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A GEOMETRIC METHOD FOR OPTIMIZE THE ACKERMANN-TYPE STEERING MECHANISM

Nicolae URSU-FISCHER, Diana Ioana POPESCU

***Abstract**: The Ackermann's four-bars linkage, having the shape of an isosceles trapezoid, is used in the steering mechanism of the cars, making rotations with unequal angles of the front wheels, in order to eliminate the slip frictions during changing the vehicle direction of travel.*

Both Ackermann's mechanism and other mechanism which perform similar tasks have come to the attention of many researchers over time and there it is a lot of scientific work in this field.

One of the important tasks is the dimensional optimization of the mechanism, to ensure compliance with a certain relationship between the angles of rotation of the two front wheels, around some axes perpendicular on the ground.

The authors propose a new method of optimization, based on geometric aspects. Theoretical problems are presented as well as a numerical example that fully justifies this new optimization methodology and its correctness.

***Key words:** four-bar mechanism kinematics, steering mechanisms, optimization, curve fitting, simultaneous equations solving*

1. INTRODUCTION

A method to avoid the lateral slip of the wheels has been known for over 200 years, this can be achieved by using a mechanism name Ackermann's mechanism (Rudolp Ackermann, 1764-1834), patented in England, in 1817, which was firstly used in the building process of carriages and now it is used for automobiles.

The Ackermann steering gear is a four-bars mechanism, ABCD, [11] (figure 1), having two links of equal length: AB and CD and two of unequal length: AD and BC, mounted at the back of the front wheels. This mechanism is the most used in the construction of vehicle steering systems, a vital element for the control and stability of vehicles during maneuvers to change the direction of travel [1],[2], [3],[7].

In a curve, all the four wheels of a vehicle should be exactly on a circular trajectory and in order to obtain this, the extensions of the four axles of the wheels must intersect in the center of the circle (Ackermann's steering principle, figure 1).

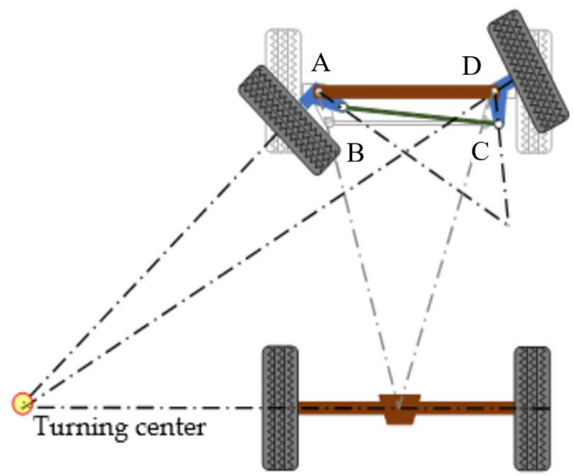

Fig. 1. Ackermann steering mechanism principle

Different aspects of the steering mechanisms presented interest for researchers over time, such as: steering geometry and performance optimization [5],[6],[12],[14], evaluation of steering errors [11], investigations on the dynamic behavior [3], geometrical and operational constraints [4],[5],[10], etc., but research topics are far from exhausted.

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This paper aims to address aspects of the geometry of the mechanism, based on which the authors propose a new method of optimization.

2. THE RELATIONSHIP BETWEEN THE ROTATION ANGLES OF THE WHEELS

By moving the four-bar mechanism's connecting rod to the left or to the right, the two rocking levers realize unequal angle rotations, such that the two wheels rotate with the same (unequal) angles around some perpendiculars on the ground. If the two (front) drive wheel's axes intersect in one point that can also be found on the rear wheel's axis, then all of the wheels are realizing rolling movement on the ground, without any side slip (figure 2).

It is clear that, in order for the axles's wheels positions represented in figures 1 and 2 to be the ones seen in the drawings, there must be a link between the two angles **φ** and **ψ**, and the value of the distance **L** between the front and rear axles's wheels and the distance between the wheels, **d**.

