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THE ACCURATE COMPUTING OF CLOTHOID COORDINATE VALUES AND OF THE DISTANCE BETWEEN A POINT AND A CLOTHOID

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Abstract: *The purpose of this paper is to present two important problems related to the numerical studies of clothoid curves: the exact calculation of their coordinates and the deduction of the distance between a point and the clothoid.*

The paper begins with a brief presentation of the history of this curve, names of great mathematicians involved over time in the research and a lot of contemporary problems such as: the use of clothoids as transition curves between a line and a circle arc, between two lines or between two circle arcs, with the direct use in the construction of railways and highways, in the trajectory path-planning of mobile robots and autonomous vehicles, CAGD and so on , with many references to the main existing works.

The clothoid curve has a very interesting kinematic property: in the case of a mobile moving along the curve with a constant velocity, the normal acceleration of the mobile varies linearly, depending on the curvilinear abscissa corresponding to the curve point, because the product between the curvilinear abscissa and the curvature radius is constant: $s\,\rho(s) = A^2$.

The paper solves the problem of calculating the exact coordinates of the clothoid points, using numerical quadrature formulas, in the context of Romberg procedure, which is based on the principle of extrapolation of Richardson. These calculations are necessary because the accuracy of the coordinates calculus induces the accuracy of all subsequent calculations for: the point-clothoid distance, the connection points coordinates, the parameters that determine the position of a clothoid in the Oxy axis system, etc.

In the second part of the paper is presented a new method to obtain the nonlinear equation whose solution is the curvilinear abscissa of the point where the perpendicular line on the clothoid from an outer point intersects it.

Many numerical results, presented mainly as graphs, obtained with our own C programs, are given in the final part of the paper, certifying the correctness of the proposed calculation methods.

Key words: *Clothoid, Fresnel's integrals, Romberg method, nonlinear equation, Newton-Raphson method.*

1. INTRODUCTION

The planar curve, named clothoid, was studied for the first time by the Swiss mathematician Jakob Bernoulli (1654-1705) linked with the problem of bending of curved elastic bars. Bernoulli has established the fundamental formula: $\mathbf{s}\,\boldsymbol{\rho}(s) = \mathbf{A}^2$, where **A** is the so-called clothoid parameter, **s** is the curvilinear abscissa of the clothoid's point and $\boldsymbol{\rho}(s)$ is the radius of the oscillating circle.

These studies have been continued by Leonhard Euler (1701-1783) who established the parametric equations as definite integrals:

$$x(s) = C(s) = \int_0^s cos\left(\frac{\pi}{2}t^2\right)dt,$$
$$y(s) = S(s) = \int_0^s sin\left(\frac{\pi}{2}t^2\right)dt \quad (1)$$

also, the limits:

$$\lim_{s\to\infty} x(s) = \frac{1}{2}, \lim_{s\to\infty} y(s) = \frac{1}{2} \quad (2)$$

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and noticed the spiral form of the curves. Consequently, the name "Euler spiral" began to be used for this curve. Considering the above-mentioned parametric equations, the value of the clothoid parameter results as $A = \frac{1}{\sqrt{\pi}} = 0.564189$.

One has to mention that the French physicist Augustin Jean Fresnel (1788-1827), studying the light diffraction problem, has find similar expressions for the coordinates of light rays trajectories.

The study of Euler's spiral was resumed a few years later by Alfred Marie Cornu (1841-1902). In 1874 he discovered other properties of this curve, by using complex numbers, so that the name "Cornu spiral" began to be known as describing the curve.

The third name – clothoid - was given by the mathematician Ernesto Cesàro (1859-1906), in 1886, after the name of goddess Clotho (Κλωθώ), of the Greek mythology.

The fourth name (and the last) "railway transition spiral" was given by Arthur Talbot, in 1899, for the curve used as a transition between a straight line and a circle arc, in the construction of railways.

Detailed information on this subject may be found in [11].

At the beginning, the study of clothoid was perform due to the theoretical importance but starting with the construction of railways and highways the practical interest grows up.

This curve has a very interesting property: if a mobile covers it with a constant speed, its tangential acceleration is null and the normal acceleration has a linear variation with respect to curvilinear abscissa of the clothoid.

If a segment of straight way is connected with a segment having a circle arc form and a vehicle passes from one to the other, when the vehicle leaves the straight section a normal acceleration will suddenly appear and also a not desirable centrifugal inertial force.

This problem may be avoided if the connection between the straight segment and the circle arc is made by using a clothoid, tangent to the straight line in its initial point (where s=0) and with the radius of the osculator circle being equal with the radius of the circle arc in the final point. When a vehicle covers such a trajectory, the acceleration is zero when the movement is on the straight line and grows up from zero to a specified value when the movement is on the clothoidal curve. Because the radius of the osculating circle corresponding to the last clothoidal point is equal with the radius of the circle arc, jerks cannot appear, and the obtained acceleration is a continuous curve.

