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VIBRATION OF AUTOMOTIVE SUSPENSIONS

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Abstract: *In this paper we study the influence of vibrations on car suspensions, first on models with one degree of freedom, then on mechanical models with two degrees of freedom. We determined the dynamic response using two modern calculation methods. One of them is the use of the unilateral Laplace integral transform with respect to time, and the second is the use of systems theory. The graphical representations that show us the evolution in time of the suspension movements caused by vibrations were obtained with the Mathematica calculation software.*
Key words: *Automotive, suspension, vibration, Laplace transform, system theory.*

1. INTRODUCTION

Vibrations are dynamic phenomena encountered in current activity, from heartbeats, running and walking, swaying trees in the wind and the tremors of buildings to earthquakes [1, 2, 3], to the vibrations of musical instruments, pneumatic perforators and oscillating conveyor belts [4]. Often "vibrations" are called unwanted movements that produce relatively high noise or mechanical stress. In this case, the effect of vibrations on humans, cars and buildings is of particular interest. Modeling vibrational phenomena involves defining the structure and parameters of vibrating bodies, the functions that describe excitation, and the levels of dynamic response [5, 6]. In a conservative system, in which there is no energy dissipation, the total mechanical energy is constant. In the position of maximum travel amplitude, the instantaneous speed is zero, the system has only potential energy. In the static equilibrium position, the deformation energy is zero and the system has only kinetic energy. The maximum kinetic energy is equal to the maximum deformation energy. By equalizing the two energies one can calculate the fundamental eigenfrequency of vibration. This is the principle of Rayleigh's method. Vibrating systems are damped due to loss of energy by dissipation or radiation. Damping decreases the amplitude of free vibrations, shifts between excitation and response, and limits the amplitude of the forced response of vibrating systems [7]. Vibration analysis is very important, there is research on vibration phenomena and in biomechanics applications, where accelerometer-type sensors are used to detect vibrations as shown in the articles [8, 9, 10, 11, 12, 13]. The most useful theoretical and experimental developments of life phenomena are those in the field of gearshift transmissions, as shown in [14]. We considered that the car's suspension is provided by four systems, first with a single degree of freedom, then with two degrees of freedom, identical, mounted between the vehicle chassis and each shaft of each wheel. The tires are considered to be completely rigid and they do not interfere with the study. Thanks to the tires, which have sufficient compressibility for the viscous friction to be omitted, it can be assumed that the damping coefficient for the tires is zero. In the model with two degrees of freedom, in addition to the excitement caused by the road, an excitatory force acting on the car chassis was also taken into account.

2. THE MODEL WITH A DEGREE OF FREEDOM

The suspension of a car is ensured by four identical systems, with a single degree of freedom, mounted between the vehicle chassis and each shaft of each wheel constituted as below:

- a helical metal spring of stiffness constant k and length in free state L_0;
- a cylindrical piston damper, with oil, fixed parallel to the spring, exerting a resistant viscous friction force of damping coefficient c.

It is assumed that the mass m of the chassis is evenly distributed between the four systems. So, a suspension only supports a quarter of the total mass of the chassis.

The tires are considered to be entirely rigid and do not interfere with the study.

The excitation of the system is caused by the movement y_0 (t) due to the road on which the car is moving.

In Lagrange's formalism, the mathematical model of motion in the form of the second order differential equation, with constant coefficients, results as follows:

$$m\ddot{y} + 4c\dot{y} + 4ky = f_0(t). \qquad (1)$$

where:

$$f_0(t) = 4ky_0(t) + 4c\dot{y}_0(t). \qquad (2)$$

The excitation is assumed to be cosine, i.e.:

$$y_0(t) = Y_0 \cos(\omega_e t). \qquad (3)$$

Where, Y_0 is the amplitude of the excitation, and ω_e is the pulsation of the excitation.

Replacing the relations (2) and (3) in equation (1), the mathematical model results in the form:

$$m\ddot{y} + 4c\dot{y} + 4ky = 4kY_0 \cos(\omega_e t) - -4cY_0\omega_e \sin(\omega_e t). \qquad (4)$$

Applying the unilateral Laplace transform with respect to the time of the equation with constant coefficients (4), results an algebraic equation, whose solution is the Laplace image $\tilde{y}(s)$ of the displacement $y(t)$:

$$\tilde{y}(s) = \frac{4kY_0 s - 4cY_0\omega_e^2}{(s^2+\omega_e^2)(ms^2+4cs+4k)}. \qquad (5)$$

