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PARAMETERISING THE GEOMETRY OF CURVED DIFFUSERS AND EXPANSION VANES USING LOGARITHMIC SPIRALS

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Abstract: *Recent research has examined how expanding corners can help to reduce the size, and thereby the cost, of wind and water tunnels. Expanding corners can be constructed using curved diffusers or expansion vanes. This paper examines traditional ways of defining the shape of curved diffusers. It is argued that these methods have shortcomings, such as non-continuity in channel curvature or changing rates of expansion. An iterative algorithm is presented to overcome these issues by defining diffuser shapes using logarithmic spirals. This approach enables precise control over diffuser expansion angle, turning angle as well as curvature. Furthermore, it is shown how the geometry of a logarithmic diffuser can be adapted and integrated into an expansion vane cascade.*
Keywords: *Curved diffusers, expansion vanes, logarithmic spiral, compact wind tunnels.*

1. INTRODUCTION

Despite the rise of computational fluid dynamics, real-world testing has retained its importance in fluid experiments. These are predominantly carried out in wind and water tunnels, which usually, have significant space requirements and construction costs. Drela et al. [1] show that the footprint of these facilities can significantly be reduced through the use of expanding corners. These corners can either be fitted with curved diffusers or expansion turning vanes. Two different types of curved diffuser geometry are commonly mentioned in literature. The first is discussed in depth by Fox and Kline [2] and involves a centre-line arc with protruding spines to create the walls. The second method is described by Chong et al. [3] and is based on an inner and a central arc, which define the outer wall. Both geometries are discussed herein, and their flaws are highlighted. In an attempt to overcome their shortcomings, an iterative way of defining curved diffusers is created. Using logarithmic spirals as a basis, the designer gains precise control over the expansion angle, the turning angle and the curvature distribution of the diffuser.

2. ARC-BASED DIFFUSERS

The seemingly most common way to construct a curved diffuser uses a circular arc as the centreline. Straight lines protrude perpendicularly from the said arc at regular intervals. The length of these lines increases at a constant rate along the length of the curve. Essentially, a regular straight-walled diffuser is mapped onto an arc. This method of constructing curved diffusers focuses on creating a constant expansion. For clarity, Fig. 1 shows a diffuser with 10 construction lines, resulting in clearly segmented walls. Smooth walls can be obtained using splines and an increased number of construction lines.

The turning angle (ω) governs the extent of the curvature, while the growth of the construction lines determines the effective expansion angle ($2\theta_{eff}$) of the diffuser. This notation might seem strange but is based on straight-walled diffuser geometry. In a straight diffuser, θ is often used to represent the expansion angle from the centreline to the side, thus 2θ is the total angle from side to side. It should be noted that $2\theta_{eff}$ used here is merely an approximation of this expansion angle [2]. The expansion ratio (D_r) is calculated using Eq.

(1) while the effective expansion angle ($2\theta_{eff}$) is found using Eq. (2).

$$D_r = \frac{D_2}{D_1} \tag{1}$$

$$2\theta_{eff} = \tan^{-1}\left(\frac{(D_2 - D_1)}{\beta R_2}\right) \tag{2}$$

Here R_2 is the centreline radius, D_1 is the width of the inlet and D_2 is the width of the outlet. By using the effective expansion angle, some findings of straight diffusers can be transferred to their curved counterparts. A detailed comparison of straight and curved diffusers is provided by Fox et al. [2].

