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THE KINEMATIC MODEL OF 3R ROBOT BASED ON FUNCTION EXPONENTIALS

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Abstract: *The purpose of this paper is to establish the equations of kinematic model for an articulated industrial robot, by symbolic calculus.The geometry and direct kinematics on the RRR-robot (with three degrees of freedom of rotation), in the nominal configuration was calculated in a previous paper. To ensure the operation of the robot, mathematical modeling is required. The exponential matrix appears in solving linear systems of differential equations. For this purpose, the matrix localization algorithm was applied to determine the direct geometry equations. To calculate the velocities and accelerations relative to the fixed system {0}, the transfer matrix algorithm was used. The results are useful to establish the equations of motion trajectory.*

Key words: *robot, function exponentials, algorithm, Jacobian matrix*

1. INTRODUCTION

A robot's arms are described by degrees of freedom. The degrees of freedom (DoF) refer to the number of basic modes in which a rigid object can move through 3D space. In this case, it is about three degrees that correspond to the rotational movement around the x, y and z axes.

To calculate the kinematics equations, the mechanical structure of a robot with n degrees of freedom is subjected to analysis.

The kinematic elements of the robot are linked by means of driving joints. The robot elements are treated as rigid bodies and the driving joints are considered mechanically perfect. In this case the static hypothesis is removed, and the column vectors of the generalized and operational coordinates become function of time. The challenge is to find a system notation as convenient as possible. In the scientific literature can be found a multitude of options to select from.

Modern vector algebra allows to describe a manipulator's kinematics by using an intrinsic formulation independent of the reference frames choice [1]. Most papers in the field approach this problem by first affixing coordinate frames to each of the links, then defining the way these frames relate to each other as one proceeds along the arm, and finally concatenating these link transforms to determine the global transformation from the base of the robot to the end-effector. In applications, the frames can be established by following the rules suggested by Denavit and Hartenberg [2], but the variant proposed by Craig [3] and Wolovich [4] seems to be very popular as well. Denavit-Hartenberg frame convention is not very useful when studying the positional accuracy of arms having nominally parallel joint axes [5,6].

In this case, the Denavit-Hartenberg convention has been adopted, and therefore each link transformation consists of a pair of consecutive screw transformations.

To represent these link transformations the well-known (4x4) homogenous transformation matrices [3,4,7,8], can be applied, although other variants are also viable.

The dual-number quaternions can be applied. These can serve as screw operators and are usually applied to the study of four-link mechanisms [8-10].

This study implies the kinematic analysis of a three degrees of freedom robot structure. The input data in the algorithm of kinematic modeling of the mechanical structure can be visualized in *Table 1*.

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1. GENERAL ASPECTS REGARDING MATRIX EXPONENTIALS ALGORITHM

When considering the mechanical structure of a robot (MSR) consisting of rotation (R) and/or translation (T) joints, the transfer matrix equations can also be expressed by using of matrix exponentials (ME). In this part are presented the algorithms that can be used for determining the forward geometry and kinematics equations based on matrix exponentials in functional analysis, according to [11-17] and [18].

These algorithms are essential in establishing the homogeneous transformations in DGM, and of the Jacobian matrix, and direct kinematics equations, based on function exponentials.

2.1 The Algorithm of Matrix Exponentials in the Direct Geometry

The matrix exponentials and their associated transformations are included in the algorithm of matrix exponentials from the direct geometric modeling, according to [11].

In the following, a brief presentation of the main steps that are followed in applying the algorithm of matrix exponentials, is performed.

First, the mechanical robot structure that is to be analyzed is defined by means of a matrix called the matrix of nominal geometry. The matrix of nominal geometry $M_{vn}^{(0)}$, defining the initial configuration $\overline{\theta}^{(0)}$ of the robot and is known:

$$M_{vn}^{(0)} = \underset{[(n+1)\times 6]}{\text{Matrix}}\left\{\left[\overline{p}_i^{(0)T} \quad k_i^{(0)T}\right], i = 1 \to n+1\right\}^T, \quad (1)$$

This matrix is completed with the screw parameters $\left\{\overline{k}_i^{(0)}\ ;\ \overline{v}_i^{(0)}\right\}$, also named the homogenous coordinates. According to [11], $\overline{k}_i$ and $\overline{v}_i$ are the screw parameters or homogenous coordinates of the driving axis which by generalization is equivalent to:

$$\overline{k}_i = \{\overline{x}_i;\overline{y}_i;\overline{z}_i\},\ \overline{v}_i = \{\overline{p}_i\times\}\cdot\overline{k}_i\cdot\Delta_i + (1-\Delta_i)\cdot\overline{k}_i, \quad (2)$$