Applying the relation (5), with the help of development theorems, the inverse of the Laplace transform, results the vertical displacement of the car in the form of time function:

$$y(t) = \frac{F_1(t)}{F_2(t)}. \qquad (6)$$

where:

$$F_1(t) = \frac{Y_0}{\sqrt{c^2-km}}\Big\{4\left(4k^2+4c^2\omega_e^2-km\omega_e^2\right)$$

$$\cos\omega_e t + 4cm\omega_e^3 \sin\omega_e t - 2e^{\frac{-2ct}{m}}\, g$$

$$g\Big[8k^2\left(\begin{array}{l}\sqrt{c^2-km}\cosh\dfrac{2\sqrt{c^2-km}}{m}t + \\ c\sinh\dfrac{2\sqrt{c^2-km}}{m}t\end{array}\right) +$$

$$2\omega_e^2\left(4c^2-km\right)g$$

$$g\left(\begin{array}{l}\sqrt{c^2-km}\cosh\dfrac{2\sqrt{c^2-km}}{m}t + \\ c\sinh\dfrac{2\sqrt{c^2-km}}{m}t\end{array}\right) +$$

$$\omega_e^4 cm^2 \sinh\frac{2\sqrt{c^2-km}}{m}t\Big]\Big\};$$

$$F_2(t) = 16k^2 + 8\omega_e^2\left(2c^2-km\right) + m^2\omega_e^4 \qquad (7)$$

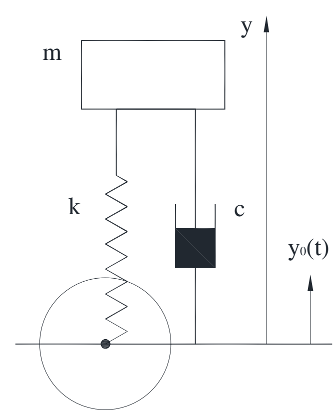


Fig. 1. Simplified suspension model.

With the numeric application:

$$m = 415[Kg]; k = 270000\left[\frac{N}{m}\right];$$

$$c = 1500\left[\frac{Ns}{m}\right]; Y_0 = 0{,}02[m];$$

$$\omega_e = 50[s^{-1}]$$

we obtained in the Mathematica program the displacement graph $y(t)$ from Fig. 2. This graphical representation shows the variation of the vertical displacement of the vehicle in relation to the time variable.

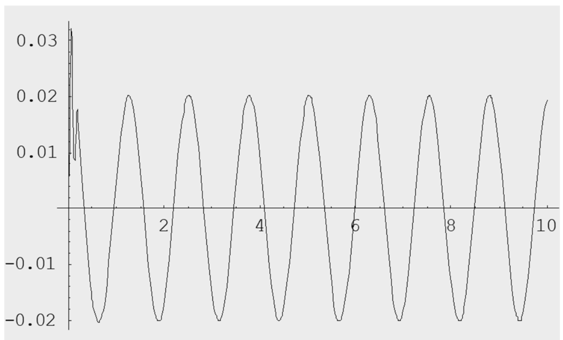
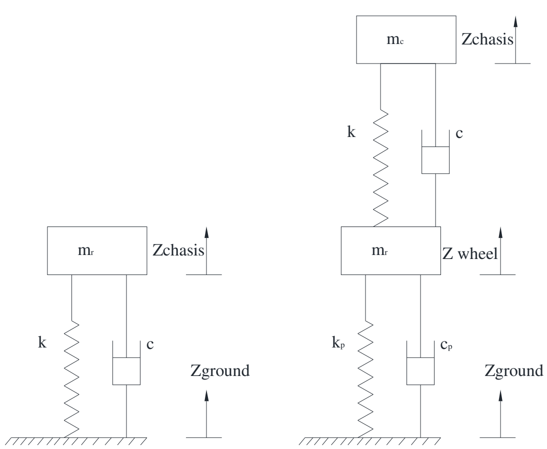


Fig. 2. Graphic representation of $y = y(t)$.

3. THE MODEL WITH TWO DEGREES OF FREEDOM

In the paper Damien Sammier, Sur la modélisation et la commande de suspension de véhicules automobiles, Institut National Polytechnique de Grenoble, Thèse pour obtenir le grad de DOCTEUR, 2001 [4], the author shows several models of the active suspension of a quarter of the vehicle.



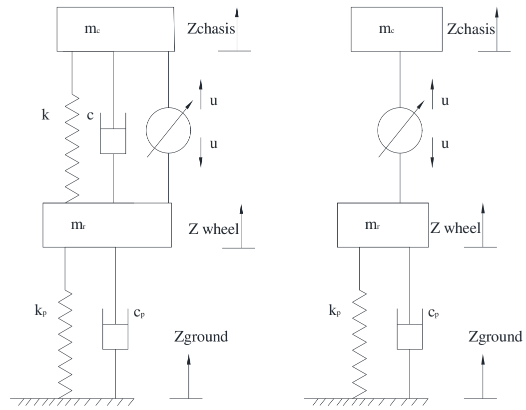


Fig. 3 Models of the active suspension of a quarter of a vehicle.

The model we chose is the one from Fig. 4, considering that the car suspension is provided by four systems, with two degrees of freedom, identical, mounted between the vehicle chassis and each shaft of each wheel as it appears, taking into account the disturbing force $\vec{F}(t)$ acting on the chassis, in Figure 4.