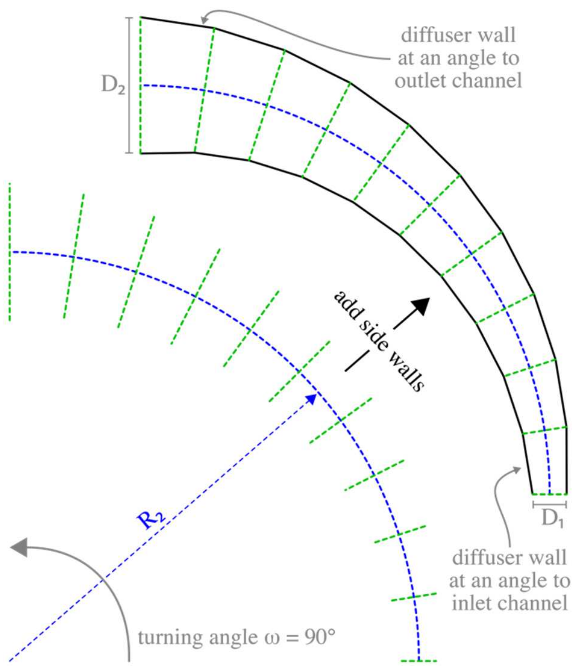


Fig. 1. Construction of a centreline-arc diffuser using a central arc (blue) and spines (green) to draw the outer walls (black).

The second type of diffuser is constructed using two ¼ arcs of different radii, as shown in Fig. 2. The purple inner arc is extended, as shown by the orange line. This ensures that it matches the length of the blue central arc. The green circles are drawn with their centre coincident with the central arc and the sides tangential to the inner arc. A spline is then constructed to be coincident and tangent with the circles to form the outer wall [3]. The expansion ratio equation is the same as previously. The radius of the central arc (R_2), the length of the extension (L_e) and the new effective expansion angle ($2\theta_{eff}$) can be calculated using Eq. (3), Eq. (4) and Eq. (5) respectively.

$$R_2 = \frac{D_2}{2} + R_1 \tag{3}$$

$$L_e = \frac{D_2 - D_1}{2} \tag{4}$$

$$2\theta_{eff} = \tan^{-1}\left(\frac{2(D_2 - D_1)}{\pi R_2}\right) \tag{5}$$

In the experience of the author, neither of these approaches is particularly simple. Especially the double-arc diffuser, with its growing circles and spline curves, take some time to be integrated into engineering design programs.

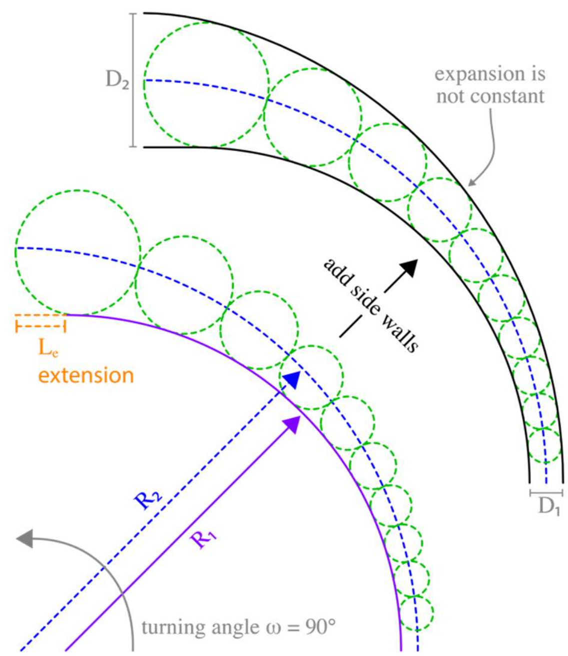


Fig. 2. Construction of a double-arc diffuser using an internal (purple), a central arc (blue) and circles (green) to construct an outer wall (black).

3. LOGARITHMIC SPIRALS

Instead of using arcs and splines, logarithmic spiral equations can be used to define the side walls of a diffuser. In addition to a guaranteed constant rate of expansion, this approach also allows for more precise control of the curvature. The flow devices defined by such spirals will henceforth be referred to as *logarithmic diffusers* and *logarithmic vanes*. As is often the case in engineering, the inspiration for a diffuser based on a logarithmic spiral came from nature. Fig. 3 shows that various natural systems form logarithmic spirals.