For the same reason, when it is necessary to connect two trajectories (straight lines or two circle arcs) other three intermediate curves are used: a clothoid, a circle arc and again a clothoid. The first practical clothoid utilization was in the domain of railways construction, later the same curve was used in the building of highways.

In the last decades, the movements of mobile robots that must avoid the obstacles and the movement of autonomous vehicles are performed covering not only straight lines, circles but also segments of clothoids.

The clothoids are usually considered in the displacement path-planning of such vehicles.

There are many scientific papers in the literature dealing with different theoretical and practical problems linked with clothoids. Here it is a short review:

- Theoretical studies of clothoidal curve and its properties [3], [5], [10],
- Computing of clothoid point coordinates [1], [3], [22],
- Fitting clothoids to the straight lines and circle arcs, [4], [5], [13], [20],
- Mobile robots and autonomous vehicles path-planning [2], [7], [21],
- The CNC machines tools trajectory optimization to obtain a low level of vibrations [16l], [17],
- Using clothoidal curves in CAGD [9],
- Approximation of clothoidal curves with other curves used in CAGD [6],[14].

Our aim is to present in this paper two problems of great interest by our opinion: the accurate computing of the clothoid coordinates and the calculation of the distance between a point and a clothoid curve.

2. THE BEST ACCURATE METHOD TO COMPUTE THE CLOTHOID COORDINATES

The coordinates of the clothoid having A as parameter are the following:

$$x(s) = C(s) = \int_0^s \cos\left(\frac{t^2}{2A^2}\right) dt\,, \qquad y(s) = S(s) = \int_0^s sin\left(\frac{t^2}{2A^2}\right) dt \tag{3}$$

There are several methods, used in the articles studying different aspects of the clothoids, to compute the coordinates, for all values of curvilinear abscissas in the interval [0; s_{max}].

The first method uses series expansions and is the most frequent cited in the literature. The series to be computed may be find in [1], [3], [4], [5], [20] a. o.

According to the second method, the series are replaced by equivalent formulas, which leads to a decrease in the number of necessary arithmetical operations, e. g. in [12].

The third possibility to compute the coordinate values is to solve simultaneous differential equations, as it is shown in [20].

The functions under the integral sign of relations (3) are presented in figures 1, 2 and 3, for different values of parameter A (2.0, 2.5 and 3.0). Looking to the right of the figures, it can be noticed that the use of quadrature formulas of generalized rectangles, trapezes, Simpson and also Chebishev or Gauss-Legendre is not appropriate due to the oscillating shape of the graphical representations.

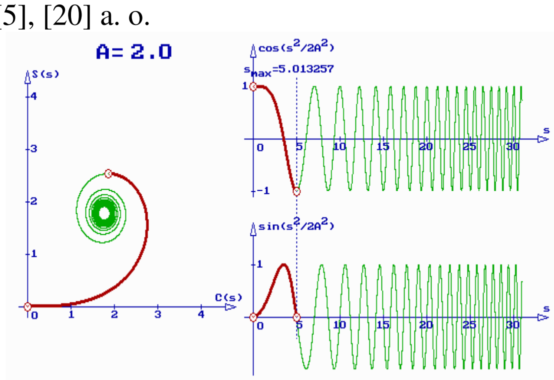


Fig. 1 Clothoid with parameter A=2.0

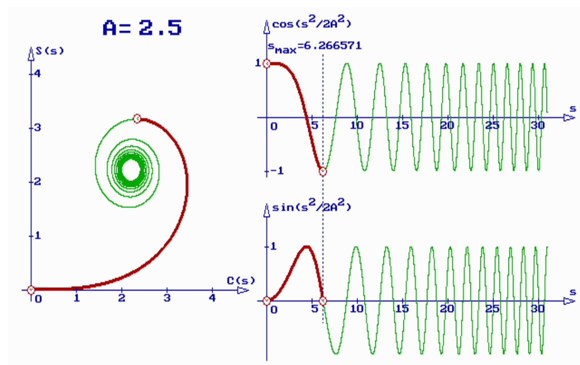


Fig. 2 Clothoid with parameter A=2.5

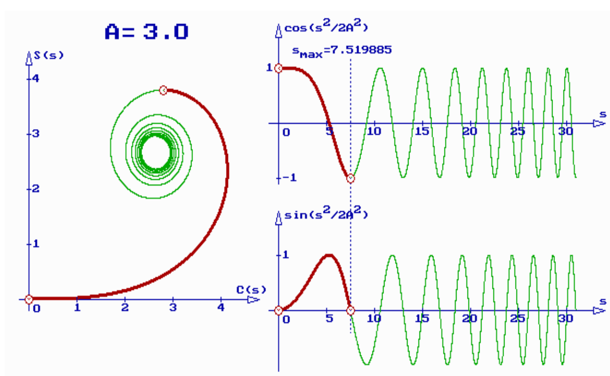


Fig. 3 Clothoid with parameter A=3.0

Remembering that the clothoid curves are used as transition curves between a line and a circle arc, line to line or circle arc to circle arc, we realize that in these cases are used only the first parts of diagrams, especially between s=0 and the value that corresponds to the point of maximal ordinate.

