TECHNICAL UNIVERSITY OF CLUJ-NAPOCA

ACTA TECHNICA NAPOCENSIS

Series: Applied Mathematics, Mechanics, and Engineering
Vol. 65, Issue IV, November, 2022

DETERMINATION AND ANALYSIS OF THE COEFFICIENT OF RESTITUTION IN THE CASE OF SOME MECHANICAL SYSTEMS

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Abstract: *The purpose of this paper is to determine and analyze the coefficient of restitution in the case of some considered mechanical systems. The paper analyzes three cases of determining the coefficient of restitution, namely: the case of the elastic ball that is allowed to drop freely on a horizontal surface from a considered height, the case of the mathematical pendulum that collides with a vertical obstacle and the case of the physical pendulum consisting of a homogeneous bar and a mass that also collides with a vertical surface.*

Key words: *collision, coefficient of restitution, bouncing ball, physical pendulum, mathematical pendulum.*

1. INTRODUCTION

The coefficient of restitution (denoted in the paper with k) is defined as the ratio between the speed after impact (final speed) and the speed before impact (initial speed) between two colliding bodies in specialized literature [5], [7], [8], [10], [12]. This coefficient of restitution is used in many collision models, especially in frictionless cases, and can take values in the range [0,1]. If the coefficient of restitution has the value 0, it is considered that the collision between the two bodies would be perfectly plastic, if it takes the value 1, the collision is perfectly elastic. If the restitution coefficient takes values from the mentioned range, the kinetic energy is dissipated or eliminated [4], [5], [8], [10], [12].

This paper analyses three cases of determining the coefficient of restitution. In the first case is considered an elastic ball that is allowed to drop freely from a initial height. The coefficient of restitution is calculated in three situations:

a. The pre-collision and post-collision heights are known;
b. The initial height of drop and the total time for dropping and rebounding the body are measured;
c. The initial height of drop and the total time of the several jumps are measured.

In the second case is considered a mathematical pendulum that collides with a vertical surface, and in the third case is considered a physical pendulum that also collides with a vertical obstacle.

2. THEORETICAL CONSIDERATIONS AND RESULTS

The first case

a) In the first case, an elastic ball of mass m is analysed. It is allowed to drop freely from an initial height h_1, after which it collides a horizontal rigid plate (Figure 1).

After the impact, the ball rebounds, making a vertical movement, reaching a height h_2, ($h_2<h_1$). In this case, the two heights are known, the one from which the drop starts and the one to which the ball reaches after the impact.

Received: 23.09.22; **Similaritie**s: 19.10.22: **Reviewed**: 09.11./07.11.22: **Accepted**:17.11.22.

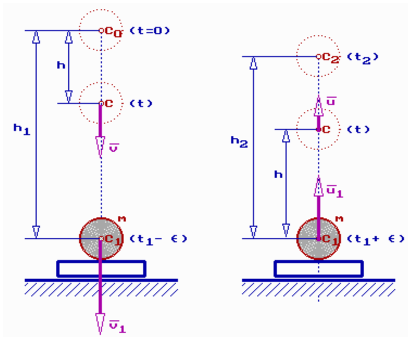

Fig.1. The mechanical system analysed in the first case

According to the definition, the coefficient of restitution, k, [3], [5], [6], [9], [11], is calculated considering the ratio of the two speeds, obtaining the relation:

$$k = \frac{u_1}{v_1} = \sqrt{\frac{h_2}{h_1}} \qquad (1)$$

b) In this case, we know the initial height h_1 from which the body drop and the time interval of dropping and rebounding the body, t_{tot}, so it can be written: $t_{tot} = t_{drop} + t_{rebound}$. The bouncing is a uniformly accelerated motion with zero initial velocity, so displacement is calculated with the formula: $h_1 = \frac{g t_{drop}^2}{2}$. The dropping time results is: $t_{drop} = \sqrt{\frac{2h_1}{g}}$. The velocity before the impact is: $v_1 = \sqrt{2gh_1}$ and it will depend on the coefficient of restitution k after the impact. The velocity u_1 becomes: $u_1 = kv_1 = k\sqrt{2gh_1}$.