Fig. 3. Logarithmic spirals found in nature. Clockwise from top-left: A nautilus shell, [4] a low-pressure system over Iceland, [5] and a section of the Mandelbrot Set [6]

There are three main reasons why this type of spiral lends itself particularly well to defining the shape of curved diffusers:

- Logarithmic spiral functions can be tweaked to yield any desired rate of expansion
- Logarithmic spirals offer a curvature-continuous transition in channel width
- Logarithmic spirals always have a constant expansion angle λ, also known as the *polar tangential angle*, as shown in Fig. 4

This last factor is particularly important as it ensures that the magnitude of rotation along the spiral is equivalent to the change in its slope. In other words, a 90° rotation along the spiral equates to a 90° change in slope.

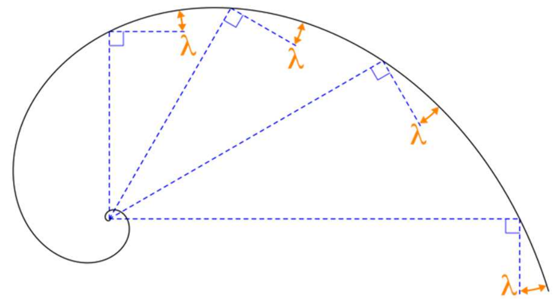


Fig. 4. Logarithmic spiral, highlighting a constant polar tangential angle (λ)

Any logarithmic spiral can be created using Eq. (6), where t is simply a variable and r is the distance from the origin. The constant b is essentially the expansion factor of the spiral, while a is a scaling factor.

$$r = ae^{bt} \tag{6}$$

Rewriting this polar equation in cartesian form results in Eq. (7) and Eq. (8):

$$x = r\cos(t) = ae^{bt}\cos(t) \tag{7}$$

$$y = r\cos(t) = ae^{bt}\sin(\mathrm{t}) \tag{8}$$

Note that the aforementioned polar tangential angle is usually denoted with the letter alpha (α). A deliberate choice was made to use λ instead, as to avoid confusion with the scaling factor (a).

4. LOGARITHMIC DIFFUSERS

The logarithmic spiral equations can be manipulated to define the side walls of a curved diffuser. Merely the coordinates and curve angles at the diffuser's inlet and outlet need to be specified. Assuming a 90° turning angle, a range of spirals is graphed in Fig. 5. Here the inlet and outlet are at respective distances of W and H from the origin. The *stretch* of the spiral can be defined as the ratio of H/W. A stretch of one creates a curve equivalent to a circular arc, where the rate of turning is constant. A lower stretch has a higher rate of turning towards the inlet, while a larger stretch results in a higher rate of turning near the outlet.

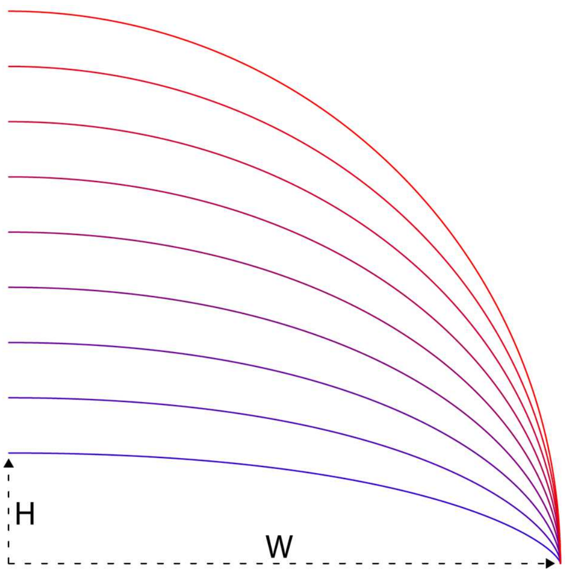


Fig. 5. Range of logarithmic spirals, ranging from a stretch of 0.2 (blue) to 1.0 (red)

This method of defining curved diffusers enables precise control over the turning angle, rate of expansion and curvature. As such, logarithmic diffusers are more versatile than either centreline-arc or double arc diffusers.