In Fig. 4 he have:

$m_r = m_1$ - unsprung mass (wheel mass);

$m_c = m_2$ - sprung mass (vehicle mass);

$k = k_1$ - the stiffness coefficient represented by a spring of the suspension;

$k_p = k_0$ - the stiffness coefficient represented by a spring for the wheels (tires);

$c = c_1$ - suspension damping coefficient;

$c_p = c_0$ - the damping coefficient related to the tires;

$z_{wheel} = z_1(t)$; $z_{chasis} = z_2(t)$ - the linear displacements of the two masses;

u - actuating force;

$\vec{F}(t)$ - the disturbing force acting on the chassis;

$Z_{ground} = z_0(t)$ - exciting movement due to the road, which is the main source of vibrations;

$Z_{ground} = z_0(t) = z_0 \sin(\omega_e t)$;

$\left\|\vec{F}(t)\right\| = F = \text{constant}$.

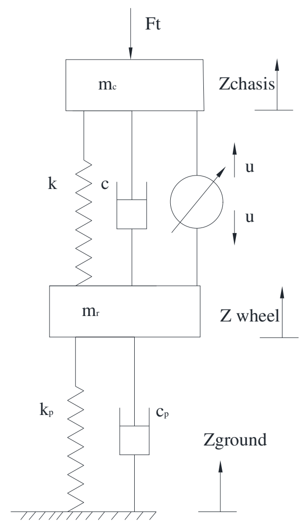


Fig. 4. Active suspension with two degrees of freedom of a quarter of a car.

Thanks to the tires, which have sufficient compressibility for the viscous friction to be omitted, it can be assumed that $c_p = c_0 = 0$.

Using, for the writing of the mathematical model of the forced and damped vibrations of the automobile, to Lagrange's formalism, we obtained the following system of two linear differential equations with constant coefficients:

$$[M]\{\ddot{z}\} + [C]\{\dot{z}\} + [K]\{z\} = \{P_e(t)\}. \quad (8)$$

where:

$$[M] = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}; [C] = \begin{bmatrix} c_1 + c_0 & -c_1 \\ -c_1 & c_1 \end{bmatrix};$$

$$[K] = \begin{bmatrix} k_1 + k_0 & -k_1 \\ -k_1 & k_1 \end{bmatrix}; \{\ddot{z}\} = \begin{Bmatrix} \ddot{z}_1 \\ \ddot{z}_2 \end{Bmatrix};$$

$$\{\dot{z}\} = \begin{Bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{Bmatrix}; \{z\} = \begin{Bmatrix} z_1 \\ z_2 \end{Bmatrix};$$

$$\{P_e(t)\} = \begin{Bmatrix} F \\ z_0 \sin(\omega_e t) \end{Bmatrix};$$

In the case of forced and non-damped vibrations, the system (8) becomes:

$$[M]\{\ddot{z}\} + [K]\{z\} = \{P_e(t)\}. \quad (9)$$

where:

Applying the unilateral Laplace transform in relation to the time of the system of differential equations (8), considering the homogeneous initial conditions, we obtain the algebraic system below, which has as unknown the Laplace images $\tilde{z}_1(s), \tilde{z}_2(s)$of the displacements $z_1(t), z_2(t)$.

$$\begin{cases} s[m_1 s^2 + (c_1 + c_0)s + k_1 + k_0]\tilde{z}_1(s) - \\ -s(c_1 s + k_1)\tilde{z}_2(s) = F \\ -(c_1 s + k_1)(s^2 + \omega_e^2)\tilde{z}_1(s) + \\ +(m_2 s^2 + c_1 s + k_1)(s^2 + \omega_e^2)\tilde{z}_2(s) = z_0\omega_e \end{cases} \quad (10)$$

With $c_0 = 0$ the algebraic system (10) become:

$$\begin{cases} s(m_1 s^2 + c_1 s + k_1 + k_0)\tilde{z}_1(s) - \\ -s(c_1 s + k_1)\tilde{z}_2(s) = F \\ -(c_1 s + k_1)(s^2 + \omega_e^2)\tilde{z}_1(s) + \\ +(m_2 s^2 + c_1 s + k_1)(s^2 + \omega_e^2)\tilde{z}_2(s) = z_0\omega_e \end{cases} \quad (11)$$

Applying the unilateral Laplace transform with respect to the time of the system of differential equations (9), considering the homogeneous initial conditions, we obtain the algebraic system below, which has as unknown the Laplace images $\tilde{z}_1(s), \tilde{z}_2(s)$of the displacements $z_1(t), z_2(t)$.

$$\begin{cases} s[m_1 s^2 + (k_1 + k_0)]\tilde{z}_1(s) - \\ -sk_1\tilde{z}_2(s) = F \\ -k_1(s^2 + \omega_e^2)\tilde{z}_1(s) + \\ +(m_2 s^2 + k_1)(s^2 + \omega_e^2)\tilde{z}_2(s) = z_0\omega_e \end{cases}. \quad (12)$$