The differential matrix A_i has the same expression for both configurations $\overline{\theta}^{(0)}$ and $\overline{\theta}$. Considering [2] and [4], this matrix is defined:

$$A_i = \left[\begin{array}{c|c} \left\{\overline{k}_i^{(0)}\times\right\}\Delta_i & \overline{v}_i^{(0)} \\ \hline 0\ \ 0\ \ 0 & 0 \end{array}\right], \quad (3)$$

where, $\overline{v}_i^{(0)} = \left\{\overline{p}_i^{(0)}\times\right\}\cdot\overline{k}_i^{(0)}\cdot\Delta_i + (1-\Delta_i)\cdot\overline{k}_i^{(0)}$. (4)

The column vector $\overline{b}_i$ is determined with this expression:

$$\begin{aligned} \overline{b}_i = {} & I_3\cdot q_i - \left\{\overline{k}_i^{(0)}\times\right\}\cdot c(q_i\cdot\Delta_i) + \\ & + \left\{\overline{k}_i^{(0)}\times\right\}^2\cdot\left[q_i - s(q_i\cdot\Delta_i)\right]\cdot\overline{v}_i^{(0)} \end{aligned} \quad (5)$$

Another matrix exponential, having a high importance for homogenous transformation, is:

$$\begin{aligned} e^{A_i\cdot q_i} &= \exp\left(\left[\begin{array}{c|c} \left\{\overline{k}_i^{(0)}\times\right\}\Delta_i & \overline{v}_i^{(0)} \\ \hline 0\ \ 0\ \ 0 & 0 \end{array}\right]\cdot q_i\right) = \\ &= \left[\begin{array}{c|c} \exp\left\{\left\{\overline{k}_i^{(0)}\times\right\}q_i\cdot\Delta_i\right\} & \overline{b}_i \\ \hline 0\ \ 0\ \ 0 & 0 \end{array}\right] \end{aligned} \quad (6)$$

and, $\exp\left\{\sum_{j=0}^{i} A_j q_j\right\} = \left[\begin{array}{c|c} \exp\{R\} & \exp\{p\} \\ \hline 0\ \ 0\ \ 0 & 1 \end{array}\right]$. (7)

Where the terms $\exp\{R\}$ and $\exp\{p\}$ have the following value:

$$\exp\{R\} = \prod_{j=o}^{j}\exp\left(\overline{k}_i^{(0)}\times\right)\left\{q_j\cdot\Delta_j\right\}, \quad (8)$$

$$\exp\{p\} = \sum_{j=0}^{i}\left\{\prod_{k=o}^{i}\exp\left\{\left(\overline{k}_k^{(0)}\times\right)\cdot q_k\cdot\Delta_k\right\}\right\} + \overline{b}_{j+1}. \quad (9)$$

In direct geometry, the matrix exponentials algorithm contains in an external loop of iterations from $(i = 1 \to n)$. Considering the initial conditions is established T_{x0}, according to mechanical structure. The obtained results are included in the resulting of rotating matrix of frame $\{n\}$ beside $\{0\}$ frame.

The exponentials expressions for the locating matrices, which define the position and *orientation* of the $\{n\}$ and $\{n+1\}$ relative to fixed frame $\{0\}$, are characterized by:

$$T_{x0} = \prod_{i=1}^{x} T_{ii-1} = \left[\begin{array}{ccc|c} & R_{x0} & & \overline{p} \\ \hline 0 & 0 & 0 & 1 \end{array}\right], \quad x = \{n; n+1\}. \quad (10)$$

The expression above can also be written in an exponential form, such as follows:

$$T_{x0} = \prod_{i=1}^{n} \left(e^{A_i \cdot q_i} \right) \cdot T_{x0}^{(0)} = \exp\left\{ \sum_{i=1}^{n} A_i \cdot q_i \right\} \cdot T_{x0}^{(0)}, \quad (11)$$

where, $$R_{x0} = \exp\left\{ \sum_{i=1}^{n} \left\{ \overline{k}_i^{(0)} \times \right\} \cdot q_i \cdot \Delta_i \right\} \cdot R_{x0}^{(0)}, \quad (12)$$

$$\begin{aligned} \overline{p} = \sum_{i=1}^{n} \left\{ \exp\left\{ \sum_{j=0}^{i-1} \left\{ \overline{k}_i^{(0)} \times \right\} \cdot q_j \cdot \Delta_j \right\} \right\} \cdot \overline{b}_i + \\ + \exp\left\{ \sum_{i=1}^{n} \cdot \left\{ \overline{k}_i^{(0)} \times \right\} \cdot q_i \cdot \Delta_i \right\} \cdot \overline{p}^{(0)} \cdot \delta_x \end{aligned} \quad (13)$$

The terms R_{x0} and $\overline{p}$ are the components of the location matrix (homogeneous transformation) between the systems in the kinematic chain of the mechanical robot structure.
R_{x0} is the rotation matrix and $\overline{p}$ is the position vector of characteristic point, and δ_x is a matrix operator, defined as $\delta_x = \{\{0;\ x = n\}; \{1;\ x = n+1\}\}$.