Manipulating the spiral equations in the desired manner can be achieved computationally through the use of an algorithm outlined herein. The generalised geometry shown in Fig. 6 is the basis of this methodology.

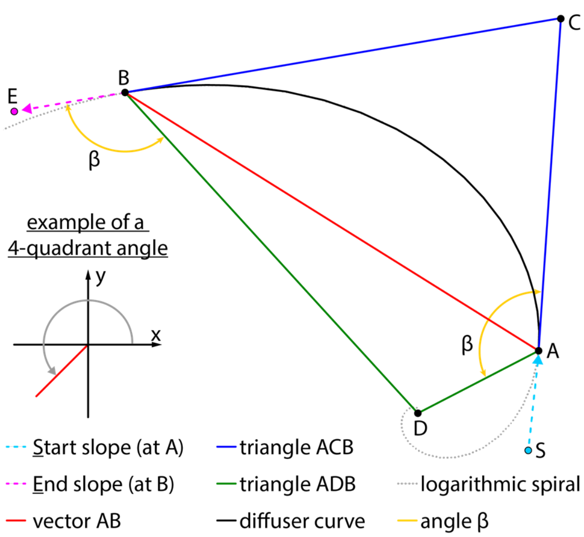


Fig. 6. The geometry used in the construction of a logarithmic diffuser. Also displayed is an example of how to measure a 4-quadrant angle (bottom left).

The various steps of the iterative algorithm are shown below:

A. Define the program parameters (the accuracy and the maximum permissible iterative steps)
B. Define the inlet diffuser angle (cyan arrow), the x- and y-coordinate at the starting point A
C. Define the outlet diffuser angle (magenta arrow), the x- and y-coordinate at the end point B
D. Calculate the 4-quadrant angle and length of vector $\overrightarrow{AB}$ (shown in red)
E. Find point C, which lies at the intersection of extended lines $\overrightarrow{SA}$ and $\overrightarrow{EB}$ (in Fig. 6, these lines are shown in blue and are clearly an extension of the start and end slopes)
F. Calculate the lengths of vectors $\overrightarrow{AC}$ and $\overrightarrow{BC}$, as well as angle $\widehat{ACB}$ (blue)
G. Determine if the spiral is contracting ($\overrightarrow{AC} > \overrightarrow{BC}$), or expanding ($\overrightarrow{AC} < \overrightarrow{BC}$), as is the case in the example
H. Check that the basic geometric requirements are met (see Fig. 8)
I. Calculate the diffuser's turning angle ω (the difference between the start and end angle)

As aforementioned, any line connecting a logarithmic spiral to its origin (Point D), always has the same angle of incidence with said spiral (see Fig. 4). Therefore, angle $\widehat{DAC}$ is the equivalent to angle $\widehat{DBE}$ (labelled β). There is still an infinite number of points which satisfy this criterion. However, there is only one unique point of origin for a logarithmic spiral that will satisfy the other geometric requirements (coordinates, start and end angles, etc.). By inspection, the initial limits of β are easily determined. The origin cannot be within the triangle ACB, thus $\beta > \widehat{BAC}$ and $\beta < \widehat{DBE}$. With these limits in mind, an accurate β can now be found iteratively:

J. Make the initial assumption that β is the average of angle $\widehat{BAC}$ and angle $\widehat{DBE}$
K. Calculate the location of the spiral origin (Point D) based on the initial value of β
L. Determine the values for t at the start and end points (t_1 and t_2 respectively)
M. Calculate values for λ, a, and b, based on the estimated value of t_1
N. Calculate the length of the segment connecting the origin and endpoint t_2

O. Evaluate the following options:
   a. If the length of the segment is within the specified tolerance of vector $\overrightarrow{BD}$, the solution has been found and the calculations have been concluded
   b. Alternatively, if the number of iterations equals the iteration limit, a solution has not been found and the calculations have been concluded
   c. Otherwise, if the segment is <u>larger</u> than vector $\overrightarrow{BD}$, the previous estimation for β was too <u>low</u>. The new <u>lower</u> limit for β should be set to this erroneous estimation. The new value for β is the average of this new lower limit and the previous upper limit. Steps K to O should be repeated.
   d. If the segment is <u>smaller</u> than vector $\overrightarrow{BD}$, the previous estimation for β was too <u>high</u>. The new <u>upper</u> limit for β should be set to this erroneous estimation. The new value for β is the average of this new upper limit and the previous lower limit. Steps K to O should be repeated.

