\}^T\{f\}dV + \{q\}^T\{\delta\}_L = = \left(\int_V \{p\}^T[\Phi]dV\right)\{\delta\}_L + \{q\}^T\{\delta\}_L \qquad (12)$$

The Lagrangian for the finite element considered is:

$$L = E_c - E_p + W + W^c \ . \qquad (13)$$

and the evolution equations were obtained with Lagrange method [14].
If you consider that:

a) $\int_V[\Phi]^T[R]^T[\ddot{R}]\begin{Bmatrix}x\\y\end{Bmatrix}\rho t dA =$

$$= \int_V\left[\left[\Phi_{(1)}\right]^T \quad F\left[\Phi_{(2)}\right]^T\right]\left(\varepsilon\begin{Bmatrix}-y\\x\end{Bmatrix} - \omega^2\begin{Bmatrix}x\\y\end{Bmatrix}\right)\rho t dA =$$
$$= \varepsilon\int_V\left(-\left[\Phi_{(1)}\right]^T y + \left[\Phi_{(2)}\right]^T x\right)\rho t dA -$$
$$- \omega^2\int_V\left(\left[\Phi_{(1)}\right]^T x + \left[\Phi_{(2)}\right]^T y\right)\rho t dA =$$
$$= \varepsilon\left(-\{m_{1y}\} + \{m_{2x}\}\right) - \omega^2\left(\{m_{1x}\} + \{m_{2y}\}\right) \qquad (14)$$

It was noted:

$$\{m_{1x}\} = \int_V \Phi_{(1)}^T x\rho t dA \quad ; \quad \{m_{1y}\} = \int_V \Phi_{(1)}^T y\rho t dA \quad ; \qquad (15)$$

$$\{m_{2x}\} = \int_V \Phi_{(2)}^T x\rho t dA \quad ; \quad \{m_{2y}\} = \int_V \Phi_{(2)}^T y\rho t dA \quad . \qquad (16)$$

b) $\int_V[\Phi]^T[R]^T[\ddot{R}][\Phi]\rho t dA =$

$$= \varepsilon\int_V\left[\left[\Phi_{(2)}\right]^T \quad -\Phi^T\right]\begin{Bmatrix}\left[\Phi_{(1)}\right]\\\left[\Phi_{(2)}\right]\end{Bmatrix}\rho t dA -$$
$$- \omega^2\int_V\left[\left[\Phi_{(1)}\right]^T \quad \left[\Phi_{(2)}\right]^T\right]\begin{Bmatrix}\left[\Phi_{(1)}\right]\\\left[\Phi_{(2)}\right]\end{Bmatrix}\rho t dA =$$
$$= \varepsilon\int_V\left(\left[\Phi_{(2)}\right]^T\left[\Phi_{(1)}\right] - \left[\Phi_{(1)}\right]^T\left[\Phi_{(2)}\right]\right)\rho t dA -$$
$$- \omega^2\int_V\left(\left[\Phi_{(1)}\right]^T\left[\Phi_{(1)}\right] + \left[\Phi_{(2)}\right]^T\left[\Phi_{(2)}\right]^T\right)\rho t dA =$$
$$= \varepsilon[c] - \omega^2[m] \qquad (17)$$

It was noted:

$$[c] = \int_V\left(\left[\Phi_{(2)}\right]^T\left[\Phi_{(1)}\right] - \left[\Phi_{(1)}\right]^T\left[\Phi_{(2)}\right]\right)\rho t dA \ ;$$
$$[m] = \int_V\left(\left[\Phi_{(1)}\right]^T\left[\Phi_{(1)}\right] + \left[\Phi_{(2)}\right]^T\left[\Phi_{(2)}\right]^T\right)\rho t dA \qquad (18)$$

c) $\int_V[\Phi]^T[\Phi]\rho t dA =$

$$= \int_V\left(\left[\Phi_{(1)}\right]^T\left[\Phi_{(1)}\right] + \left[\Phi_{(2)}\right]^T\left[\Phi_{(2)}\right]^T\right)\rho t dA =$$
$$= [m_{11}] + [m_{22}] = [m] \qquad (19)$$