On the left, figures 4, 5 and 6 represents segments of clothoids used as transition curves and on the right there are the functions under the integral sign. It is obvious that for such shapes the use of quadrature formulas is possible, with a high degree of accuracy.

Consequently, the well-known formulas of numerical quadrature will be used, and to obtain results close to the exact values, the number of intervals must be big enough ([15], [19]).

Evidently, it comes out that a great number of intervals increases the accuracy but also grows the number of necessary computations, to obtain six ~ eight exact significant digits are necessary thousands of intervals.

Our idea is to use the Romberg procedure based on Richardson extrapolation, set out in some numerical analysis books [15] pp. 144-146, [9] pp. 351-356 a. o. The main advantage of this method is high accuracy resulting with minimal computational effort.

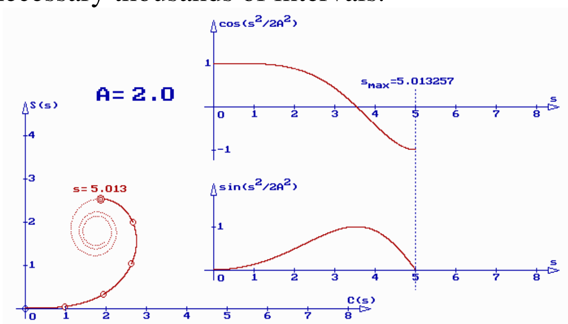


Fig. 4 Clothoid with parameter A=2.0

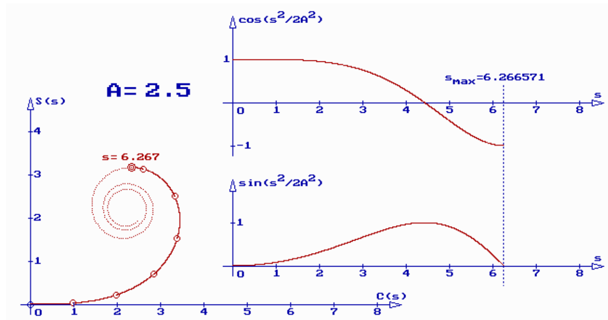


Fig. 5 Clothoid with parameter A=2.5

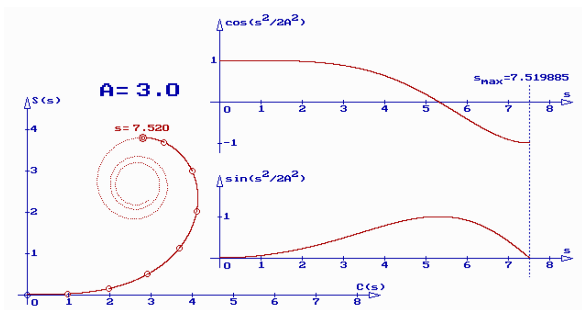
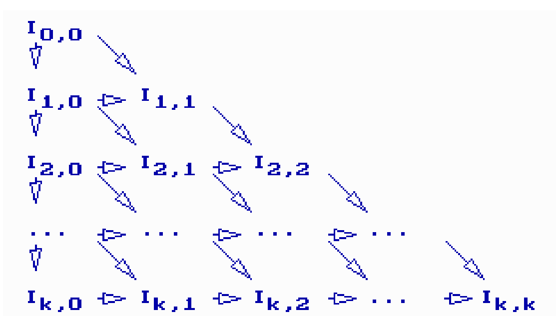


Fig. 6 Clothoid with parameter A=3.0

To find the value of definite integral $I = \int_a^b f(x)dx$ we need to calculate the elements of the following triangular Romberg matrix:



starting with elements on the first column, using a recurrent formula, followed with elements on the next columns (starting with elements on the main diagonal), using another recurrence formula.

From line to line, it is necessary to double the number of intervals used in the quadrature formula.

The values of the elements in the first column, except for the first, are obtained with a recurrence formula based on the generalized trapezoidal method for 2^m respectively 2^{m+1} intervals, using the following relations:

$$I_{0,0} = \frac{b-a}{2}[f(a) + f(b)] \tag{4}$$

$$I_{m+1,0} = \frac{I_{m,0}}{2} + \\ + \frac{b-a}{2^{m+1}} \sum_{i=1}^{2^m} f\left(a + (2i-1)\frac{b-a}{2^{m+1}}\right), \\ m = 0,1,\dots,k-1 \tag{5}$$

the accuracy depending on the square of the interval length.

The following recurrence formula is repeatedly used for the computing of elements on the next columns,

$$I_{m,j} = \frac{I_{m-1,j-1} - 4^j I_{m,j-1}}{1 - 4^j}, \quad m = 2,3,4 \dots, k\,; \quad j = 2,3, \dots, m \tag{6}$$

and passing from column to column the accuracy of computed elements grows with square of interval length.