Solving the algebraic system elementary (11) results in Laplace images $\tilde{z}_1(s), \tilde{z}_2(s)$ of displacements $z_1(t), z_2(t)$, as below:

$$\begin{cases} z_1(t) = \dfrac{P_1(s)}{P(s)} \\ z_2(t) = \dfrac{P_2(s)}{P(s)}, \end{cases}. \quad (13)$$

where:

$$P_1(s) = F\left(m_2 s^2 + c_1 s + k_1\right)\left(s^2 + \omega_e^2\right) + z_0\omega_e s\left(c_1 s + k_1\right);$$

$$P_2(s) = z_0\omega_e s\left[m_1 s^2 + \left(c_1 + c_0\right)s + \left(k_1 + k_0\right)\right] + F\left(c_1 s + k_1\right)\left(s^2 + \omega_e^2\right);$$

$$P(s) = s\left[m_1 s^2 + \left(c_1 + c_0\right)s + k_1 + k_0\right] \left(m_2 s^2 + c_1 s + k_1\right)\left(s^2 + \omega_e^2\right)$$

Applying the development theorems, we inverted the Laplace transforms from relations (13), where we did not take into account the force $\vec{F}(t)$, resulting in displacements $z_1(t)$, $z_2(t)$, in the case of forced and damped vibrations produced by exciting displacement due to the road, as below:

$$\begin{cases} z_1(t) = \dfrac{F_1(t)}{F_2(t)} + \dfrac{F_3(t)}{F_4(t)} + \dfrac{F_5(t)}{F_6(t)} \\ z_2(t) = \dfrac{F_7(t)}{F_8(t)}, \end{cases} \quad (14)$$

where:

$$F_1(t) = z_0 \left[\begin{pmatrix} k_0k_1^2 + k_1^3 + c_1^2k_0\omega_e^2 + c_1^2k_1\omega_e^2 - k_1\omega_e^2m_1 - \\ k_0k_1\omega_e^2m_2 - k_1^2\omega_e^2m_2 - c_1^2m_1\omega_e^4 - c_1^2m_2\omega_e^4 \\ +k_1\omega_e^2m_1m_2 \end{pmatrix} \cdot \right.$$

$$\left. \cdot \sin(\omega_e t) + \begin{pmatrix} -c_1k_1^2\omega_e - c_1^3\omega_e^3 - \\ c_1k_0m_2\omega_e^3 + c_1m_1m_2\omega_e^5 \end{pmatrix} \cdot \cos(\omega_e t) \right];$$

$$F_2(t) = \begin{pmatrix} k_0^2 + 2k_0k_1 + k_1^2 + c_1^2\omega_e^2 - 2k_0m_1\omega_e^2 \\ -2k_1m_1\omega_e^2 + m_1^2\omega_e^4 \end{pmatrix} \cdot$$

$$\left(k_1^2 + c_1^2\omega_e^2 - 2k_1m_2\omega_e^2 + m_2\omega_e^4\right);$$

$$F_3(t) = z_0\omega_e\left[\cosh\left(\omega_n^1 t\right) - \sinh\left(\omega_n^1 t\right)\right] \cdot$$

$$\left[c_1^4k_0(m_1 - m_2)f_1(t) + c_1^3k_0(m_1 - m_2)m_1\omega_n f_2(t) + \right.$$

$$+2k_1m_1^2(-k_1m_1 + k_0m_2 + k_1m_2)\left(k_0 + k_1 - m_1\omega_e^2\right)f_2(t)$$

$$+\left(k_1^2m_1 + k_0^2m_2 - k_1^2m_2 - k_0m_1m_2\omega_e^2\right)c_1m_1^2\omega_n \cdot f_2(t) +$$

$$\left. c_1^2m_1 \begin{pmatrix} -2k_0^2m_1 - 2k_0k_1m_1 + k_1^2m_1 + 3k_0^2m_2 \\ +2k_0k_1m_2 - k_1^2m_2 + 2k_0m_1^2\omega_e^2 - k_0m_1m_2\omega_e^2 \end{pmatrix} f_1(t) \right];$$

$$F_4 = 2\sqrt{c_1^2 - 4k_0m_1} \begin{pmatrix} k_0^2 + 2k_0k_1 + k_1^2 + c_1^2\omega_e^2 - \\ 2k_0m_1\omega_e^2 - 2k_1m_1\omega_e^2 + m_1^2\omega_e^4 \end{pmatrix} \cdot$$

$$\begin{pmatrix} c_1^2k_0m_1 + k_1^2m_1^2 - c_1^2k_0m_2 - 2k_0k_1m_1m_2 - 2k_1^2m_1m_2 \\ +k_0^2m_2^2 + 2k_0k_1m_2^2 + k_1^2m_2^2 \end{pmatrix};$$

$$F_5(t) = m_2z_0\omega_e\left[\cosh\left(\Omega_n^1 t\right) - \sinh\left(\Omega_n^1 t\right)\right]$$

$$\left[2k_1m_2(-k_1m_1 + k_0m_2 + k_1m_2)\left(-k_1 + m_2\omega_e^2\right)f_1^1(t) - \right.$$

$$-c_1^2\left(k_1^2m_1 - k_1^2m_2 + k_0m_2^2\omega_e^2\right)f_1^1(t)$$

$$\left. +c_1m_2\Omega_n\left(-k_1^2m_1 + k_1^2m_2 + k_0m_2^2\omega_e^2\right)f_2^1(t) \right];$$