After the impact, the material point performs a uniformly decelerated movement, its speed becomes zero at the highest point, so $u_2 = 0 = u_1 - g t_{rebound}$, and the rebound time result $t_{rebound} = \frac{u_1}{g} = \frac{k\sqrt{2gh_1}}{g} = k\sqrt{\frac{2h_1}{g}}$. The sum of the two times is as follows:

$$t_{total} = \sqrt{\frac{2h_1}{g}} + k\sqrt{\frac{2h_1}{g}} = \sqrt{\frac{2h_1}{g}}(1+k) \qquad (2)$$

The expression of the restitution coefficient depends on the initial height, h_1, and the total time, t_{tot}. In this case, the formula of the coefficient of restitution is:

$$k = t_{total}\sqrt{\frac{g}{2h_1}} - 1 \qquad (3)$$

c) In this situation, the height from which the body drops, and the total time taken by the few jumps on the horizontal plane until the final stop are considering.

The total time measured is the sum of the times it takes to drop, then rebound, drop again, etc. until these movements stop. It is observed that the rebound time in a certain stage is equal to the drop time in the next stage,

$$t_{total} = t_{drop\_1} + \underbrace{t_{rebound\_1} + t_{drop\_2}}_{the\ times\ are\ equal} + \underbrace{t_{rebound\_2} + t_{drop\_3}}_{the\ times\ are\ equal} + \underbrace{t_{rebound\_3} + t_{drop\_4}}_{the\ times\ are\ equal} + \cdots \qquad (4)$$

It was previously shown that in the case of a drop of the body followed by a rebound, the total time is calculated with the expression:

$$t_{total} = t_{drop} + t_{rebound} = \sqrt{\frac{2h_1}{g}} + k\sqrt{\frac{2h_1}{g}} \qquad (5)$$

so, the rebound time is obtained by multiplying the drop time by the restitution coefficient, consequently it can be written:

$$t_{total} = \sqrt{\frac{2h_1}{g}} + \left(k\sqrt{\frac{2h_1}{g}} + k\sqrt{\frac{2h_1}{g}}\right) + \left(k^2\sqrt{\frac{2h_1}{g}} + k^2\sqrt{\frac{2h_1}{g}}\right) + \left(k^3\sqrt{\frac{2h_1}{g}} + k^3\sqrt{\frac{2h_1}{g}}\right) + \ldots \qquad (6)$$

or

$$t_{total} = \sqrt{\frac{2h_1}{g}}(1 + 2k + 2k^2 + 2k^3 + \cdots) \qquad (7)$$

The relation (7) can be written in the form:

$$t_{total} = \sqrt{\frac{2h_1}{g}}[2(1 + k + k^2 + \ldots) - 1] \qquad (8)$$

In the general case it can be written

$$\frac{a^n - 1}{a - 1} = a^{n-1} + a^{n-2} + \ldots + a^3 + a^2 + a + 1 \qquad (9)$$

In the case when a is small (a<<1) it is obvious that the relationship is valid

$$\lim_{n\to\infty} \frac{a^n - 1}{a - 1} = -\frac{1}{a-1} = \frac{1}{1-a} \qquad (10)$$

The restitution coefficient, k, is always less than 1, so the total time expression can be written in the form:

$$t_{total} = \sqrt{\frac{2h_1}{g}}\left(\frac{2}{1-k} - 1\right) \Rightarrow t_{total} = \sqrt{\frac{2h_1}{g}}\frac{1+k}{1-k} \qquad (11)$$

From the relation (11) results the formula for calculating the restitution coefficient depending on the height from which the body is allowed to drop freely and the total duration of the jumps,

$$k = \frac{\sqrt{2h_1} - t_{total}\sqrt{g}}{\sqrt{2h_1} + t_{total}\sqrt{g}} \qquad (12)$$

Examining the relation (11), between $\sqrt{h}$ and t_{total} there is a linear dependence, which can be represented by a straight line passing through the origin of the coordinate system,

$$t_{total} = \sqrt{\frac{2}{g}\frac{1+k}{1-k}}\sqrt{h} \quad (13)$$

The figure 2 shows several such lines, corresponding to restitution coefficients with values between 0.50 and 0.90.