where it was noted:

$$[m_{ij}] = \int_V \Phi_{(i)}^T\Phi_{(j)}\rho t dA \qquad (20)$$

d) $\int_V[\Phi]^T[R]^T[\dot{R}][\Phi]\rho t dA =$

$$= \omega\int_V\left[\left[\Phi_{(2)}\right]^T \quad -\left[\Phi_{(1)}\right]^T\right]\begin{Bmatrix}\left[\Phi_{(1)}\right]\\\left[\Phi_{(2)}\right]\end{Bmatrix}\rho t dA = \omega[c] \qquad (21)$$

It is also noted:

$$m = \int_V \rho t dA \quad ; \quad [m_o^i] = \int_V[\Phi]^T\rho t dA \quad ;$$
$$\{m_\omega^i\} = \{m_{1x}\} + \{m_{2y}\} \quad ; \{m_\varepsilon^i\} = -\{m_{1y}\} + \{m_{2x}\} \ . \qquad (22)$$

The stiffness matrix is found in [5]. The evolution equations can be written:

$$\left(\int_V \left([\Phi_{(1)}]^T[\Phi_{(1)}] + [\Phi_{(2)}]^T[\Phi_{(2)}]^T\right)\rho t dA\right)\{\ddot{\delta}\}_L +$$
$$+2\omega\left(\int_V \left([\Phi_{(2)}]^T[\Phi_{(1)}] - [\Phi_{(1)}]^T[\Phi_{(2)}]\right)\rho t dA\right)\{\dot{\delta}\}_L +$$
$$+[[k] + \varepsilon\left(\int_V ([\Phi_{(2)}]^T[\Phi_{(1)}] - [\Phi_{(1)}]^T[\Phi_{(2)}])\rho t dA\right) -$$
$$-\omega^2\left(\int_V ([\Phi_{(1)}]^T[\Phi_{(1)}] + [\Phi_{(2)}]^T[\Phi_{(2)}]^T)\rho t dA\right)]\{\delta\}_L =$$
$$= \{q\}_L + \int_V [\Phi]^T\{p\}\rho t dA -$$
$$-\left(\int_V [[\Phi_{(1)}]^T \quad [\Phi_{(2)}]^T]\rho t dA\right)[R]\begin{Bmatrix}\ddot{X}_o\\ \ddot{Y}_o\end{Bmatrix} -$$
$$-\varepsilon\int_V [-[\Phi_{(1)}]^T y + [\Phi_{(2)}]^T x]\rho t dA +$$
$$+\omega^2\int_V [[\Phi_{(1)}]^T x + [\Phi_{(2)}]^T y]\rho t dA \qquad (23)$$

or:

$$([m_{11}] + [m_{22}])\{\ddot{\delta}\}_L + 2\omega([m_{21}] - [m_{12}])\{\dot{\delta}\}_L +$$
$$+[[k_e] + \varepsilon([m_{21}] - [m_{12}]) - \omega^2([m_{11}] + [m_{22}])]\{\delta\}_L =$$
$$= \{q\}_L + \{q^*\}_L - [[m]_{(1)}^T[m]_{(2)}^T]\begin{Bmatrix}\ddot{x}_o\\ \ddot{y}_o\end{Bmatrix} -$$
$$-\varepsilon(-[m_{1y}] + [m_{2x}]) + \omega^2([m_{1x}]^T + [m_{2y}]^T) \qquad (24)$$

or:

$$[m]\{\ddot{\delta}\}_L + [c]\{\dot{\delta}\}_L + [[k_e] + \varepsilon[c] - \omega^2[m]]\{\delta\} =$$
$$= \{q\}_L + \{q^*\}_L - [m_o^i]\begin{Bmatrix}\ddot{x}_o\\ \ddot{y}_o\end{Bmatrix} - \varepsilon\{m_\varepsilon^i\} + \omega^2\{m_\omega^i\}$$
(25)

Now it is possible to determine the matrix coefficients accordingly.