In keeping with this, the matrix, corresponding to initial conditions, can be determined without establishing moving frame.

The Matrix Exponentials Algorithm, due to computational advantages and independent of the reference can be applied for any robot. As a conclusion, an important advantage of using of matrix exponential is the lack of reference frames, thus improving the accuracy of calculus.

2.2 The Algorithm of Matrix Exponentials in the Direct Kinematics

Considering the Matrix Exponentials in the Direct Geometry, in this section is presented a new method of determining the Jacobian matrix, also named the transfer matrix of velocities, as well as of its first order time derivative with respect time. According to scientific literature [12-18], the Jacobian matrix consists of two main components: ${}^{0}J_{V}(\overline{\theta})$ and ${}^{0}J_{\Omega}(\overline{\theta})$.

The first one is the linear transfer matrix of velocities and the last one represents the angular transfer matrix of velocities. The matrix transfer equations, characterizing a certain kinematic chain with rotational and translational driving joints can be expressed by means of matrix exponential functions.

In this purpose, the results obtained by applying the Matrix Exponentials Algorithm in the Direct Geometry are used.

Also, the exponential functions from direct kinematics [12,] are called. It notices that for obtaining the Jacobian Matrix based on matrix exponentials, three calculus variants can be approached. In the following are presented the main steps in applying the Algorithm of Matrix Exponentials in the Direct Kinematics (MEK).

When applying the first calculus variant, the matrix exponentials appled directly, resulting the expression for the Jacobian matrix:

$${}^{0}J(\overline{\theta}) = \left\{ \left[{}^{0}J_{iv}(\overline{\theta}) \quad {}^{0}J_{i\Omega}(\overline{\theta}) \right]^{T}, \ \{i = 1 \rightarrow n\} \right\}^{T}, \quad (14)$$

where

$$\begin{aligned} {}^{0}J_{iv}(\overline{\theta}) = \exp\left\{ \sum_{j=0}^{i-1} \left\{ \overline{k}_j^{(0)} \times \right\} q_j \cdot \Delta_j \right\} \cdot \overline{v}_i + \\ + \Delta_i \cdot \exp\left\{ \sum_{j=0}^{i-1} \left\{ \overline{k}_j^{(0)} \times \right\} \cdot q_j \cdot \Delta_j \right\} \cdot \left\{ \overline{k}_i^{(0)} \times \right\} \cdot \\ \cdot \left\{ \sum_{k=i}^{n} \left\{ \exp\left\{ \sum_{m=i-1}^{k-1} \left\{ \overline{k}_m^{(0)} \times \right\} \cdot q_m \cdot \delta_m \right\} \cdot \overline{b}_k \right\} + \\ + \exp\left\{ \sum_{k=i}^{n} \left\{ \overline{k}_k^{(0)} \times \right\} \cdot q_k \cdot \Delta_k \right\} \cdot \overline{p}_n^{(0)} \end{aligned} \quad (15)$$

and

$${}^{0}J_{i\Omega}(\overline{\theta}) = \exp\left\{ \sum_{j=0}^{i-1} \left\{ \overline{k}_j^{(0)} \times \right\} q_j \cdot \Delta_j \right\} \cdot \overline{k}_i^{(0)} \cdot \Delta_i. \quad (16)$$