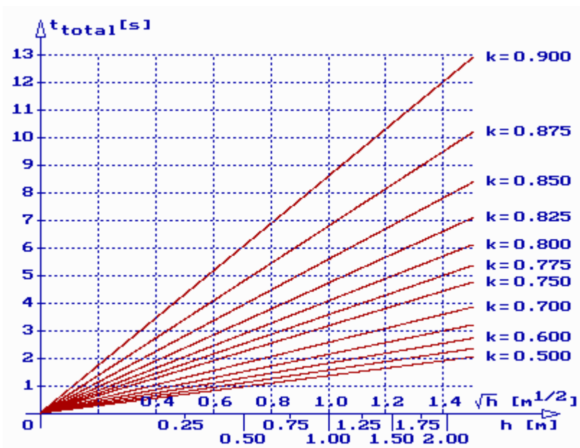

Fig.2. The linear dependence between h and t_{total}

The second case. A situation like the first example also occurs in the case of the impact of an elastic ball with a vertical obstacle (Fig. 3). The ball is fixed of a bar of length L (the mathematical pendulum).

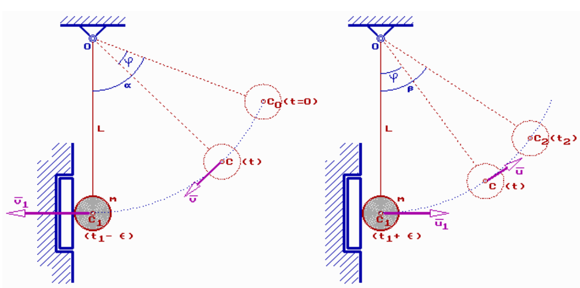

Fig.3. The mathematical pendulum colliding with a vertical obstacle

The determination of the speed v_1 (before the collision) of the pendulum is done using the kinetic energy theorem in finite form. The initial position of the pendulum is determined by the angle α

$$E_{c1} - E_{c0} = L_{0-1} \Rightarrow \frac{mv_1^2}{2} = 2mgL\,sin^2\frac{\alpha}{2} \quad (14)$$

hence the speed before impact is:

$$v_1 = 2\sqrt{gL}\,sin\frac{\alpha}{2} \quad (15)$$

The determination of the speed u_1 (after collision) of the mathematical pendulum, so that it reaches the position determined by the angle β, is done using the same theorem:

$$E_{c2} - E_{c1} = L_{1-2} \Rightarrow \frac{mu_1^2}{2} = 2mgL\,sin^2\frac{\beta}{2} \quad (16)$$

The velocity after impact becomes:

$$u_1 = 2\sqrt{gL}\,sin\frac{\beta}{2} \quad (17)$$

The collision is central and right, so the coefficient of restitution is calculated with the known formula $k = u_1/v_1$. In this case we have:

$$k = \frac{u_1}{v_1} = \frac{sin\frac{\beta}{2}}{sin\frac{\alpha}{2}} \quad (18)$$

The third case. Consider the physical pendulum consisting of a homogeneous bar of length L and the mass m_1 on which the mass m_2 is fixed.

The rotation of a rigid solid around a fixed axis is studied [8], [10], [12]. The differential equation of the rotational motion, the expressions of the mechanical moments of inertia in relation to the axis of rotation of the pendulum, of the bar and of the ball are written:

$$J_{O1} = \frac{m_1L^2}{3}, J_{O2} = m_2L^2 \quad (19)$$

The differential equation of the oscillatory movements in the case of the physical pendulum is:

$$\left(\frac{m_1}{3}L^2 + m_2L^2\right)\ddot{\phi} = -\left(\frac{1}{2}m_1L + m_2L\right)g\,sin\,\phi \quad (20)$$

After calculations, equation (20) becomes:

$$\ddot{\phi} + \frac{3(m_1+2m_2)g}{2(m_1+3m_2)L}sin\,\phi = 0 \quad (21)$$

and in the case of small oscillations $sin\,\phi \approx \phi$ is approximated and obtained:

$$\ddot{\phi} + \frac{3(m_1+2m_2)g}{2(m_1+3m_2)L}\phi = 0 \quad (22)$$

Since the differential equation (22) corresponds to a harmonic motion, the expression of the pulsation and the period can be written:

$$\omega^2 = \frac{3(m_1+2m_2)g}{2(m_1+3m_2)L}$$

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{2(m_1+3m_2)L}{3(m_1+2m_2)g}} \quad (23)$$