By using the elements from figure 2, this link can be easily established. The lines OA_1 and OA_2, which pass through the origin and have known angular factors, have the following equations:

$$(D_1)\ \ y = x\ \tan\varphi, \qquad (D_2)\ \ y = x\ \tan\psi$$

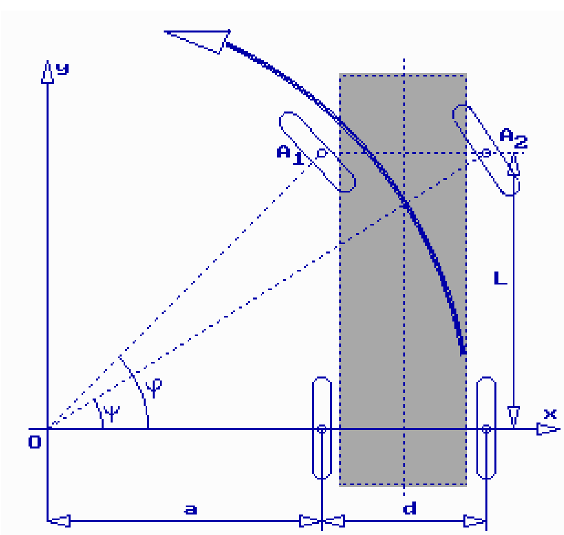

Fig. 2. Working principle, geometry and notations

Given the fact that the points A_1 and A_2, of known coordinates, are on these lines, we can write:

$$\begin{aligned} &A_1[a;L] \quad L = a\ \tan\varphi, \\ &A_2[a+d;L] \quad L = (a+d)\tan\psi \end{aligned} \tag{1}$$

The parameter **a** can be eliminated, resulting into a relationship between the four variables:

$$\tan\psi = \frac{L\ \tan\varphi}{L + d\ \tan\varphi}\ , \tag{2}$$

having the final form:

$$\psi = a\tan\left(\frac{\xi\ \tan\varphi}{\xi + \tan\varphi}\right), \quad \xi = \frac{L}{d} \tag{3}$$

Its graphical representation in figure 3 uses four different values of the **L/d** ratio.

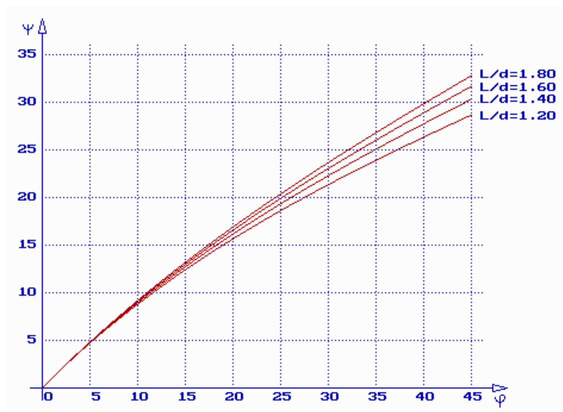

Fig. 3. Dependency between ψ, φ and L/d

Starting from the **L** and **d** dimensions, which characterize the vehicle, the rotation angle's variation curve of a wheel based on the rotation angle of the other wheel has been determined, such that **the first part** of the problem was solved.

The solution for the **second part** of the problem is a bit more difficult, as we don't have an exact way of solving it – a quadrilateral trapezoid isosceles type mechanism with which we can obtain values for angle **ψ** for the given values of angle **φ** based on the relation (3) for a certain value of the ratio **L/d** can not be determined.

From figure 4, the relation between angles **φ** and **ψ** can be established, which will be based on the values of the other elements of the mechanism, lengths **d** and **r** and the angle α.