This procedure may be used to compute the definite integral values (3) for the clothoid having as parameter A=3.0 and for the point of clothoid defined by the curvilinear abscissa value s=3.0.

The expressions of integrals are as follows

$$C(3.0) = \int_0^{3.0} cos\left(\frac{t^2}{18}\right) dt \qquad S(3.0) = \int_0^{3.0} sin\left(\frac{t^2}{18}\right) dt \tag{7}$$

The Romberg's matrix for the first definite integral is:

1	**2**.8163738428							
2	**2**.8964834223	**2.92**31866154						
4	**2.9**184074623	2.9257154756	**2.9258**840663					
8	**2.92**39925049	2.9258541858	2.9258634332	**2.925863**1056				
16	**2.925**3950126	2.9258625152	2.9258630704	2.9258630647	**2.92586306 4**5			
32	**2.925**7460259	2.9258630303	2.9258630647	2.9258630646	2.9258630646	**2.9258630646**		
64	**2.9258**338033	2.9258630625	2.9258630646	2.9258630646	2.9258630646	2.9258630646	**2.9258630646**	
128	**2.9258**557492	2.9258630645	2.9258630646	2.9258630646	2.9258630646	2.9258630646	2.9258630646	**2.9258630646**

We notice that the value of sixth element on the main diagonal remains unchanged on the following positions on the diagonal, this stands that after dividing the interval [0;3.0] in only **32 parts** the resulting integral value has 11 exact digits, therefore one may write:

$$C(3.0) = \int_0^{3.0} cos\left(\frac{t^2}{18}\right) dt = 2.9258630646 \tag{8}$$

In order to realize the advantages of Romberg procedure, it can be mentioned that the result with eight exact significant digits is obtained with the generalized trapezoidal method if the interval [0;3.0] is divided in **1362 parts**, as can be seen below:

1355	2.9258629993	
1356	2.9258629994	
1357	2.9258629995	
1358	2.9258629996	
1359	2.9258629997	
1360	2.9258629998	
1361	2.9258629999	
1362	**2.9258630**000	(eight exact significant digits)
1363	2.9258630001	
1364	2.9258630002	
1365	2.9258630003	

According to the Romberg procedure, the result with the same accuracy is obtained on the fifth matrix line, after dividing the interval in 16 parts.

In the case of definite integral **S(3.0)** the Romberg matrix is as follows:

1	0.7191383079							
2	0.5465812540	**0.4**890622361						
4	0.5048918762	0.4909954170	**0.4911**242957					
8	**0.49**45725520	0.4911327772	0.4911419346	**0.491142**2145				
16	**0.491**9993035	0.4911415541	0.4911421392	0.4911421424	**0.491142142**2			
32	**0.491**3564049	0.4911421053	0.4911421421	0.4911421421	0.4911421421	**0.4911421421**		
64	**0.4911**957061	0.4911421398	0.4911421421	0.4911421421	0.4911421421	0.4911421421	**0.4911421421**	
128	**0.4911**555330	0.4911421420	0.4911421421	0.4911421421	0.4911421421	0.4911421421	0.4911421421	**0.4911421421**

and the result with ten exact significant digits is obtained on the sixth matrix line,

$$S(3.0) = \int_0^{3.0} sin\left(\frac{t^2}{18}\right) dt = 0.491\,142\,142\,1 \tag{9}$$

The obtained results certify that this method, based on the Romberg procedure, may be successfully applied, the values of the definite integrals are obtained with extremely good accuracy with minimal number of arithmetic operations.

The proposed method may be considered **the fourth possible method** to compute the clothoid's coordinate values and has many advantages by comparison to the previously mentioned methods.

3. THE DISTANCE BETWEEN A POINT AND A CLOTHOID CURVE

The computing of clothoid curve coordinates was done only for the points at the beginning of this curve, belonging to a segment used as transition curve. The same considerations will apply when calculating the distance between a point and a clothoid curve.

Such a problem arises when the clothoidal trajectory of a mobile robot or autonomous vehicle must avoid an obstacle or when a deviation from the prescribed trajectory occurs.

It is obvious that in both cases we deal with the initial section of clothoid, the spiral part is not considered.

The cloidoid curve has as its point of symmetry the origin of the coordinate system, so if the point is located in the second or third quadrant, we can consider its symmetrical position in the fourth or first quadrant.

In figures 1 to 6 the sections considered useful of the clothoid are drawn with thick line, the point A has the greatest abscissa value, the angle between the tangent through A and the horizontal axis is $\varphi = \frac{\pi}{2}$.

The ordinate of point B has the highest value and the tangent through B is horizontal $\varphi = 0$.

The slope of the tangent line is computed with the formula:

$$m = tan\,\varphi = \frac{y'(s)}{x'(s)} = tan\left(\frac{s^2}{2A^2}\right), \qquad (10)$$

$$\varphi = \frac{s^2}{2A^2}$$

and results

$$\varphi_A = \frac{s_A^2}{2A^2} = \frac{\pi}{2}, s_A = A\sqrt{\pi}, \qquad (11)$$

$$\varphi_B = \frac{s_B^2}{2A^2} = \pi, s_B = A\sqrt{2\pi}$$

obtaining the values of the curvilinear abscissas s_A and s_B corresponding to the points A and B.