The distance from the rotation axis to the collision center is denoted by y. It is known that this distance is equal to the length of the mathematical pendulum synchronous with that of the physical pendulum,

$$\ddot{\phi} + \frac{g}{y}\phi = 0 \qquad T_0 = 2\pi\sqrt{\frac{y}{g}} \qquad (24)$$

the value of y results, depending on the values of the two masses and the length L of the homogeneous bar:

$$y = \frac{2(m_1+3m_2)}{3(m_1+2m_2)}L \qquad (25)$$

The notation $r = m_2/m_1$ is made and then the ratio y/L can be expressed in the form:

$$\frac{y}{L} = \frac{2(1+3r)}{3(1+2r)} \qquad (26)$$

From the graphic representation (Fig. 4) of the function corresponding to relation (26) the collision center moves towards the end of the bar when the ratio m_2/m_1 increases but will not coincide with the fixing point of the mass m_2 at the end of the bar. Consequently, since the collision does not occur in the collision center and percussions will also occur in the bearings fixing the pendulum, this case cannot be assimilated to that of the mathematical pendulum when the calculation relation of the coefficient of restitution (18) is rigorously exact.

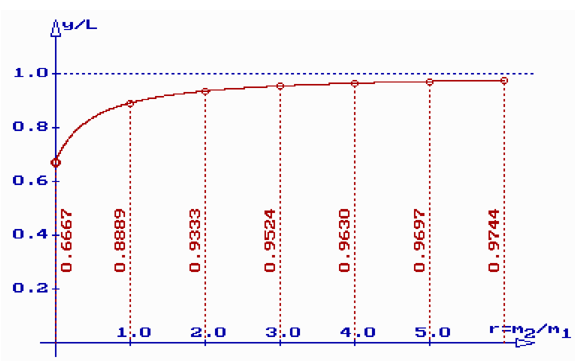

Fig.4. The graphical representation of the function y/L

In this case, the definition of the restitution coefficient as a ratio of two percussions is not correct.

If the distance ℓ at which the mass m_2 is placed on the homogeneous bar of length L and the mass m_1 is known (Fig.5), the differential equation of the rotational motion of the physical pendulum is as follows:

$$\ddot{\phi} + \frac{3(m_1L+2m_2\ell)g}{2(m_1L^2+3m_2\ell^2)}\sin\phi = 0 \qquad (27)$$

and in the case of a mathematical pendulum with length y the equation of motion is: $\ddot{\phi} + \frac{g}{y}\phi = 0$.

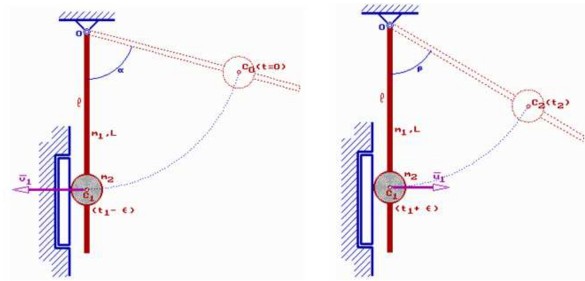

Fig.5. The physical pendulum

After identifying the two equations, we can write:

$$y = \frac{2(m_1L^2+3m_2\ell^2)}{3(m_1L+2m_2\ell)} => \frac{y}{L} = \frac{2\left(1+3r\frac{\ell}{L}\right)}{3\left(1+2r\frac{\ell}{L}\right)} \qquad (28)$$

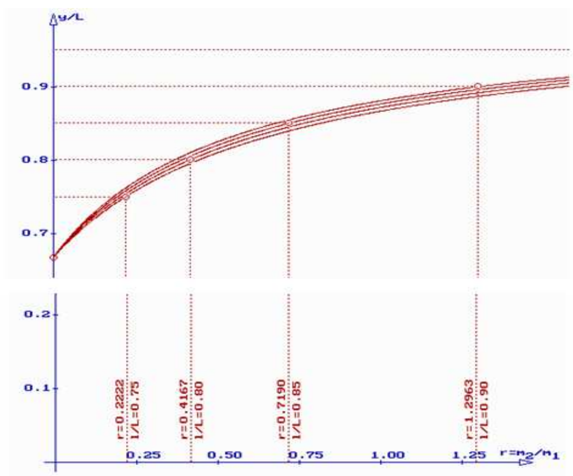

Fig.6. The graphical representation of the function y/L when the length ℓ is known