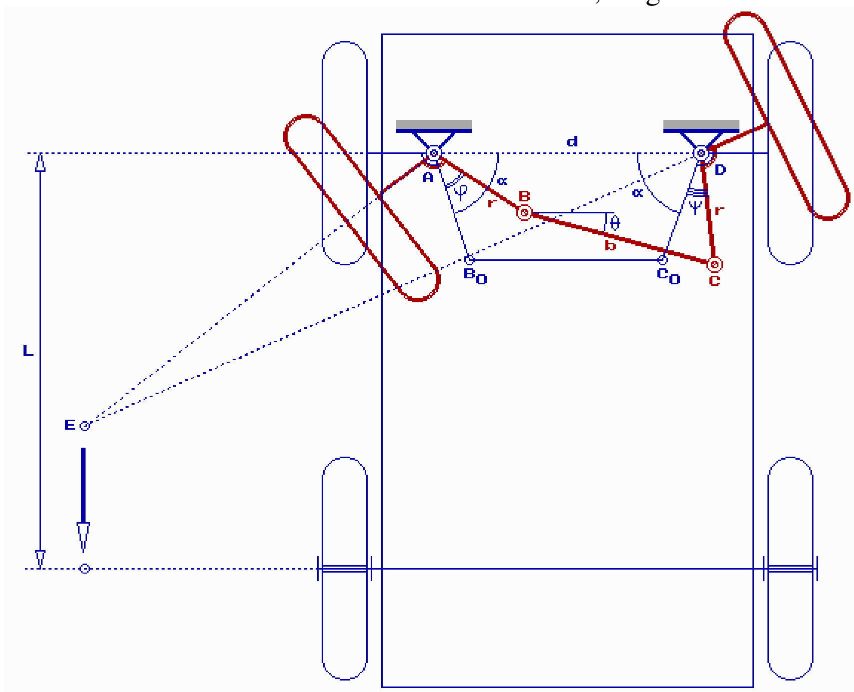

Fig. 4. Ackermann's four-bars linkage – working position

The length of the rod, **b**, results from the relation:

$$d = b + 2r\cos\alpha, \quad b = d - 2r\cos\alpha \qquad (4)$$

The vector equation of the closing contour of the mechanism is projected horizontally and vertically, obtaining:

$$\overline{AB} + \overline{BC} + \overline{CD} + \overline{DA} = 0$$

$$\begin{cases} r\cos(\alpha - \varphi) + b\cos\theta + r\cos(\alpha + \psi) - d = 0 \\ r\sin(\alpha - \varphi) + b\sin\theta - r\sin(\alpha + \psi) = 0 \end{cases} \qquad (5)$$

relations from which the unknown angle $\boldsymbol{\theta}$ can be eliminated, obtaining a trigonometric equation having the unknown $\boldsymbol{\psi}$:

$$\begin{aligned} &(d - 2r\cos\alpha)^2 = \\ &[d - r\cos(\alpha - \varphi) - r\cos(\alpha + \psi)]^2 + \\ &+ r^2[\sin(\alpha + \psi) - \sin(\alpha - \varphi)]^2 \end{aligned} \qquad (6)$$

After some calculations the equation becomes:

$$A\sin(\alpha + \psi) + B\cos(\alpha + \psi) + C = 0 \qquad (7)$$

The A, B and C coefficients have the expressions:

$$\begin{aligned} &A = -2r^2\sin(\alpha - \varphi), \\ &B = -2r[d - r\cos(\alpha - \varphi)] \\ &C = r^2\sin^2(\alpha - \varphi) + [d - r\cos(\alpha - \varphi)]^2 + \\ &+ \ r^2 - (d - 2r\cos\alpha)^2 \end{aligned} \qquad (8)$$

With the substitutions:

$$\begin{aligned} &t = \tan\frac{\alpha + \psi}{2}, \quad \sin(\alpha + \psi) = \frac{2t}{1 - t^2}, \\ &\cos(\alpha + \psi) = \frac{1 - t^2}{1 + t^2} \end{aligned} \qquad (9)$$

equation (7) turns into a second degree equation,

$$(C - B)t^2 + 2At + B + C = 0 \qquad (10)$$

having the solutions:

$$\begin{aligned} &t_1 = \frac{-A - \sqrt{A^2 - C^2 + B^2}}{C - B}, \\ &t_2 = \frac{-A + \sqrt{A^2 - C^2 + B^2}}{C - B} \end{aligned} \qquad (11)$$

The value of the angle ψ is calculated as:

$$\psi = 2\,\mathrm{a}\tan(t_2) - \alpha \tag{12}$$

If we consider a certain specified value of the angle φ, which corresponds to a computed value of the angle ψ and the extensions of the two front wheel's axes are drawn, we can see that their point of intersection, E, is situated somewhere in the plan (figure 4), but only by chance it will be situated on the rear wheel's axis, as it should be in order not to have any wheel side slip.

By repeatedly using the relations (8), (11) and (12), the values of the angles φ and ψ can be graphically represented, overlapping the obtained curve with the one of relation (3), noting that the two graphical representations don't coincide, but are "close" one to another. It can be seen that after some modifications of the parameters **r** (=AB) and **α** (the angle between the crank and the horizontal) a graphical representation closer to the desired one can be obtained, and if we refer to figure 4, the intersection point of the extensions of the two front wheel's axles will be closer to the rotation axis of the rear wheels, for different values of angle φ.