For the second and the third variant calculating the Jacobian matrix, the expressions are applied:

$$\underset{(3\times3)}{ME}\left(V_{i1}\right)=\exp\left\{\sum_{j=0}^{i-1}\left\{\bar{k}_j^{(0)}\times\right\}q_j\cdot\Delta_j\right\}, \quad (17)$$

$$\underset{(3\times6)}{ME}\left(V_{i2}\right)=\left[\begin{array}{cc} I_3 & \Delta_i\cdot\left\{\bar{k}_i^{(0)}\times\right\}\end{array}\right], \quad (18)$$

$$\underset{\left\{6x\left[9+3\cdot(n-i)\right]\right\}}{ME}\left(V_{i3}\right)=\begin{bmatrix} I_3 & [0] & [0] \\ [0] & A_1 & A_2 \end{bmatrix}, \quad (19)$$

where, $$A_1=\begin{bmatrix}\exp\left\{\sum_{m=i-1}^{k-1}\left\{\bar{k}_m^{(0)}\times\right\}q_m\cdot\delta_m\cdot\Delta_m\right\} \\ \text{unde} \quad k=i\to n\end{bmatrix}, \quad (20)$$

$$A_2=\exp\left\{\sum_{k=i}^{n}\left\{\bar{k}_k^{(0)}\times\right\}q_k\cdot\Delta_k\right\},$$

$$\delta_m=\left\{\{0;\ m=i-1\};\ \{1;\ m\geq i\}\right\}. \quad (21)$$

Considering [11] the relations can be written:

$$\underset{(6\times6)}{ME}\{J_{i1}\}=\left[\begin{array}{c|c} ME\{V_{i1}\} & [0] \\ \hline [0] & ME\{V_{i1}\}\end{array}\right], \quad (22)$$

$$\underset{(6\times9)}{ME}\{J_{i2}\}=\left[\begin{array}{c|c} ME\{V_{i2}\} & [0] \\ \hline [0] & I_3\end{array}\right], \quad (23)$$

$$\underset{\left\{9\times\left[12+3\cdot(n-i)\right]\right\}}{ME}\{J_{i3}\}=\left[\begin{array}{c|c} ME\{V_{i3}\} & [0] \\ \hline [0] & I_3\end{array}\right]. \quad (24)$$

The three matrices presented above are included in another matrix of the following form:

$$\underset{\left\{6\times\left[12+3\cdot(n-i)\right]\right\}}{ME}\left\{{}^0J_i\right\}=ME\{J_{i1}\}\cdot ME\{J_{i2}\}\cdot ME\{J_{i3}\}. \quad (25)$$

The column vector is expressed in an exponential form according to:

$$\underset{\left\{\left[12+3\cdot(n-i)\right]\times1\right\}}{M_{iv\omega}}= =\left[\bar{v}_i^{(0)T}\quad\left[\bar{b}_k,\ k=i\to n\right]^T\quad\bar{p}_n^{(0)T}\quad\Delta_i\cdot\bar{k}_i^{(0)T}\right]^T \quad (26)$$

Finally, the Jacobian matrix written in an exponential form, for the first and second calculus variants, is obtained:

$$\underset{(6\times n)}{{}^0J}\left(\bar{\theta}\right)=\left[\underset{(6\times1)}{{}^0J_i},\ \left(i=1\to n\right)\right]=ME\left\{{}^0J_i\right\}\cdot M_{iv\omega}, \quad (27)$$

$$\underset{(6\times n)}{{}^0J}\left(\bar{\theta}\right)=$$

$$\begin{bmatrix} ME\{V_{i1}\} & [0] \\ [0] & ME\{V_{i1}\}\end{bmatrix}\cdot\begin{bmatrix} ME\{V_{i2}\}\cdot ME\{V_{i3}\} & [0] \\ [0] & I_3\end{bmatrix}\cdot M_{iv\omega}$$

The angular velocities and accelerations, in exponential form can be defined as follows:

$${}^0\bar{\omega}_i=\left\{\sum_{j=1}^{i}\left\{exp\left\{\sum_{k=1}^{j-1}\left\{\bar{k}_k^{(0)}\times\right\}q_k\Delta_k\right\}\right\}\bar{k}_j^{(0)}\dot{q}_j\Delta_j\right\} \quad (29)$$

$$\bar{\omega}_1^0=\begin{pmatrix}0\\ \dot{q}_1\\ 0\end{pmatrix};\ \bar{\omega}_2^0=\begin{pmatrix}\dot{q}_2\cos(q_1)\\ \dot{q}_1\\ -\dot{q}_2\sin(q_1)\end{pmatrix};$$

$$\bar{\omega}_3^0=\begin{pmatrix}\dot{q}_2\cos(q_1)+\dot{q}_3\cos(q_2)\ \sin(q_1)\\ \dot{q}_1\cos(q_2)^2+\dot{q}_1\sin(q_2)^2-\dot{q}_3\sin(q_2)\\ \dot{q}_3\cos(q_1)\ \cos(q_2)-\dot{q}_2\sin(q_1)\end{pmatrix}$$

$$\bar{\omega}_4^0=\begin{pmatrix}\dot{q}_2\cos(q_1)+\dot{q}_3\cos(q_2)\ \sin(q_1)\\ \dot{q}_1-\dot{q}_3\sin(q_2)\\ \dot{q}_3\cos(q_1)\ \cos(q_2)-\dot{q}_2\sin(q_1)\end{pmatrix}$$