Depending on the value of the clothoid parameter A and position of point P (figures 7 and 8), there are two intervals [0; s_A] or [s_A; s_B] which may contain the value of the unknown curvilinear abscissa of point N, the foot point, representing the intersection between the curve and the perpendicular drawn on it from point P.

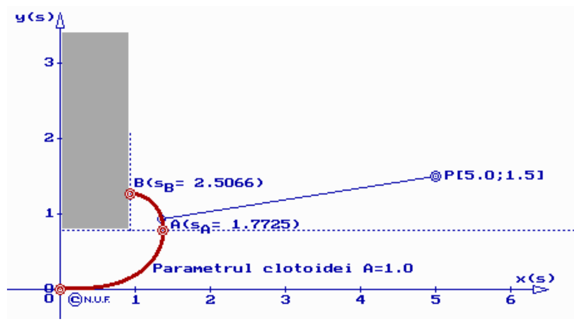


Fig. 7 The distance from point to the clothoid

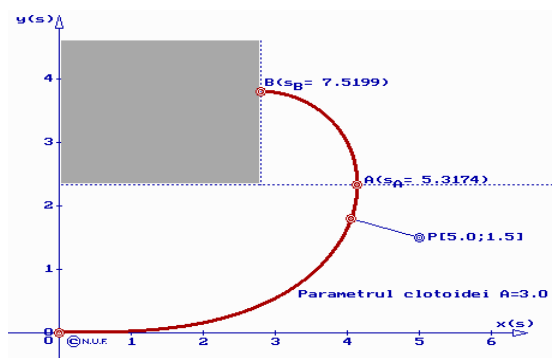


Fig. 8 The distance from point to the clothoid

Note that if $y_P < y_A$, the perpendicular from point P on the clothoid intersects the curve in a foot point having the curvilinear abscissa belonging to the interval [0; s_A], as shown in figure 8.

Figure 7 shows that if the coordinates of point P meet the conditions $x_P > x_B$ and $y_P > y_A$, the value of the curvilinear abscissa of the foot point belongs to the interval [s_A; s_B].

It is obvious that the distances result immediately in two particular cases:

1. if $y_P = y_A$ and $x_P > x_B$ then $d = |x_P - x_A|$,
2. if $x_P = x_B$ and $y_P > y_A$ then $d = |y_P - y_B|$,

The points situated in the first quadrant defined by the inequalities $0 < x < x_B$ and $y > y_A$

are without any importance because the clothoid's sections with points having **s > sB** aren't used as transition curves between straight lines and circle arcs.

The first method that may be used to compute the distance between a point P and a clothoid is based on finding the value of the curvilinear abscissa s that minimize an expression, the square of the distance from point P to a curve ([8]),

$$D^2(s) = [x_P - C(s)]^2 + \\ + [y_P - -S(s)]^2 \qquad (12)$$

In order to obtain the value of **s** determining an extreme value of this expression, it is necessary to equal with zero the derivative:

$$\frac{d[D^2(s)]}{ds} = \\ -[x_P - C(s)]cos\left(\frac{s^2}{2A^2}\right) - \\ -[y_P - S(s)]sin\left(\frac{s^2}{2A^2}\right) = 0 \qquad (13)$$

and to solve the resulting nonlinear equation.

This equation has a single solution, located in the interval **[0; sA]** or in the interval **[sA; sB]**, depending on the position of point P.

After determining the value of s, we may calculate the coordinates of the cloidoid point, followed by the calculation of the distance.

According to the second method, and here is our contribution, the nonlinear equation is established using another method, its importance consisting in the fact that it can also be used in solving another theoretical problem related to clothoids.

Let's consider in figure 9, the clothoid, the osculating circle, the curvature radius ΓN and an external point P, whose distance from the clothoid will be calculated.

Note that the angle **φ** between the tangent in N to the clothoid and the horizontal direction is equal to the angle between the vertical direction and the PN segment, perpendicular to the clothoid (angles with perpendicular sides).

According to relation (10), one may write $\varphi = \frac{s^2}{2A^2}$ and there are the following relations between the coordinates of points P and N:

$$x_P = x_N - d\sin\phi, \\ y_P = y_N + d\cos\phi \qquad (14)$$

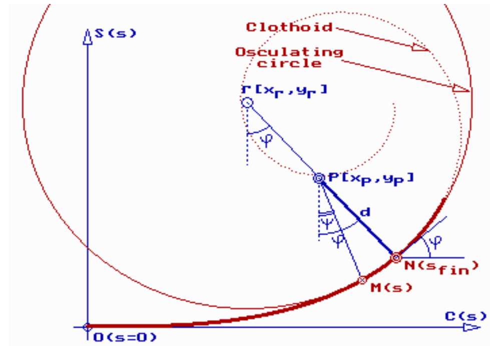


Fig. 9 The clothoid, osculating circle and distance

If **ψ** is the angle between the vertical direction and a segment PM, M being an arbitrary point on the clothoid, we write the following relations

$$d\sin\psi = x_M - x_P, d\cos\psi = y_P - y_M$$

and after dividing them:

$$\tan\psi = \frac{x_M - x_P}{y_P - y_M} = \frac{C(s) - x_P}{y_P - S(s)} \qquad (15)$$