$$F_6(t) = 2m_2\Omega_n \begin{pmatrix} c_1^2k_0m_1 + k_1^2m_1^2 - c_1^2k_0m_2 - \\ 2k_0k_1m_1m_2 - 2k_1^2m_1m_2 + k_0^2m_2^2 \\ +2k_0k_1m_2^2 + k_1^2m_2^2 \end{pmatrix} \cdot$$

$$\cdot\left(k_1^2 + c_1^2\omega_e^2 - 2k_1m_2\omega_e^2 + m_2\omega_e^4\right);$$

$$F_7(t) = 2z_0\omega_e\left\{-2\cos(\omega_e t) + \frac{2k_1}{\omega_e}\sin(\omega_e t)\right.$$

$$-2m_2\omega_e\sin(\omega_e t) +$$

$$\left[c_1^2\left(\cosh\left(\Omega_n^1 t\right) - \sinh\left(\Omega_n^1 t\right)\right)f_1^1(t) + \right.$$

$$\left. \left. +c_1m_2\Omega_n f_2^1(t) + 2m_2f_1^1(t)\right]\left(-k_1 + m_2\omega_e^2\right)\right\}$$

$$\left(k_1^2 + c_1^2\omega_e^2 - 2k_1m_2\omega_e^2 + m_2\omega_e^4\right);$$

$$F_8(t) = m_2\Omega_n;$$

$$\omega_n = \frac{\sqrt{c_1^2 - 4(k_0 + k_1)m_1}}{m_1}; \quad \omega_n^1 = \frac{c_1 + m_1\omega_n}{2m_1};$$

$$\Omega_n = \frac{\sqrt{c_1^2 - 4k_1m_2}}{m_2}; \quad \Omega_n^1 = \frac{c_1 + \sqrt{c_1^2 - 4k_1m_2}}{2m_2} =$$

$$= \frac{c_1 + m_2\Omega_n}{2m_2};$$

$$f_1(t) = -1 + \cosh(\omega_n t) + \sinh(\omega_n t);$$

$$f_2(t) = 1 + \cosh(\omega_n t) + \sinh(\omega_n t);$$

$$f_1^1(t) = -1 + \cosh(\Omega_n t) + \sinh(\Omega_n t);$$

$$f_2^1(t) = 1 + \cosh(\Omega_n t) + \sinh(\Omega_n t)$$

Based on the numerical application, the functions $z_1(t)$, $z_2(t)$ in the relations (14), resulting from the inversion of the formulas (13), where we did not take into account the force $\vec{F}(t)$, have the graphical representations from Fig. 5 and Fig. 6.

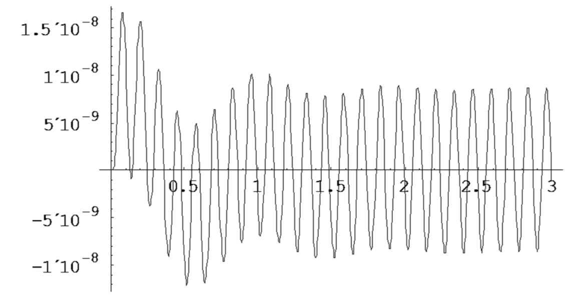


Fig. 5. Representation of $z_1 = z_1(t)$

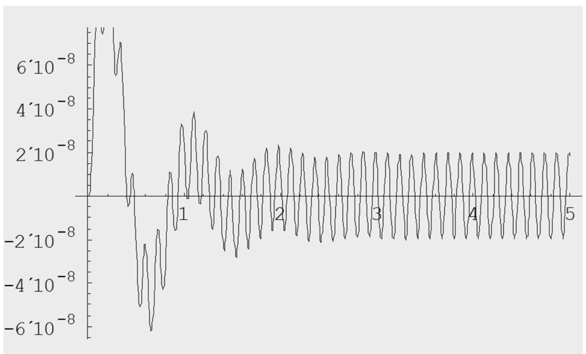


Fig. 6. Representation of $z_2 = z_2(t)$

Based on the numerical application, the functions $z_1(t), z_2(t)$, resulting from the inversion of the formulas (13), where we also considered the force $\vec{F}(t)$, have the graphical representations from Fig. 7 and Fig. 8, with the observation that we considered this force to be constant.

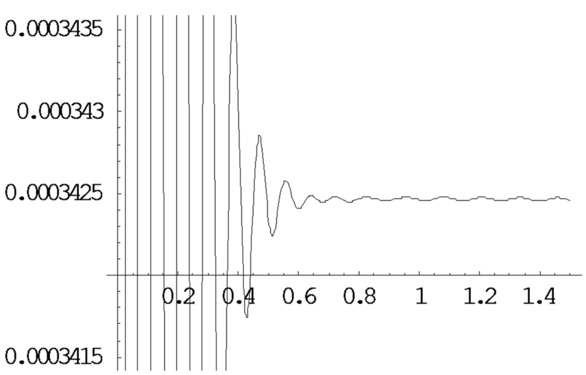


Fig. 7. Representation of $z_1 = z_1(t)$

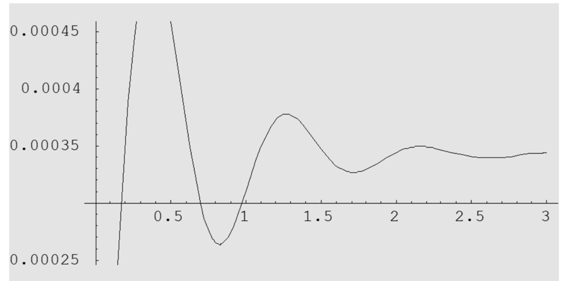


Fig. 8. Representation of $z_2 = z_2(t)$

Applying the development theorems, we inverted the Laplace transforms from relations (13), where we did not take into account the force $\vec{F}(t)$, resulting in displacements $z_1(t), z_2(t)$, in the case of forced and non-damped vibrations produced by exciting displacement due to the road, as below:

$$\begin{cases} z_1(t) = \dfrac{k_1 m_1 \cosh(\omega_n t)}{(-k_1 m_1 + k_0 m_2 + k_1 m_2)\cdot(k_0 + k_1 - m_1\omega_e^2)} + \\ \dfrac{k_1 \cos(\omega_e t)}{(k_1 - m_2\omega_e^2)\cdot(k_0 + k_1 - m_1\omega_e^2)} \\ + \dfrac{k_1 m_2 \cos\left(\sqrt{\dfrac{k_1}{m_2}}\, t\right)}{(-k_1 + m_2\omega_e^2)\cdot(-k_1 m_1 + k_0 m_2 + k_1 m_2)} \\ z_2(t) = \dfrac{\sin(\omega_e t) - \omega_e\sqrt{\dfrac{m_2}{k_1}}\sin\left(\sqrt{\dfrac{k_1}{m_2}}\, t\right)}{k_1\omega_e - m_2\omega_e^3} \end{cases} \quad .(15)$$

Based on the numerical application, the functions $z_1(t), z_2(t)$ in the relations (15), resulting from the inversion of the formulas (13), where we did not take into account the force $\vec{F}(t)$, have the graphical representations from Fig. 9 and Fig. 10.

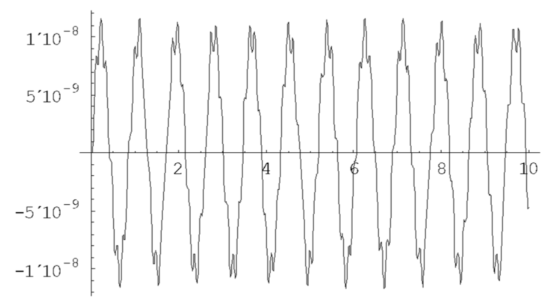


Fig. 9 Representation of $z_1(t)$

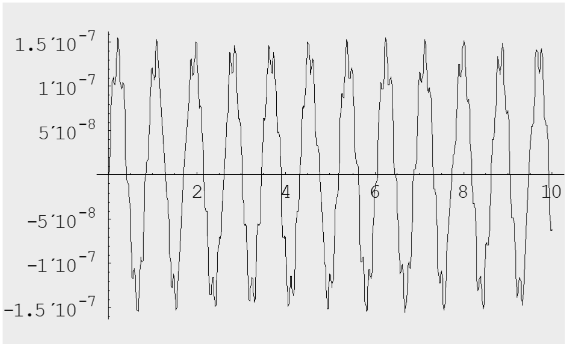


Fig. 10 Representation of $z_2(t)$

Applying the development theorems, we inverted the Laplace transforms from relations (13), where we took into account the force $\vec{F}(t)$, considered constant, resulting in displacements $z_1(t), z_2(t)$, in the case of forced and unamortized vibrations produced by exciting displacement due to the road, as below:

$$\begin{cases} z_1 = \dfrac{F}{k_o+k_1} - \dfrac{F_1}{F_2} - \dfrac{F_3}{F_4} - \\ \dfrac{z_0\omega_e m_2\sqrt{k_1 m_2}\sin\left(\sqrt{\dfrac{k_1}{m_2}}\,t\right)}{\left(k_1-m_2\omega_e^2\right)g\left[k_0 m_2+k_1\left(-m_1+m_2\right)\right]} - \\ \dfrac{F_5}{F_6} - \dfrac{F_7}{F_8} + \dfrac{F_9}{F_{10}} + \dfrac{F_{11}}{F_{12}} \\ z_2 = k_1^2\sqrt{k_1}\begin{Bmatrix} m_2\left[Ff_1(t)-z_0\sin(\omega_e t)\right]+ \\ m_1\left[-Ff_2(t)+z_0\sin(\omega_e t)\right] \end{Bmatrix} + \\ +k_0^2 m_2\sqrt{m_2}z_0\omega_e\sin\left(\sqrt{\dfrac{k_1}{m_2}}\,t\right) \\ +k_1^2 z_0\omega_e\sqrt{m_2}\left(m_2-m_1\right)\sin\left(\sqrt{\dfrac{k_1}{m_2}}\,t\right)+ \\ +k_0k_1\sqrt{m_2}z_0\omega_e\left(2m_2-m_1\right)\sin\left(\sqrt{\dfrac{k_1}{m_2}}\,t\right)- \\ k_0\sqrt{k_1}m_2\left[k_0z_0\sin(\omega_e t)+Fm_2\omega_e^2 f_1(t)\right]+ \\ +k_1\sqrt{k_1}k_0\begin{Bmatrix} m_1z_0\sin(\omega_e t)+ \\ m_2\left[Ff_1(t)-2z_0\sin(\omega_e t)\right] \end{Bmatrix}+ \\ \dfrac{Fm_2\omega_e^2\left(m_1f_2-m_2f_2\right)}{\sqrt{k_1}\left(k_0+k_1\right)\left[k_1\left(m_1-m_2\right)-k_0m_2\right]\left(k_1-m_2\omega_e^2 t\right)}. \end{cases} \tag{16}$$