In the specialty literature this problem is widely represented, the authors using different optimization methods to achieve the optimal solution. In the next section an optimization solution which belongs to the authors will be presented.

3. THE NEW OPTIMIZATION METHOD

Figure 5 shows the axles of the front wheels that intersect the axis of the rear wheels at points T1 and T2, and the point T3 is their common point of intersection.

By writing the equations of the three lines and solving the three equation systems, each with two unknowns, the cartesian coordinates of the three points T_1 , T_2 and T_3 are determine.

$$\begin{cases} y = x\tan\varphi, & (AT_1) \\ y = -L, & (T_1T_2) \end{cases} \qquad x_1 = -\frac{L}{\tan\varphi},\ y_1 = -L \tag{13}$$

$$\begin{cases} y = (x-d)\tan\psi, & (DT_2) \\ y = -L, & (T_1T_2) \end{cases}$$
$$x_2 = d - \frac{L}{\tan\psi}, \qquad y_2 = -L \tag{14}$$

$$\begin{cases} y = x\tan\varphi, & (AT_1) \\ y = (x-d)\tan\psi, & (DT_2) \end{cases}$$
$$x_3 = -\frac{d\tan\psi}{\tan\varphi - \tan\psi},\ y_3 = -\frac{d\tan\varphi\tan\psi}{\tan\varphi - \tan} \tag{15}$$

It is clear that, in order not to have any wheel lateral slip, those three points would have to coincide for any possible value of angle φ, which is impossible.

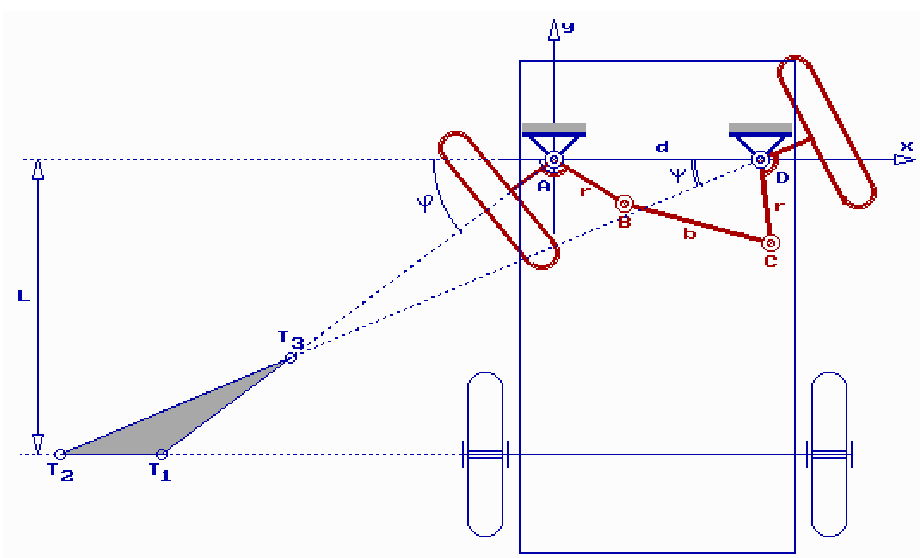

Fig. 5. The steering mechanism and the triangle to be minimized

A possible way to determine the optimal values of the parameters **r** and **α** is to find their values for which the sum of the areas of the triangles which are obtained for every value of the angle **φ** from the range $[0^0; \varphi_{max}]$, taken with constant step, to be minimal [13].

The area of each triangle can be calculated by using the formula:

$$S=\frac{1}{2}\left|(x_2-x_1)(y_1-y_3)\right|= \qquad (16)$$

$$=\frac{1}{2}\left|\left(d-\frac{L}{\tan\psi}+\frac{L}{\tan\varphi}\right)\left(-L+\frac{d\tan\varphi\tan\psi}{\tan\varphi-\tan\psi}\right)\right|$$

and their sum is realized, which is attributed to the elements of a matrix. The number of the matrix's lines being equal to the number of values which **r** can take and the number of its columns being equal to the number of values which the angle **α** can take.

The matrix (18) was obtained for the following values: **L = 2700 [mm]** and **d = 1300 [mm]** (Figure 2), its elements being the calculated sums of the areas of the triangles in the case of the values of **r** and **α**.