The segment PM must be perpendicular to the clothoid, therefore the angles φ and ψ will be equal, **φ = ψ**. This is possible when the value of the curvilinear abscissa s satisfies the relation:

$$\tan\frac{s^2}{2A^2} = \frac{C(s) - x_P}{y_P - S(s)} \qquad (16)$$

which is obtained by matching expressions (10) and (15). The previous relation may be written as a nonlinear equation:

$$f(s) = [y_P - S(s)]\sin\left(\frac{s^2}{2A^2}\right) - \\ -[C(s) - x_P]\cos\left(\frac{s^2}{2A^2}\right) \qquad (17)$$

that has the same form as (13), obtained with the first method ([8]).

The numerical solving of equations (17) or (13) may be performed using the bisection or Newton-Raphson method, in the specified intervals: [0; sA] or [sA; sB] existing only one root.

In the case of Newton-Raphson method, the following formulas are repeatedly used:

$$s_{fin} = s_{init} - \frac{f(s_{init})}{f'(s_{init})}, s_{init} \leftarrow s_{fin}$$

and the number of necessary iterations is about 10~15. In this case, due to the shape of f(s) expression, it is not necessary to use other formulas belonging to Halley, Ostrowski, a. o., containing the second function derivatives which would increase the number of arithmetic operations [15], [19] pg. 67-68).

The final value **s_{fin}** of the curvilinear abscissa is thus obtained, corresponding to the point N of the clothoid, where the perpendicular from P crosses the curve, the coordinates of foot point N being:

$$x_N = C(s_{fin}), y_N = S(s_{fin}) \qquad (18)$$

As final step, we may calculate the distance between points P and N,

$$d_{PN} = \sqrt{(x_P - x_N)^2 + (y_P - y_N)^2} \qquad (19)$$

the problem being accomplished.

If the value of **s_{fin}** (curvilinear abscissa corresponding to point N) is known, there are other possibilities for calculating the distance: first the calculus of angle **φ^*** between the horizontal direction and the tangent through N and then the calculation of distance:

$$\varphi^* = a\,tan\left(\frac{C(s_{fin})-x_P}{y_P-S(s_{fin})}\right),\ d_{PN} = \frac{y_P-S(s_{fin})}{cos\ \phi^*}$$

4. NUMERICAL RESULTS

Example no. 1. The clothoid with parameter A=2.0 is considered and five points in the plane (in the allowed regions) having the coordinates: **P_1**[1.0; -1.0], **P_2**[1.5; 1.25], **P_3**[4.0; 1.0], **P_4**[2.25; 1.75], **P_5**[3.5; 3.0]

When the clothoid parameter is A=2.0 the following values for the curvilinear abscissas of points A and B are obtained:

$$s_A = A\sqrt{\pi} = 3.544908,$$
$$s_B = A\sqrt{2\pi} = 5.013257$$

resulting the Cartesian coordinates of these points:

$A[C(s_A); S(s_A)]$, $\quad x_A = C(s_A) = 2.764150,$
$\quad y_A = S(s_A) = 1.553181$

$B[C(s_B); S(s_B)]$, $\quad x_B = C(s_B) = 1.874128,$
$\quad y_B = S(s_B) = 2.530966$