Also, the linear velocities and accelerations can be expressed in exponential form according to:

$${}^0\bar{v}_i=\sum_{j=1}^{i}M\ exp\{J_{j1}\}\cdot M\ exp\{J_{j2}\}\cdot M\ exp\{J_{j3}\}\cdot M_{jv}\cdot\dot{q}_j, \quad (30)$$

For the 3R robot, the linear velocities are:

$$\bar{v}_1^1=\begin{pmatrix}0\\0\\0\end{pmatrix};\qquad \bar{v}_2^2=\begin{pmatrix}\dot{q}_2\\ -l_1\dot{q}_1\sin(q_2)\\ -l_1\dot{q}_1\cos(q_2)\end{pmatrix};$$

$$\bar{v}_3^3=\begin{pmatrix}\cos(q_3)\left(\dot{q}_2+l_2\dot{q}_1\sin(q_2)\right)-l_1\dot{q}_1\sin(q_2)\ \sin(q_3)\\ -\sin(q_3)\left(\dot{q}_2+l_2\dot{q}_1\sin(q_2)\right)-l_1\dot{q}_1\cos(q_3)\ \sin(q_2)\\ l_2\dot{q}_2-l_1\dot{q}_1\cos(q_2)\end{pmatrix} \quad (31)$$

$$\bar{v}_4^4=\begin{pmatrix}\sin(q_3)\,\sigma_1+l_3(\dot{q}_2\cos(q_3)+\dot{q}_1\cos(q_2)\ \sin(q_3))\\ +l_1\dot{q}_1\cos(q_3)\ \sin(q_2)\\ \cos(q_3)\,\sigma_1-l_3(\dot{q}_2\sin(q_3)-\dot{q}_1\cos(q_2)\ \cos(q_3))\\ -l_1\dot{q}_1\sin(q_2)\ \sin(q_3)\\ l_2\dot{q}_2-l_1\dot{q}_1\cos(q_2)\end{pmatrix}$$

$$A_3 = \begin{bmatrix} M\,exp\{\dot{J}_{j1}\} \cdot M\,exp\{J_{j2}\} \cdot M\,exp\{J_{j3}\} \\ M\,exp\{J_{j1}\} \cdot M\,exp\{J_{j2}\} \cdot M\,exp\{\dot{J}_{j3}\} \\ M\,exp\{J_{j1}\} \cdot M\,exp\{J_{j2}\} \cdot M\,exp\{J_{j3}\} \end{bmatrix}. \quad (32)$$

The matrix exponential algorithm (MEG and MEK) presents several advantages given the compact form, the simple geometric visualization of the exponential functions and the fact that their determination does not depend on the reference systems of each kinetic element. Although their application is difficult, matrix exponentials have an essential role, due to the advantages mentioned above, in the analysis of geometry and direct kinematics, geometric, kinematic and dynamic control functions, as well as kinematic and dynamic precision.

The formalism based on matrix exponentials leads to the evaluation of the performance for the robot mechanical structure under study, regardless of its complexity level.

3. THE MODELING OF A 3R ROBOT USING MATRIX EXPONENTIALS

3.1 The Equations of Direct Geometry of a 3R Robot using Matrix Exponentials

In this section, the Algorithm of Matrix Exponentials in direct geometry and kinematics will be applied. First, the robot mechanical structure that is to be analyzed is defined by means of the matrix of nominal geometry which is filled up with the screw parameters, as shown in *Table 1*.

Table 1

The matrix of nominal geometry

i	$\{R;T\}$	$\bar{k}_i^{(0)T}$			$\bar{p}_i^{(0)T}$			$\bar{v}_i^T$		
1	R	0	1	0	0	0	l_0	$-l_0$	0	0
2	R	1	0	0	l_1	0	l_0	0	l_0	0
3	R	0	0	1	l_1	l_2	l_0	l_2	$-l_1$	0
4	-	1	0	0	l_1	l_2	$l_2 +$	-	-	-