P. Translate the curves and graph the results (assuming the spiral parameters have been determined successfully).

The methodology outlined above is also summarised as a flowchart in Fig. 9. This method should be repeated for the inner, central and outer curves of the desired diffuser. Fig. 10 shows the successful execution of a python script using this algorithm.

Granted that a spiral can be fitted to the given points and angles, there is one final step before it can be graphed. As shown by the dotted lines in Fig. 7, the spiral curves need to be rotated and translated before they resemble a diffuser. These steps are built into the modified parametric equations (9) and (10) below, where once again λ is the polar tangential angle.

$$x = ae^{bt}\cos(t+\lambda) + ae^{bt}\sin(\lambda) \tag{9}$$

$$y = ae^{bt}\cos(t+\lambda) - a\sin(\lambda) \tag{10}$$

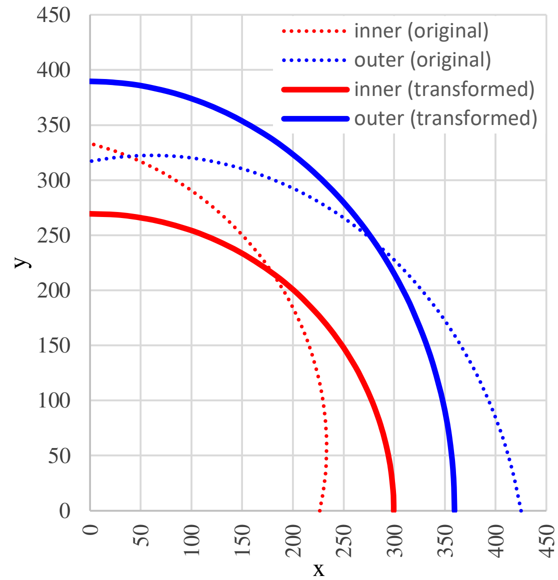


Fig. 7. Plots of inner and outer curves, from 0 to π/2, before and after transformation.

Beyond merely fitting a logarithmic spiral to a set of geometric conditions, the algorithm will also detect various invalid initial conditions. A logarithmic spiral cannot be created if both the start (cyan) and end (magenta) angles are directed towards the same side of vector $\overrightarrow{AB}$, as is the case in part 1 of Fig. 8. Equally, neither start nor end angle can be parallel to vector $\overrightarrow{AB}$, as is the case in parts 2 and 3. A valid orientation for the angles is shown in part 4. The initial check for valid geometry is represented in the box labelled *"check if geometric requirements met"* in the flowchart shown in Fig. 9.

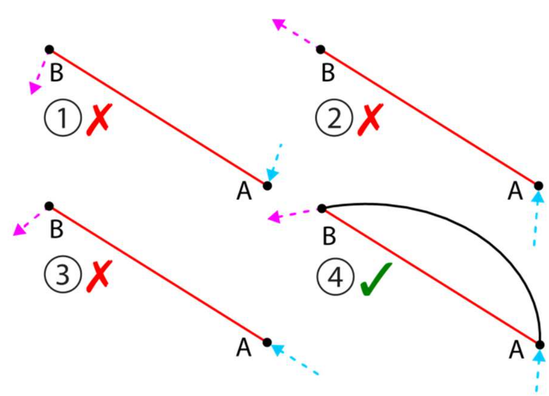


Fig. 8. Various invalid (1 to 3) and one valid (4) geometric condition for constructing curved diffusers