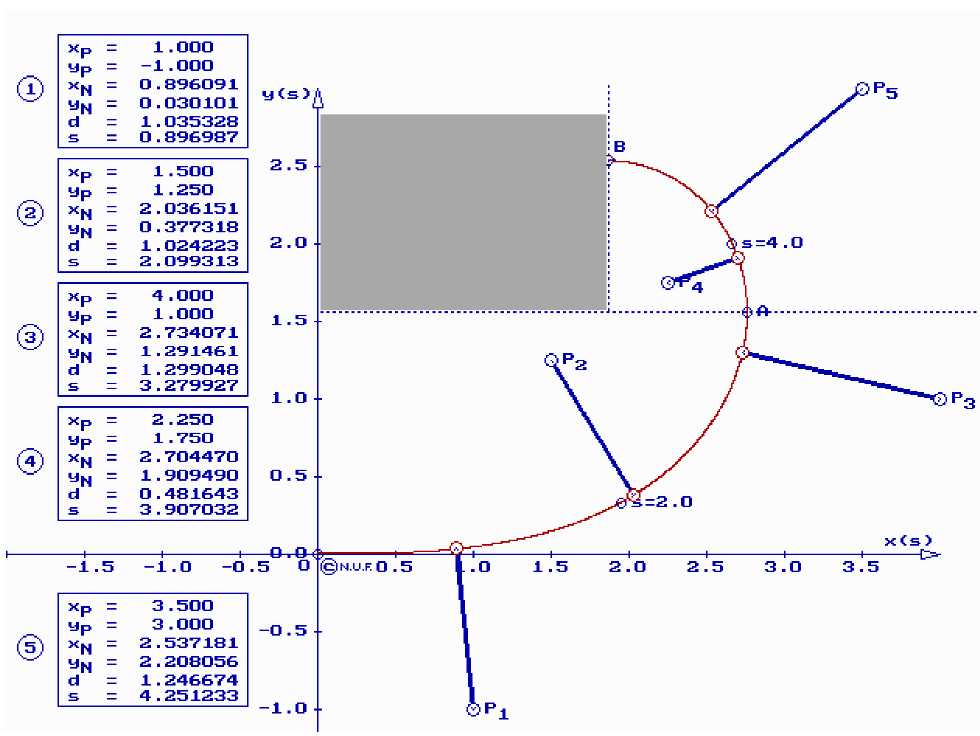


Fig. 10. The clothoid (parameter A=2.0), five points in the plane and the computed distances between points and the clothoid. The points A and B have the coordinates: A[2.764150;1.553181], B[1.874128;2.530966]

For each considered point, the nonlinear equation (17) is solved, at the beginning choosing the interval where the root is located, starting with an initial value. The root is determined by the Newton-Raphson method and finally the distance is computed.

Numerical results are presented in figure 10: the clothoid, positions of points **P**$_1$ to **P**$_5$ and - in the rectangles - the obtained coordinates of the foot points N, the distances from points **P**$_1$ to **P**$_5$ to the clothoid and also the value of the curvilinear abscissa for each point N.

Example no. 2. Three clothoids are considered having different parameter values $\left(0.40, \frac{1}{\sqrt{\pi}} = 0.564189, 0.70\right)$ and a point **P** of coordinates [0.95; -0.15]. The Newton-Raphson method was used three times, resulting the curvilinear abscissas corresponding to the points **N**$_1$, **N**$_2$ and **N**$_3$ followed by the computing of coordinates and distances.

In all three cases the solving of the nonlinear equation starts with the initial value $s_{init} = 0.6$, the following obtained values being presented in Table 1 and the numerical values in figure no. 11.

Table 1

Nr. iter	A_1 =0.40 s	$f_1(s)$	A_2=0.564189 s	$f_2(s)$	A_3=0.70 s	$f_3(s)$
0	0.60		0.60		0.60	
1	0.578056	-0.322145	0.693633	-0.203569	0.792369	-0.272327
2	0.564364	-0.146264	0.692309	-0.004559	0.772988	-0.044831
3	0.557556	-0.061064	0.692001	-0.001053	0.770042	-0.006369
4	0.554678	-0.023790	0.691929	-0.000244	0.769687	-0.000759
5	0.553571	-0.008848	0.691925	-0.000014	0.769676	-0.000025
6	0.553158	-0.003254	0.691922	-0.000009	0.769668	-0.000017
7	0.553005	-0.001205	0.691921	-0.000006	0.769663	-0.000011
8	0.552937	-0.000530	0.691920	-0.000003	0.769659	-0.000008
9	0.552920	-0.000136	0.691919	-0.000002	0.769657	-0.000005
10	0.552910	-0.000079	**0.691919**	**-0.000001**	0.769655	-0.000003
11	0.552904	-0.000046			0.769654	-0.000002
12	**0.552901**	**-0.000027**			0.769653	-0.000002
13					**0.769653**	**-0.000001**
	N$_1$[0.504608;0.164955] **d$_1$=0.545500**		**N$_2$[0.653872;0.166586]** **d$_2$=0.433495**		**N$_3$[0.742052;0.151071]** **d$_3$=0.365905**	