3. EXAMPLE

Consider a plate with the dimension length = 0.2 m and width = 0.16 m. The thickness = 0.001 m and the Young's modulus = 210 GPa. The mass density is 7800 kg/m^3 (Fig.2).

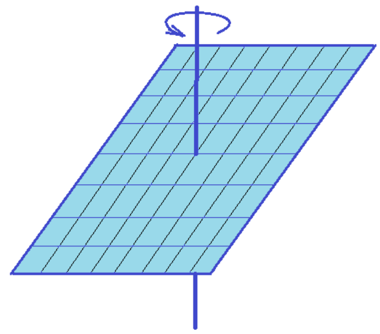

Fig. 2. Rectangular plane plate in rotation

Using FEA, meshing the plate and written the motion eqautions for the entire shell is possible to obtain eigenvalues. In Fig.3 we present the first 15 eigenvalues. The angular speed is varied between 5,000 and 15.000 rad/s.

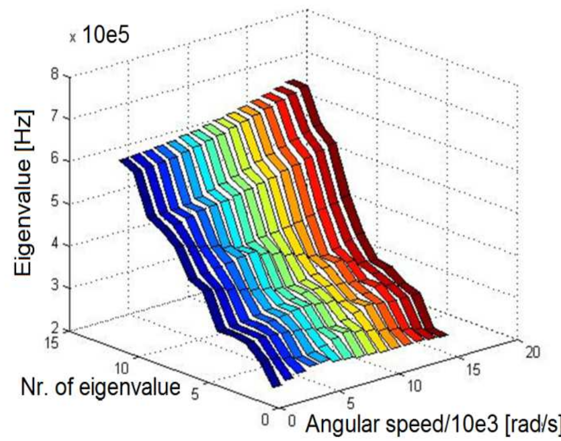

Fig.3. First 15 eigenvalues for different angular speed

In Figure 4 are presented the first 15 eigenvalues for an angular speed of 14,000 rad/s.

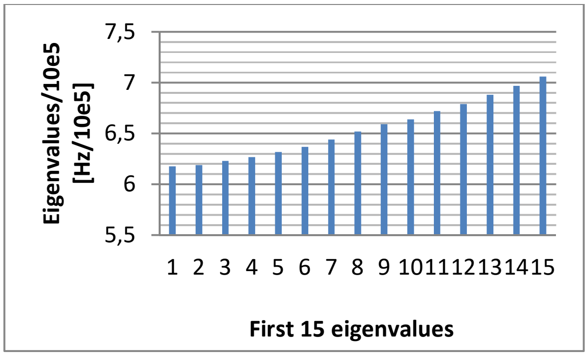

Fig.4. First 15 eigenvalues for angular speed 14,000 rad/s

4. CONCLUSIONS

In the paper are obtained the evolution equations for a rectangular element used for the study of multi-body mechanical systems with elastic elements. Lagrange method were used to obtain these. Specific shape functions were used for this type of finite element, known from the static or steady state analysis. It is presented, for example, the calculation for a plate in rotation. The main problem for such an analysis is the volume of calculation required to obtain the matrix coefficients of the equations of motion.

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Element finit dreptunghiular în stare de membrană pentru analiza dinamică a unui sistem multicorp cu o mișcare plană

Rezumat *Lucrarea își propune dezvoltarea unui element finit dreptunghiular pentru studiul mișcărilor plane ale unei plăci în stare de membrană. Problema utilizării elementelor finite bidimensionale în analiza dinamică a membranelor este puțin aboordată în literatură, lucru datorat mai ales formalismului implicat, care necesită un efor de calcul deosebit. În lucrare sunt stabilite ecuațiile de mișcare pentru elementul finit studiat și sunt calculați coeficienții matriceali. Un exemplu de calcul al valorilor proprii pentru o placă dreptunghiulară este prezentat, care utilizează formalismul dezvoltat în lucrare.*

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