In the first step, corresponding to the first kinetic link of the robot, the following expressions for the matrices and matrix exponential functions are determined:

$$A_1 = \begin{bmatrix} \{\bar{k}_1^{(0)} \times\} & \bar{v}_1^{(0)} \\ 0\;0\;0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & -l_0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad (33)$$

$$e^{\{\bar{k}_1^{(0)} \times\} q_1} = \begin{bmatrix} \cos q_1 & 0 & \sin q_1 \\ 0 & 1 & 0 \\ -\sin q_1 & 0 & \cos q_1 \end{bmatrix},$$

$$\bar{b}_1 = \begin{bmatrix} -l_0 \cdot \sin q_1 \\ 0 \\ -l_0 \cdot \cos q_1 + l_0 \end{bmatrix}, \quad (28)$$

$$e^{A_1 q_1} = \begin{bmatrix} \exp\{\{\bar{k}_1^{(0)} \times\} q_1 \cdot \Delta_1\} & \bar{b}_1 \\ 0 \quad 0 \quad 0 & 1 \end{bmatrix} =$$

$$= \left[\begin{array}{ccc:c} \cos q_1 & 0 & sq_1 & -l_0 \cdot \sin q_1 \\ 0 & 1 & 0 & 0 \\ -\sin q_1 & 0 & cq_1 & -l_0 \cdot \cos q_1 + l_0 \\ \hdashline 0 & 0 & 0 & 1 \end{array}\right] \quad (29)$$

For the second kinetic link the matrix exponential functions are also calculated.

$$A_2 = \begin{bmatrix} \{\bar{k}_2^{(0)} \times\} & \bar{v}_2^{(0)} \\ 0\;0\;0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & l_0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad (30)$$

$$e^{\{\bar{k}_2^{(0)} \times\} q_2} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos q_2 & -\sin q_2 \\ 0 & \sin q_2 & \cos q_2 \end{bmatrix},$$

$$\bar{b}_2 = \begin{bmatrix} 0 \\ l_0 \cdot sq_2 \\ l_0 - l_0 \cdot cq_2 \end{bmatrix}, \quad (31)$$

$$e^{A_2 q_2} = \begin{bmatrix} \exp\left\{\left\{\bar{k}_2^{(0)} \times\right\} q_2 \cdot \Delta_2\right\} & \bar{b}_2 \\ 0 \quad 0 \quad 0 & 1 \end{bmatrix} =$$

$$= \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & \cos q_2 & -\sin q_2 & l_0 \cdot \sin q_2 \\ 0 & \sin q_2 & \cos q_2 & l_0 - l_0 \cdot \cos q_2 \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \quad (32)$$

The third driving joint is also analyzed using matrix exponentials. The results for 3R robot are:

$$A_3 = \begin{bmatrix} \left\{\bar{k}_3^{(0)} \times\right\} & \bar{v}_3^{(0)} \\ 0\,0\,0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 & l_2 \\ 1 & 0 & 0 & -l_1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad (33)$$

$$e^{\left\{\bar{k}_3^{(0)} \times\right\} q_3} = \begin{bmatrix} \cos q_3 & -\sin q_3 & 0 \\ \sin q_3 & \cos q_3 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad (34)$$

$$\bar{b}_3 = \begin{bmatrix} l_2 \cdot \sin q_3 + l_1 \cdot (1 - \cos q_3) \\ -l_1 \cdot \sin q_3 + l_2 \cdot (1 - \cos q_3) \\ 0 \end{bmatrix}. \quad (35)$$

$$e^{A_3 q_3} = \begin{bmatrix} \exp\left\{\left\{\bar{k}_3^{(0)} \times\right\} q_3 \cdot \Delta_3\right\} & \bar{b}_3 \\ 0 \quad 0 \quad 0 & 1 \end{bmatrix} =$$

$$= \left[\begin{array}{ccc|c} \cos q_3 & -\sin q_3 & 0 & l_2 \cdot \sin q_3 + l_1 \cdot (1 - \cos q_3) \\ \sin q_3 & \cos q_3 & 0 & -l_1 \cdot \sin q_3 + l_2 \cdot (1 - \cos q_3) \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \quad (36)$$

According to [11] the homogenous transformation matrix between the mobile systems $\{0\} \rightarrow \{4\}$ are determined using matrix exponentials, as follows:

$${}^0_4[T] = \left\{\prod_{i=1}^{3} \exp(A_i \cdot q_i)\right\} \cdot T_{40}^{(0)} =$$