One may observe that the minimal value was obtained for the values

$$r=200\,[\text{mm}],\quad \alpha=70^0 \qquad (17)$$

In each of the figures 6, 7 and 8 the diagram which corresponds to the theoretical case has been graphically represented, based on relation (3), showing the dependency between the values of the angles **ψ** and **φ**, when **L** and **d** have the values: **2700** and respectively **1300**.

In figure 6 another diagram was represented, containing the values of angle **ψ** based on angle **φ**, obtained after solving the trigonometric equation (7), considering the values for **r** and **α** the ones from (17), which realizes the minimal value of the sum of the triangles areas.

One may observe that in this case the two diagrams are very close, which denotes the correctness and efficiency of the used process.

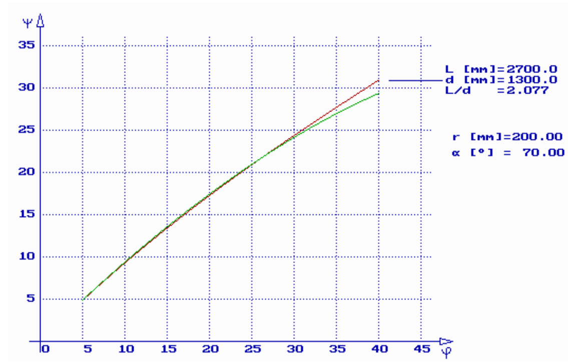

Fig. 6. Results for optimal values of r and α

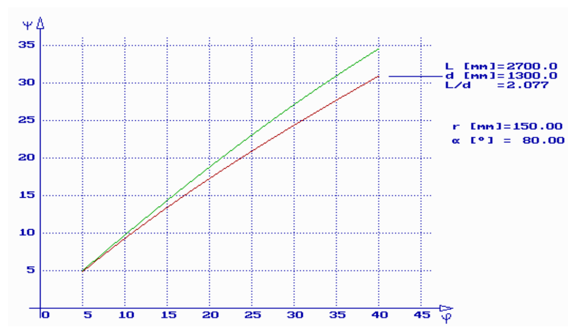

Fig. 7. Results for nonoptimal values of r and α

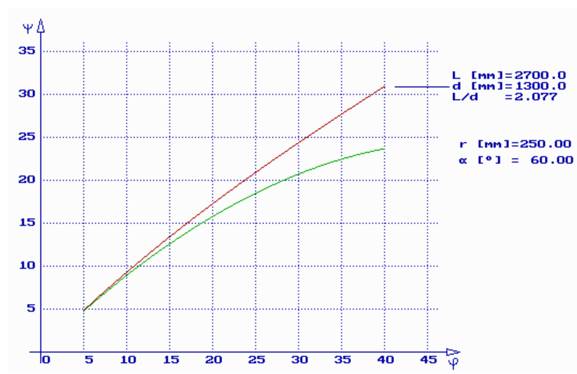

Fig. 8. Results for nonoptimal values of r and α

(18)

	r	**55⁰**	**60⁰**	**65⁰**	**70⁰**	**75⁰**	**80⁰**	**85⁰**
1	100	265450.9	116411.7	27392.1	11635.4	113741.7	480849.8	1867342.3
2	125	290995.4	130301.3	32523.1	10250.5	107658.5	471596.1	1856245.1
3	150	318757.7	145460.0	38329.5	9185.1	101698.5	462372.9	1845148.4
4	175	348966.6	161997.3	44859.2	8457.3	95866.7	453181.3	1834052.1
5	**200**	381885.0	180036.6	52164.9	**8086.2**	90168.7	444022.1	1822956.3
6	225	417816.4	199717.0	60304.7	8092.9	84610.2	434896.1	1811861.0
7	250	457113.1	221195.9	69342.9	8499.8	79197.4	425804.4	1800766.1
8	275	500187.4	244652.6	79351.1	9331.6	73936.7	416748.1	1789671.7
9	300	547524.7	270292.1	90408.7	10615.0	68835.2	407728.0	1778577.8
10	325	599702.0	298350.1	102604.8	12379.0	63900.1	398745.4	1767484.5
11	350	657410.8	329099.4	116039.2	14655.6	59139.4	389801.3	1756391.6