where:

$$f_1(t) = -1+\cos\left(\sqrt{\frac{k_1}{m_2}}\right);$$

$$f_2(t) = -1+\cosh(\omega_n t);\ \omega_n = \sqrt{\frac{-k_0-k_1}{m_1}};$$

$$F_1(t) = Fk_0^2 m_2\cosh(\omega_n t);$$

$$F_2(t) = (k_0+k_1)\left[k_0m_2+k_1(m_2-m_1)\right]\left(k_0+k_1-m_2\omega_e^2\right);$$

$$F_3(t) = k_1^2\begin{bmatrix} Fm_1\sqrt{m_1}\omega_n\cosh(\omega_n t)- \\ Fm_2\omega_n\sqrt{m_1}\cosh(\omega_n t)+ \\ m_1\sqrt{m_1}z_0\omega_e\sinh(\omega_n t) \end{bmatrix};$$

$$F_4(t) = m_1\sqrt{m_1}\omega_n^3\left[k_0m_2+k_1(m_2-m_1)\right]\left(k_0+k_1-m_1\omega_e^2\right);$$

$$F_5(t) = k_0\begin{cases} F\omega_n\sqrt{m_1}m_1m_2\omega_e^2\cosh(\omega_n t)+ \\ k_1\left[F\omega_n\sqrt{m_1}m_1\cosh(\omega_n t)\right. \end{cases}$$
$$-2F\omega_n\sqrt{m_1}m_2\cosh(\omega_n t)+$$
$$\left.+m_1\sqrt{m_1}z_0\omega_e\sinh(\omega_n t)\right]\Big\};$$

$$F_6(t) = m_1\sqrt{m_1}\omega_n^3\left[k_0m_2+k_1(m_2-m_1)\right]\left(k_0+k_1-m_1\omega_e^2\right);$$

$$F_7(t) = k_1Fm_1^2\omega_e^2\cosh(\omega_n t);$$

$$F_8(t) = (k_0+k_1)\left[k_0m_2+k_1(m_2-m_1)\right];$$

$$F_9(t) = Fm_1m_2\omega_e^2\cosh(\omega_n t);$$

$$F_{10}(t) = (k_0+k_1)\left[k_0m_2+k_1(m_2-m_1)\right];$$

$$F_{11}(t) = z_0\sin(\omega_e t);$$

$$F_{12} = \left(k_0+k_1-m_1\omega_e^2\right)\left(k_1-m_2\omega_e^2\right);$$

$$\omega_n = \sqrt{\frac{k_0+k_1}{m_1}};\ \omega_n\sqrt{m_1} = \sqrt{k_0+k_1};$$

$$m_1\sqrt{m_1}\omega_n^3 = \left(k_0+k_1\right)^{\frac{3}{2}}$$

Based on the numerical application, the functions $z_1(t), z_2(t)$ in the relations (16), resulting from the inversion of the formulas (13), where we took into account the force $\vec{F}(t)$, have the graphical representations from Fig. 11 and Fig. 12.

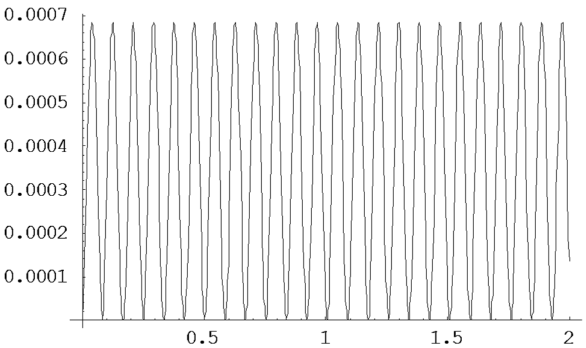


Fig. 11. Representation of $z_1 = z_1(t)$

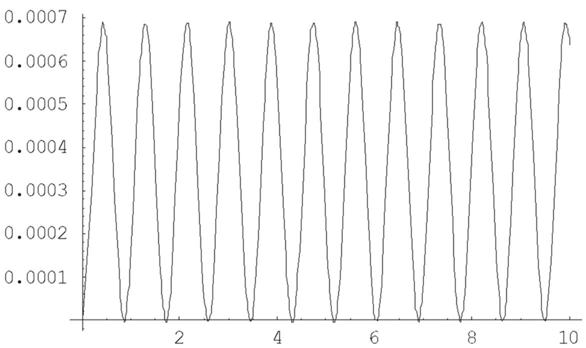


Fig.12. Representation of $z_2 = z_2(t)$

Numerical application:

$$m_1 = 52\,[Kg];$$
$$m_2 = 415\,[Kg];$$
$$k_0 = 270000\left[\frac{N}{m}\right];$$
$$k_1 = 22000\left[\frac{N}{m}\right];$$
$$c_0 = 7500\left[\frac{Ns}{m}\right];$$
$$c_1 = 1500\left[\frac{Ns}{m}\right];$$
$$z_0 = 0{,}02\,[m];$$
$$F = 100[N];$$
$$\omega_e = 50\left[s^{-1}\right]$$