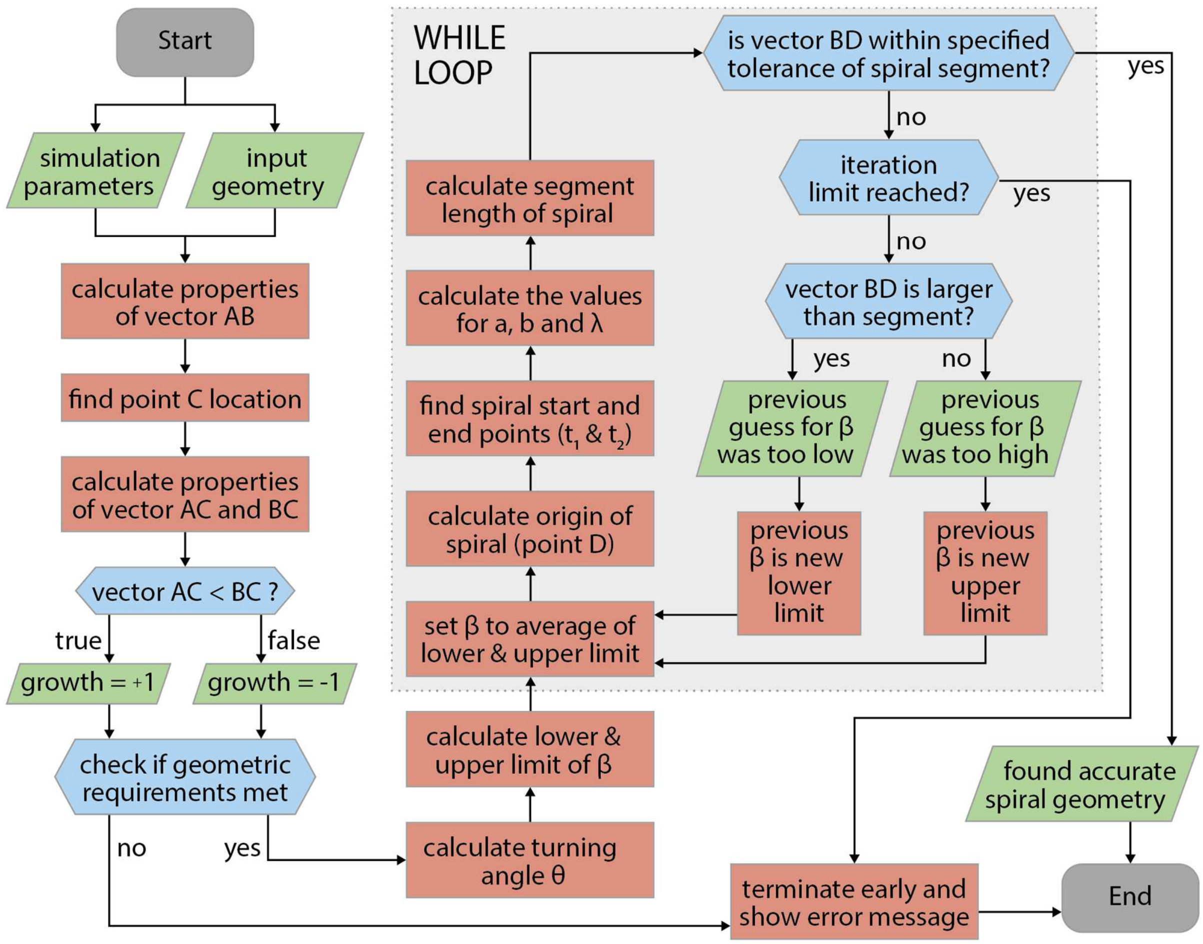
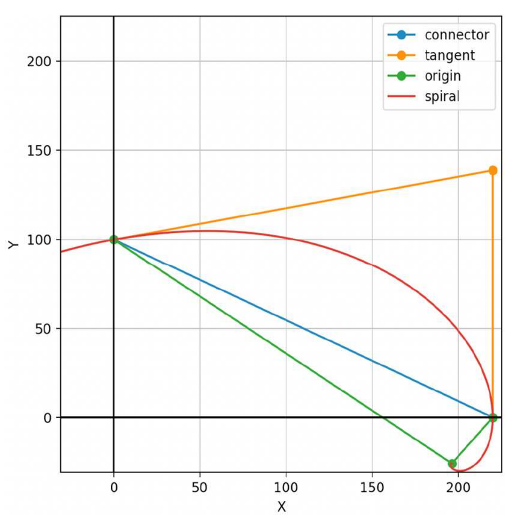