The y/L function (Fig.6) is represented graphically for various values of the ℓ/L ratio (0.75, 0.80, 0.85 and 0.90) for values of the parameter r = m_2/m_1 between 0 and 1.25.

The problem is to determine if, in the case of the respective curves, there are values of the ratios r for which the collision center coincides with the point where the mass m_2 is fixed, when y = ℓ.

The condition is imposed: $\frac{\ell}{L} = \frac{2\left(1+3r\frac{\ell}{L}\right)}{3\left(1+2r\frac{\ell}{L}\right)}$ and results in a relationship between ℓ/L and r in which r is expressed as a function of ℓ/L:

$$r = \frac{3\frac{\ell}{L}-2}{6\frac{\ell}{L}\left(1-\frac{\ell}{L}\right)} \qquad (29)$$

if the fixing point of the mass m_2 on the homogeneous bar is at a distance ℓ from the axis of rotation (the ratio ℓ/L has a known value) then the ratio r will have a specified value so that the collision center is at the point of fixing the mass m_2, as a result.

The graphic representation of the function $r = r(\ell/L)$ is given in figure 7.

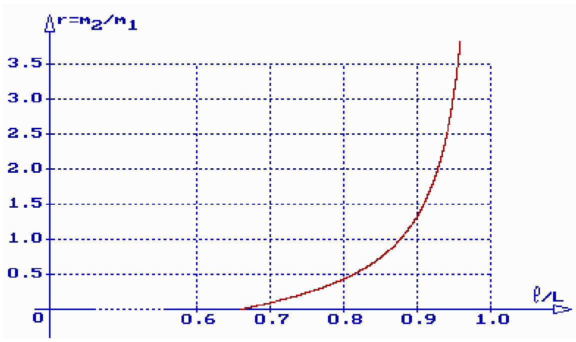

Fig.7. The graphical representation of the function r

In figure 6 (the case when ℓ is unknown), some values are specified that meet the above conditions: for example, in the case of the third curve (numbered from bottom to top) which corresponds to the case when the mass is fixed at the distance $\ell = 0.85\cdot L$ results that the ratio $r = m_2/m_1$ must have the value 0.7190.

In this case, in case of collision with an obstacle at a distance $\ell=0.85\cdot L$ from the axis of rotation, no impulsive reactions occur in its bearings. In this case, the collision of the physical pendulum with an obstacle can be assimilated with the collision of a mathematical pendulum with an obstacle. The problem has a solution only if $r > 0$, so when $\ell/L > 2/3$.

Table 1. The numerical values corresponding to the figure 7

No.crt.	ℓ/L	$r = m_2/m_1$
1	0.700	0.079365
2	0.725	0.146290
3	0.750	0.222222
4	0.775	0.310633
5	0.800	0.416667
6	0.825	0.548341
7	0.850	0.718954
8	0.875	0.952381
9	0.900	1.296296
10	0.925	1.861862
11	0.950	2.982456
12	0.975	6.324786

Some numerical values corresponding to the points of the graph drawn in figure 7 are given in the table 1.

For the collision center to be close to the end of the bar, it is necessary that the ratio $r = m_2/m_1$ be very big, which does not happen in the case of the experimental devices.

3. CONCLUSIONS

In this paper is performed a theoretical study to determine the coefficient of restitution by calculating the velocities before and after the impact and taking into account the height from which the body drops and rebounds, the time interval for a drop and rebound respectively the total time of all jumps. The study continues with the analysis of the cases of the mathematically and physically pendulums which collide with a vertical surface.