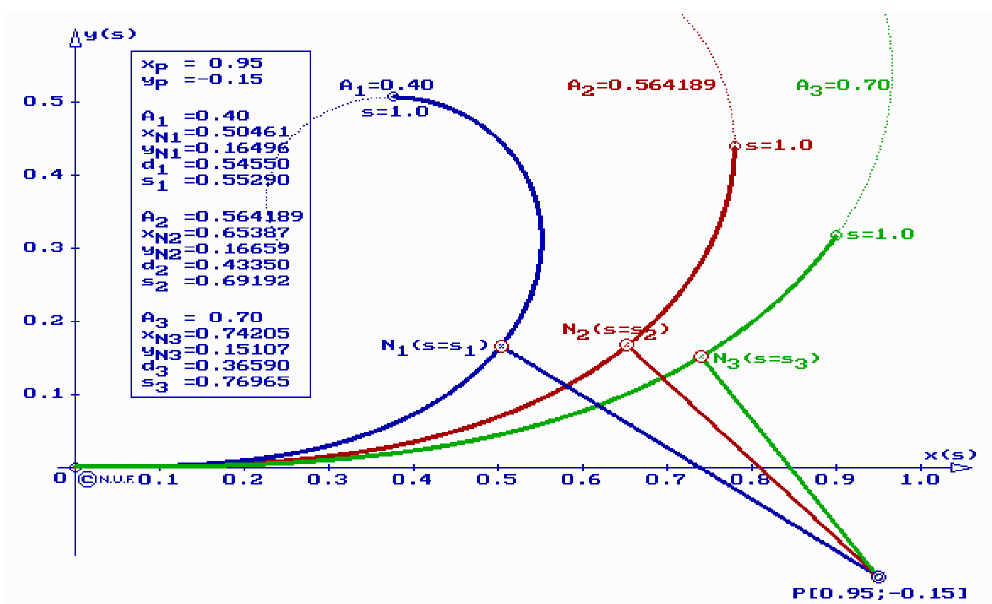


Fig. 11. The clothoids with parameters A=**0.40**, A=**0.564189** and A=**0.70** and the distances between a point **P** and clothoid

5. CONCLUSIONS

The main problem when the clothoid curves are studied is the accurate computing of point coordinates. In all cases involving calculus performed on clothoids, such as: the computing of the distance between a point and a clothoid, the coordinates of points where clothoid fits with a line or with a circle arc when are used as transiton curves, coordonates of clothoids deplaced and rotated in the plane, it is necessary to have accurate values of the clothoid points.

The above presented method for the definite integral computing, using the Romberg procedure based on the Richardson extrapolation, solves the problem of accuracy and requires a minimal number of aritmetic operations, as can be seen from the numerical examples.

Practical aspects were considered when the distance between a point and a clothoid was calculated. In the current practical activity (railways and highways construction), the distance between a point and a spiral segments of clothoid has no application, that's why our attention focused on the clothoid segments really used in practical situations.

Consequently, an interval containing a single root results and the solving of the nonlinear equation can be done with the well known method of Newton-Raphson or bisection method.

The root of this nonlinear equation is the value of the curvilinear abscissa of the clothoid's point where the perpendicular form the outer point P intersects the curve.

In paper [8] the authors have determined the nonlinear equation by searching the value of **s** that minimize the lenght of the segment connecting the point P and a point on the clothoid.

The present paper proposes another method, obviously obtaining the same results, based on geometrical considerations.

Our original method may be used not only for solving the problem of computing the distance, but also in the frame of other problems, linked with the clothoids, especially when the fitting of clothoid with other planar curves is studied.

The paper contains many numerical results, diagrams obtained by running original C programs, and tables, all of them illustrating and sustaining the studied theoretical and practical aspects.

We hope that this paper may be included, together with the those contained in the reference section, in the category of scientific works that highlight important aspects of this domain and bring new achievements.

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CALCULUL EXACT AL COORDONATELOR CURBELOR CLOTOIDE ȘI DETERMINAREA DISTANȚEI DINTRE UN PUNCT ȘI O CLOTOIDĂ

Această lucrare tratează două probleme importante legate de studiul numeric al curbelor clotoide: calculul exact al coordonatelor acestora şi determinarea distanţei dintre un punct şi clotoidă.

După o succintă prezentare a istoriei a acestei curbe sunt trecute în revistă principalele probleme legate de curba clotoidă: realizarea racordărilor dintre două curbe (dreaptă cu arc de cerc, arc de cerc cu arc de cerc, etc.) utilizate la construcţia de căi ferate şi autostrăzi, utilizarea lor în path-planning-ul deplasărior roboţilor mobili şi a vehiculelor autonome, utilizarea lor în CAGD, etc., sunt citate principalele lucrări existente.

Curba clotoidă are o proprietate cinematică foarte importantă: în cazul parcurgerii acesteia cu o viteză constantă acceleraţia normală a mobilului variază liniar, deoarece produsul dintre abscisa curbiline a punctelor clotoidei şi raza cercului osculator corespunzător punctului este constant, $\mathbf{s}\,\boldsymbol{\rho}(s) = \mathbf{A}^2$.

Este rezolvată problema calculului exact al coordonatelor punctelor clotoidei, utilizând formule de cuadratură numerică, încadrate în metoda lui Romberg, care este fundamentată pe principiul extrapolării al lui Richardson. Aceste calcule sunt necesare deoarece precizia de calcul a coordonatelor determină precizia tuturor calculelor ulterioare, calculul distanţei punct-clotoidă, determinarea coordonatelor punctelor de racordare, valoarea parametrilor care determină poziţia unei clotoide în sistemul de axe Oxy, etc.

În cadrul celei de-a două părţi a lucrării se determină distanţa dintre un punct şi clotoidă, utilizând o metodă proprie pentru determinarea ecuaţiei neliniare a cărei soluţie este abscisa piciorului perpendicularei dusă pe clotoidă din punctul exterior acesteia.

Lucrarea conţine multe rezultate numerice, prezentate în special sub formă de grafice, obţinute în urma utilizării unor programe proprii, rezultate care certifică corectitudinea relaţiilor de calcul folosite.

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