Fig. 9. Flowchart showing the necessary steps to construct an arbitrarily curved diffuser based on a logarithmic spiral. This algorithm has been implemented in a Python script, which produced the console readout shown below.

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Console Readout:

Position of point A = (220, 0)
Position of point B = (0, 100)
vector AC angle = 90
vector BC angle = 190
vector AB angle = 155.556
basic geometric requirements have been met
achieved accurate solution after 31 iterations

horizontal spiral origin       x = 196.231
vertical spiral origin         y = -25.793
polar slope angle          alpha = 0.826
polar slope                    b = 1.085
scaling factor                 a = 14.31
polar angle at point A,        t = 0.826

from t = 0.826 to t = 2.571
x = 14.310 * exp(1.085 * t) * cos(t) + 196.231
y = 14.310 * exp(1.085 * t) * sin(t) - 25.793
```

Fig. 10. Graphical results and console readout after running a Python script using the algorithm outlined herein Geometric parameters of the logarithmic spiral are shown in the readout.

The algorithm outlined herein is capable of generating an endless variety of diffuser shapes and sizes, but it does have some inherent limitations. While it has been sufficient for the scope of this project, future endeavours could benefit from improvements that overcome some of the known limitations. Currently, all initial coordinates should be positive. In other words, they should be located in the first cartesian quadrant. Furthermore, input angles should also be positive. Lastly, the inlet should be smaller than the outlet, i.e., the geometry should be a diffuser rather than a nozzle.

As with previous diffuser types, to accurately classify logarithmic diffusers, the equivalent expansion angle should be determined. Therefore, the length of the logarithmic centreline must be calculated. It is known that the arclength (s), between a point of angle 't' and the origin of a logarithmic spiral, is determined using Eq. (11) [7].

$$s = \frac{ae^{bt}\sqrt{1+b^2}}{b} \tag{11}$$

The length of the central (L_s) arc is simply the arc length from its end to the origin, minus the length from its beginning to the origin:

$$L_s = \frac{ae^{bt_2}\sqrt{1+b^2}}{b} - \frac{ae^{bt_1}\sqrt{1+b^2}}{b} \tag{12}$$

$$L_s = (e^{bt_2} - e^{bt_1}) \cdot \frac{a\sqrt{1+b^2}}{b} \tag{13}$$

If the diffuser in question has a turning angle of 90°, the equation can be simplified further:

$$L_s = \left(e^{b\pi/2} - e^{b0}\right) \cdot \frac{a\sqrt{1+b^2}}{b} \tag{14}$$

$$L_s = \left(e^{b\pi/2} - 1\right) \cdot \frac{a\sqrt{1+b^2}}{b} \tag{15}$$

If the inlet width D_1, the outlet width D_2 and the length of the central curve L_s are known, the equivalent expansion angle can be determined:

$$2\theta_{eff} = \tan^{-1}\left(\frac{D_2 - D_1}{L_s}\right) \tag{16}$$

Substituting Eq. (15) for L_s results in the following equation:

$$2\theta_{eff} = \tan^{-1}\left(\frac{D_2 - D_1}{e^{b\pi/2} - 1} \cdot \frac{b}{a\sqrt{1+b^2}}\right) \tag{17}$$

5. LOGARITHMIC EXPANSION VANES