$$= \left[\begin{array}{ccc|c} \begin{matrix}\cos q_3 \sin q_1 \sin q_2 - \\ -\cos q_1 \sin q_3\end{matrix} & \begin{matrix}-\sin q_1 \sin q_2 \sin q_3 - \\ -\cos q_1 \cos q_3\end{matrix} & \cos q_2 \sin q_1 & \\ \cos q_2 \cos q_3 & -\cos q_2 \sin q_3 & -\sin q_2 & \bar{p}_4 \\ \begin{matrix}\cos q_1 \sin q_2 \cos q_3 + \\ +\sin q_1 \sin q_3\end{matrix} & \begin{matrix}-\cos q_1 \sin q_2 \sin q_3 + \\ +\sin q_1 \cos q_3\end{matrix} & \cos q_1 \cos q_2 & \\ \hline 0 & 0 & 0 & 1 \end{array}\right], \quad (37)$$

where

$$\bar{p}_4 = \begin{bmatrix} l_1 \cos q_1 + l_2 \sin q_1 \sin q_2 + l_3 \sin q_1 \cos q_2 \\ l_2 \cos q_2 - l_3 \sin q_2 \\ l_0 - l_1 \sin q_1 + l_2 \cos q_1 \sin q_2 + l_3 \cos q_1 \cos q_2 \end{bmatrix}. \quad (38)$$

The expressions presented above (42) and (43) represent the resultant orientation matrix and the position vector, both included in the resultant locating matrix. These matrices are esential in determining of the column vector of operational variables also known as the equations of the direct geometry.

3.2 The Equations of Direct Kinematics of a 3R Robot using Matrix Exponentials

For the same robot structure, defined in its initial configuration by means of the matrix of nominal geometry, presented in *Table 1*, the direct kinematics equations are defined. For this, the Algorithm of Matrix Exponentials in Direct Kinematics is applied. First, the Jacobian matrix is defined using the second or the third calculus variant [11] and for this, for $i = 1 \rightarrow 3$, the following matrix exponentials are defined:

$$\underset{(3\times3)}{ME}(V_{11}) = \exp\left\{\sum_{j=0}^{1-1}\left\{\bar{k}_j^{(0)} \times\right\} q_j \cdot \Delta_j\right\} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad (39)$$

$$\underset{(6\times6)}{ME}\{J_{11}\} = \left\{\left[\begin{array}{c|c} ME\{V_{11}\} & [0] \\ \hline [0] & ME\{V_{11}\} \end{array}\right] = \left[\begin{array}{c|c} I_3 & [0] \\ \hline [0] & I_3 \end{array}\right]\right\}, \quad (40)$$

$$\underset{(3\times6)}{ME}(V_{12}) = \begin{bmatrix} I_3 & \Delta_1 \cdot \left\{\bar{k}_1^{(0)} \times\right\} \end{bmatrix} =$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \end{bmatrix} \quad (41)$$

$$\underset{(6\times9)}{ME}\{J_{12}\} = \left[\begin{array}{c|c} ME\{V_{12}\} & [0] \\ \hline [0] & I_3 \end{array}\right] =$$

$$= \left[\begin{array}{cccccc|c} 1 & 0 & 0 & 0 & 0 & 1 & \\ 0 & 1 & 0 & 0 & 0 & 0 & [0] \\ 0 & 0 & 1 & -1 & 0 & 0 & \\ \hline & & & [0] & & & I_3 \end{array}\right] \quad (42)$$

$$\underset{(6x15)}{ME}\left(V_{13}\right) =$$

$$\left[\begin{array}{c:c:c} I_3 & [0]_{9\times3} & [0]_{3\times3} \\ \hdashline [0]_{3\times3} & e^{\sum\limits_{m=1-1}^{k-1}\left\{\bar{k}_m^{(0)}\times\right\} q_m\cdot\delta_m\cdot\Delta_m} & e^{\sum\limits_{k=1}^{3}\left\{\bar{k}_k^{(0)}\times\right\} q_k\cdot\Delta_k} \end{array}\right]. \quad (43)$$

<u>where,</u>

$$k = 1 \to 3, \quad \delta_m = \{\{0;\ m = i-1\};\ \{1;\ m \geq i\}\}$$

$$e^{\sum\limits_{m=1-1}^{k-1}\left\{\bar{k}_m^{(0)}\times\right\} q_m\cdot\delta_m\cdot\Delta_m} =$$

$$= \left[\begin{array}{ccc:ccc:ccc} 1 & 0 & 0 & \cos q_1 & 0 & \sin q_1 & \cos q_1 & 0 & \sin q_1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -\sin q_1 & 0 & \cos q_1 & -\sin q_1 & 0 & \cos q_1 \end{array}\right], \quad (49)$$