	r	67⁰	68⁰	69⁰	**70⁰**	71⁰	72⁰	73⁰
1	180	19973.3	12092.8	8095.6	8353.9	13333.5	23618.3	39944.0
2	185	20732.0	12549.1	8269.6	8264.9	12999.8	23057.5	39172.7
3	190	21514.5	13025.7	8460.9	8190.5	12678.4	22506.8	38409.7
4	195	22321.1	13522.8	8669.7	8130.9	12369.4	21966.4	37655.2
5	200	23152.0	14040.8	8896.2	8086.2	12072.8	21436.3	36909.2
6	205	24007.7	14580.0	9140.6	8056.7	11788.9	20916.8	36171.7
7	**210**	24888.4	15140.5	9403.2	**8042.3**	11517.7	20407.8	35443.0
8	215	25794.5	15722.6	9684.1	8043.5	11259.5	19909.5	34723.0
9	220	26726.4	16326.7	9983.6	8060.3	11014.3	19422.0	34011.9

(19)

	r	55⁰	60⁰	65⁰	**70⁰**	75⁰	80⁰	85⁰
1	100	37688.3	25113.6	9643.4	12614.8	43850.7	105445.7	283964.3
2	125	39239.2	26562.1	11023.0	11670.6	42544.8	104155.7	282680.8
3	150	40823.8	28030.2	12413.5	10780.8	41236.3	102864.7	281396.9
4	175	42445.3	29519.6	13815.6	9960.9	39925.2	101572.6	280112.7
5	200	44107.9	31032.1	15230.1	9203.8	38611.4	100279.5	278828.2
6	225	45815.9	32570.0	16658.0	8520.1	37294.6	98985.3	277543.3
7	250	47574.9	34135.5	18100.2	7917.9	35974.9	97689.9	276258.0
8	275	49390.8	35731.4	19557.8	7403.1	34652.1	96393.4	274972.3
9	300	51271.1	37360.9	21032.2	6984.9	33326.0	95095.7	273686.3
10	325	53224.3	39027.5	22524.7	6684.2	31996.5	93796.7	272400.0
11	**350**	55260.9	40735.4	24036.8	**6511.3**	30663.5	92496.5	271113.2

(20)

In figures 7 and 8, diagrams which correspond to other values for r and α are displayed:

r=150 [mm], =800 (Figure 7),
r=250 [mm], =600 (Figure 8).

One may observe from figures 7 and 8 that the diagrams computed based on the solving procedure are much more different than the theoretical diagram, compared to the case of figure 6.

The optimal variant has been obtained when the sum of the areas of the triangles was minimal.

In order to obtain a better result, the calculation can be repeated, decreasing the ranges from which the values **r** and **α** are taken, and also decreasing their variation step [13].

After a new iteration of the procedure, the matrix (19) was obtained. For the values:

$$r = 210\ [\text{mm}], \quad \alpha = 70^0 \qquad (21)$$

the computed sum of the areas decreases from **8086.2** to **8042.3** .

The above procedure was based on the determination of the values of **r** and **α** for which the sum of the areas of the triangles **$T_1T_2T_3$** (Figure 5) is minimal, so all of the obtained triangles should be as small as possible.

In conclusion, another way for obtaining this optimisation can be taken into account: small triangles mean smaller lengths of the sides, so instead of searching for **r** and **α** values which minimise the sum of the area of the triangles, we could search those values which would minimise the sum of the perimeters of the triangles.

The perimeter of a triangle **$T_1T_2T_3$** can be computed by using the following relation:

$$P = |x_1 - x_2| + \sqrt{(x_3 - x_1)^2 + (y_3 - y_1)^2} + \\ + \sqrt{(x_3 - x_2)^2 + (y_3 - y_2)^2} \qquad (22)$$

For each pair of values **r** and **α**, the sum of perimeters is computed, determining the elements of a matrix, which, as in the case of the previous method, will have the number of lines equal to the number of values of **r** and the number of columns given by the values taken by the angle **α**.

As a result of performing these calculations the matrix (20) resulted. The matrix's smallest element is obtained for the values:

$$r = 350\ [\text{mm}], \quad \alpha = 70^0 \qquad (23)$$

Performing the graphical representation of the variation of the angle **ψ** depending on the angle **φ** (Figure 9) it can be seen that in the previous case, of minimizing the sum of surfaces, a better result was obtained, the curves in Figure 6 being closer than those in Figure 9, so the first method returns better results.