The first case (a) represents a typical case of calculating the coefficient of restitution according to the formula introduced by Newton in 1687 [1], [2]. The case b) has a disadvantage, namely determining the total time it takes to drop and then rebound the ball to the maximum height position cannot be done with sufficient precision. Another variant, much more accurate (the case c), from this point of view of time measurement, is the one in which the initial data is the height from which the body drops and the total time it takes for the few jumps on the horizontal plane, up to the final stop. But this case also presents two disadvantages: the time measurement may not be sufficiently accurate and the deduction of the calculation formula for the coefficient of restitution, k, was made under the assumption that the number of jumps is infinite, but after a limited number of jumps the movement stops.

The diagrams shown in figure 2 can be easily used to determine the coefficient of restitution. In an experiment, a ball is dropped from an initial height h and the total time t_{total} is measured. Place a point within the figure that has the coordinates $[\sqrt{h}; t_{total}]$ and depending on the position of the point in relation to the drawn lines, you can decide what is the value of the restitution coefficient.

In the case of the mathematical pendulum (the second case) the coefficient of restitution was calculated using the ratio between the two speeds, before and after hitting the vertical surface. The determination of the coefficient of restitution, k, depends on the energy losses during the collision.

In the case of a collision of a physical pendulum with an obstacle (the third case), the determination of the coefficient of restitution is possible only in the case of the specified conditions, respecting the data corresponding to the diagram drawn in figure 7.

This study can be continued for different bodies of various shapes and materials as well as for different environments and impact surfaces.

4. REFERENCES

[1] Aguiar, C. E., Laudares, F. *Listening to the coefficient of restitution and the gravitational acceleration of a bouncing ball*. Am. J. Phys. 71 (5) May 2003. pp. 499-501.

[2] Bernstein, A. D., *Listening to the coefficient of restitution*, American Journal of Physics, 1977, Vol. 45, pp. 41-44

[3] Cross, R., *The bounce of the ball*, American Journal of Physics, 1999, Vol. 67, pp. 222-227

[4] Farkas, N., Ramsier, R. D., *Measurement of coefficient of restitution made easy*, Physics Education, 2006, 41(1), pp. 73-75

[5] Harris, C.M., Piersol, A.G. (ed.), Shock and Vibration Handbook, 5th Edition, McGRAW-HILL, New York, 2002, pp. 1456

[6] Haron, A., Ismail, K. A., *Coefficient of restitution of sports balls: A normal drop test*, First International Conference on Mechanical Engineering Research, 2011, pp. 8

[7] Popescu, P. D., *Mechanics. Dynamics (in Romanian)*, Institutului Politehnic, Cluj-Napoca, 1981, pp. 274

[8] Rădoi, M., Deciu, E., *Mechanics (in Romanian)*, Editura Didactică şi Pedagogică, Bucureşti, 1977, pp. 650

[9] Stensgaardand, I., Lægsgaard, E., *Listening to the coefficient of restitution—revisited*, American Journal of Physics, 2001, Vol. 69, pp. 301–305

[10] Ursu-Fischer, N, *Mechanical Vibrations, Theory and Applications (in Romanian),* Editura Casa Cărţii de Ştiinţă, Cluj-Napoca, 1998

[11] Ursu-Fischer, N., Radu, I., Ursu, M., *Contribution to the study of the mechanical system vertical movements with bounces,* Proceedings of the International Symposium "Research and Education in an Innovation Era", 2nd Edition, "Aurel Vlaicu" University of Arad, 2008, Engineering Sciences, Ed. Universităţii "Aurel Vlaicu" Arad, pp. 333-343

[12] Vâlcovici, V., Bălan, St., Voinea, R., *Theoretical Mechanics (in Romanian),* ed. II, Editura Tehnică, Bucureşti, 1963, pp.1007

Determinarea și analizarea coeficientului de restituire în cazul unor sisteme mecanice

Scopul acestei lucrări este determinarea și analizarea coeficientului de restituire în cazul unor sisteme mecanice considerate. Lucrarea analizează trei cazuri de determinare a coeficientului de restituire, și anume: cazul bilei elastice care este lăsată să cadă liber pe o suprafață orizontală de la o înălțime considerată, cazul pendulului matematic care se ciocnește de un obstacol vertical și cazul pendulului fizic format dintr-o bară omogenă și o masă care de asemenea se ciocnește de o suprafață verticală.

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