From the analysis of these graphs it is clearly observed the effect of damping on the displacements caused by vibrations, these being much smaller in the case of damped vibrations.

Also, ignoring the disturbing force, which acts on the chassis, leads to much smaller displacements of the chassis, compared to the situation in which this force is not neglected.

Next, we used another modern method, such as systems theory. We will resume the mathematical model of forced and damped vibrations (7). This mathematical model places vibrations with n> 1 degrees of freedom in the category of open multivariable linear systems. The input vector is:

$$\vec{i} = \{Q_e\},$$

and the output one

$$\vec{e} = \{q\}.$$

From the matrix form (9) is deduced the canonical form of this mathematical model:

$$\begin{cases} \dot{\vec{x}}(t) = [A_0]\vec{x} + [B_0]\vec{i}_0 \\ \{q\} = [C_0]\vec{x} \end{cases}. \qquad (17)$$

where:

$\vec{x}$ is the state vector;

$$\vec{x} = \left[\frac{\{q\}}{\{\dot{q}\}}\right]. \qquad (18)$$

$$[A_0]_{2n\times 2n} = \left[\begin{array}{c|c} [0]_{n\times n} & [I]_{n\times n} \\ \hline -[A]^{-1}[K] & -[A]^{-1}[C] \end{array}\right]. \qquad (19)$$

$$[B_0]_{2n\times 2n} = \left[\begin{array}{c|c} [0]_{n\times n} & [0]_{n\times n} \\ \hline [I]_{n\times n} & [0]_{n\times n} \end{array}\right]. \qquad (20)$$

$$[C_0]_{n\times 2n} = \left[[I]_{n\times n}\,[0]_{n\times n}\right]. \qquad (21)$$

$$\vec{i}_0 = \left[\frac{[A]^{-1}\{P_e(t)\}}{[0]_{n\times 1}}\right]. \qquad (22)$$

$$[I]_{n\times n} = \begin{bmatrix} 1 & 0 & ... & 0 \\ 0 & 1 & ... & 0 \\ ... & ... & ... & ... \\ 0 & 0 & 0 & 1 \end{bmatrix}. \qquad (23)$$

The dynamic answer, the solution of the canonical system, is given by:

$$\begin{cases} \vec{x}(t) = e^{[A_0]t}\,\vec{x}_0 + \int_0^t e^{[A_0](t-\tau)}[B_0]\vec{i}_0(\tau)\,d\tau \\ \{q\} = [C_0]\,\vec{x}(t) \end{cases}. \qquad (24)$$

The canonical form of the system (9), which represents the mathematical model of forced and non-damped vibrations, is given by the system (17) in which, this time,

$$[A_0]_{2n\times 2n} = \left[\begin{array}{c|c} [0]_{n\times n} & [I]_{n\times n} \\ \hline -[A]^{-1}[K] & [0]_{n\times n} \end{array}\right]. \qquad (25)$$

After determining the matrix function and solving the integral in relation (24), we found the time functions (13), obtained by the first method.

4. CONCLUSIONS

We considered that the car's suspension is provided by four systems, first with a single degree of freedom, then with two degrees of freedom, identical, mounted between the vehicle chassis and each shaft of each wheel. From the analysis of the graphs of variation in time of the movements of the suspension, as a result of the vibrations to which it is subjected, it is clearly observed the effect of the damping on the movements, these being much smaller in the case of the damped vibrations. Also, ignoring the perturbing force, which acts on the chassis, leads to much smaller displacements of the chassis, compared to the situation in which this force is not neglected. It is imperative that the movements caused by vibrations be taken into account when dimensioning the suspension. The use of two methods for determining the dynamic response, one being the use of the unilateral Laplace transform with respect to time and the other the application of modern systems theory, allowed the comparison of results, which are the same in both situations.

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Vibraţiile supensiilor automobilelor

Rezumat: În această lucrare se studiază influența vibrațiilor asupra suspensiilor automobilelor, mai întâi, pe modele cu un grad de libertate, apoi pe modele mecanice cu două grade de libertate.

Răspunsul dinamic l-am determinat folosind două metode moderne de calcul. Una ditre ele este aceea a folosirii transformatei integrale Laplace unilaterală în raport cu timpul, iar cea de a doua este aceea a folosirii teoriei sistemelor.

Reprezentările grafice care ne arată evoluția în timp a deplasărilor suspensiei provocate de vibrații au fost obținute cu programul de calcul Mathematica.

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