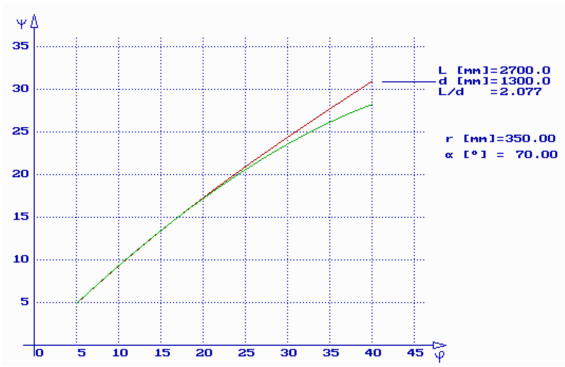

Fig. 9. Results obtained by minimizing the sum of triangle perimeters

4. CONCLUSIONS

The problem of determining the optimal dimensions of the elements of the trapezoidal mechanism used in the vehicle steering system has been in the attention of many researchers over time. At present, the Ackerman-type steering mechanism has returned to the attention of researchers, especially in the field of construction of mobile robots, namely in the study of the rotation of their wheels in order to follow a prescribed path or to avoid certain obstacles.

The method chosen to optimize the dimensions of the mechanism - so that Ackermann's law is satisfied for the largest possible ranges of values of the considered variables - is based on geometric considerations, regarding the reduction of the dimensions of a triangle formed by the intersection of three directions: rear wheel axis and the axles of the two front wheels, both in terms of its surface and perimeter, an idea that has not been used so far. The numerical calculations and graphs presented in the paper were made by authors with programs written in the C programming language [8],[9],[13].

Like other optimization methods, the proposed method has a degree of uncertainty, because it was considered that the axes of the joints of the trapezoidal quadrilateral mechanism are vertical, which is not achieved in practice.

For the previous calculations it was considered that a front wheel rotates around a vertical axis (perpendicular on the ground). In reality, this axis, the pivot axis, has an inclination to the vertical, a so call escape angle, in a vertical plan perpendicular on the wheel's axis, and an inclination angle, in a vertical plan which contains the wheel's axis.

In practice, the wheelbase **L** and track gauge are specified in the technical characteristics of the vehicles. The distance **d** between the pivot axes is obviously smaller than the gauge and must be measured to work with real values.

This study will be continued in the future, both in the case of the analyzed mechanism and in the case of the other variant used (two mechanisms driven by an element performing a translational movement) determining the optimal dimensions of the elements so that Ackermann's law is satisfied for different positions of the steering mechanisms.

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O METODĂ GEOMETRICĂ DE OPTIMIZARE A MECANISMULUI DE DIRECŢIE TIP ACKERMANN

Rezumat: Patrulaterul lui Ackermann, de forma unui trapez isoscel este utilizat în cadrul mecanismului de direcţie a autovehiculelor, realizând rotiri cu unghiuri neegale a roţilor din faţă, pentru ca să se elimine frecările de alunecare în timpul schimbării direcţiei de mers al autovehiculului.

Atât mecanismul lui Ackermann cât şi alte variante care sunt utilizate efectiv în construcţia dispozitivelor de direcţie a autovehiculelor au stat în atenţie cercetătorilor, existând multe lucrări în acest domeniu.

Una din problemele importante este cea a optimizării dimensionale a mecanismului astfel încât să fie asigurată respectarea relaţiei de legătură între unghiurile de rotaţie, în jurul unor axe verticale, a celor două roţi din faţă.

Autorii propun o nouă metodă de optimizare, bazată pe aspecte geometrice. Sunt prezentate problemele teoretice precum şi un exemplu numeric care justifică pe deplin metodologia de optimizare propusă.

Nicolae URSU-FISCHER, Prof. dr. eng .math., Technical University of Cluj-Napoca, Department of Mechanical Systems Engineering, E-mail: nic_ursu@yahoo.com, Phone: 0264-401659

Diana Ioana POPESCU, Prof. dr. eng., Technical University of Cluj-Napoca, Department of Mechanical Systems Engineering, E-mail: diana.popescu@mep.utcluj.ro, Phone: 0264-401783