<u>and</u>

$$e^{\sum\limits_{k=1}^{3}\left\{\bar{k}_k^{(0)}\times\right\} q_k\cdot\Delta_k} =$$

$$= \left[\begin{array}{c:c:c} \begin{array}{c} \cos q_3 \sin q_1 \sin q_2 - \cos q_1 \sin q_3 \\ -\sin q_1 \sin q_2 \sin q_3 - \cos q_1 \cos q_3 \\ \cos q_2 \sin q_1 \end{array} & & \\ \hdashline \begin{array}{c} \cos q_2 \cos q_3 \\ -\cos q_2 \sin q_3 \\ -\sin q_2 \end{array} & & \\ \hdashline \begin{array}{c} \cos q_1 \sin q_2 \cos q_3 + \sin q_1 \sin q_3 \\ -\cos q_1 \sin q_2 \sin q_3 + \sin q_1 \cos q_3 \\ \cos q_1 \cos q_2 \end{array} & & \end{array}\right], \quad (44)$$

$$\underset{(9\times18)}{ME}\{J_{13}\} = \left[\begin{array}{c:c} ME\{V_{13}\} & [0] \\ \hdashline [0] & I_3 \end{array}\right], \quad (45)$$

$$\underset{(6\times18)}{ME}\left\{{}^0J_1\right\} = ME\{J_{11}\}\cdot ME\{J_{12}\}\cdot ME\{J_{13}\}. \quad (46)$$

$$\underset{\{18\times1\}}{M_{1v\omega}} = \left[\ \bar{v}_1^{(0)T} \quad \left[\bar{b}_k\ ;\ k = 1 \to 3\right]^T \quad \bar{p}_3^{(0)T} \quad \Delta_1\cdot\bar{k}_1^{(0)T}\ \right]$$

By performing the matrix product between the two matrix functions, finally the first column of the Jacobian matrix corresponding to the 3R robot structure is obtained.

$$\underset{(6\times1)}{{}^0J_1} = ME\left\{{}^0J_1\right\}\cdot M_{1v\omega} = \begin{bmatrix} l_2\cdot\cos q_1\cdot\sin q_2 - l_1\cdot\sin q_1 \\ 0 \\ -l_2\cdot\sin q_1\cdot\sin q_2 - l_1\cdot\cos q_1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad (47)$$

Applying the same steps for the rest of the driving joints, the following results are obtained:

$$\underset{(6\times1)}{{}^0J_2} = ME\left\{{}^0J_2\right\}\cdot M_{2v\omega} = \quad , \quad (48)$$
$$= \left[l_2\cdot\cos q_2\cdot\sin q_1 \quad l_2\cdot\sin q_2 \quad l_2\cdot\cos q_1\cdot\cos q_2 \quad \cos q_1 \quad 0 \quad -\sin q_1\right]^T$$

$$\underset{(6\times1)}{{}^0J_3} = ME\left\{{}^0J_3\right\}\cdot M_{3v\omega} = \quad (49)$$
$$\left[0 \quad 0 \quad 0 \quad \sin q_1\cdot\cos q_2 \quad -\sin q_2 \quad \cos q_1\cdot\cos q_2\right]^T$$

The importance of Jacobian matrix is to determine the direct kinematic equations (absolute linear and angular velocities).

4. CONCLUSION

The present paper, aims for defining the Jacobian matrix corresponding to a three degrees of freedom robot structure, by using the matrix exponentials. Regarding matrix exponentials can be highlighted some important remarks regarding the number of mathematical operations, which is lower than compared to classical algorithms. Another conclusion revealed by the present study is that the use of screw parameters, defined in the matrix of input data, makes from the use of mobile frames a nonsense, hence, the geometrical errors introduced by the reference systems are highly diminished. The use of matrix exponentials allows a compact representation of the necessary information for defining the direct geometry of a mechanical system with an open or close chain.

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Modelarea cinematică a robotului de tip 3R bazată pe funcții exponențiale

Scopul acestei lucrări este de a dezvolta un studiu asupra robotului 3R, utilizând modelarea cinematică prin aplicarea exponențialelor matriceale. Geometria și cinematica directă pe robotul RRR (cu trei grade de libertate de rotație), în configurația nominală a fost calculată într-o lucrare anterioară. Pentru a asigura funcționarea robotului, este necesară modelarea matematică. Matricea exponenţială apare în rezolvarea sistemelor liniare de ecuaţii diferenţiale. În acest scop, a fost aplicat algoritmul de localizare a matricei pentru a determina ecuațiile de geometrie directă. Pentru a calcula vitezele și accelerațiile în raport cu sistemul fix {0}, a fost utilizat algoritmul matricei de transfer. Rezultatele sunt utile pentru a stabili ecuațiile traiectoriei mișcării.

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