Once a suitable diffuser shape has been computed, it can easily be used to create a cascade of expansion vanes, as shown in Fig. 11. The outer diffuser side (blue) is used to define the pressure side of each vane. The inside of the diffuser (red) is extended with straight sections (green) to form the suction side of each vane. The vane thickness should be taken into consideration when choosing the inlet and outlet width. The thickness is likely to be dictated by the material the vanes will be made of.

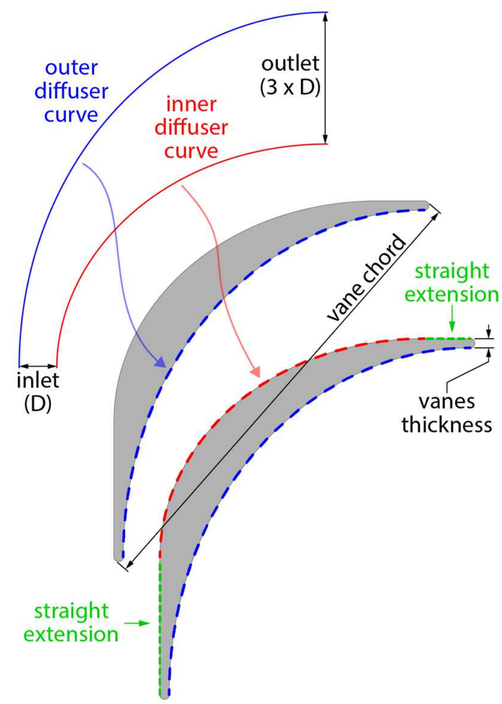

Fig. 11. Constructing a cascade of logarithmic expansion vanes using a logarithmic diffuser as a template

6. COMPARING DIFFUSER TYPES

As stated previously, the centreline-arc diffuser does not offer a smooth curvature-continuous transition to the upstream and downstream channels, thereby increasing the likelihood of flow separation. This is unsurprising, as it is based on a straight wall diffuser, which presents similar drawbacks. The double-arc diffuser offers smooth transitions but lacks a constant rate of expansion. The expansion rate in this type of diffuser is low at the inlet and increases significantly towards the outlet of the diffuser. This too, makes flow separation more likely. In comparison, the logarithmic diffuser offers a constant rate of expansion, smooth transitions and curvature control. Its shape is entirely based on equations, which allows for easy parameterisation, and streamlined optimisation using computational fluid dynamics. Some of these findings are summarised in the table below:

Table 1. Comparing curved diffuser geometries

Diffuser type	Smooth transition	Constant expansion	Curvature control
centreline-arc	no	yes	no
double-arc	yes	no	no
logarithmic	yes	yes	yes

7. CONCLUSIONS

An iterative method of defining curved diffusers has been introduced and its benefits were highlighted. In summary:

- An algorithm to mould logarithmic spirals into a diffuser shape has been developed and demonstrated to work effectively within the limitations stated.
- The algorithm works to a user-determined accuracy and care should be taken as an increased accuracy will require a higher number of calculation iterations.
- Equations for calculating equivalent diffuser angle and length are presented, to facilitate the comparison of future experimental results with the scientific literature
- A simple method of adapting logarithmic diffuser shapes into a cascade of expansion vanes is introduced.
- A comparison between different methods used to construct diffusers has been made and the advantages of logarithmic diffusers shown.

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PARAMETRIZAREA GEOMETRIEI DIFUZOARELOR CURBATE ȘI A PALETELOR DE EXPANSIUNE FOLOSIND SPIRALE LOGARITMICE

Cercetări recente au evidențiat modul în care colțurile extinse pot ajuta la reducerea dimensiunii și, prin urmare, a costurilor tunelurilor de vânt și apă. Colțurile expansive pot fi construite folosind difuzoare curbate sau palete de expansiune. Această lucrare examinează modalitățile tradiționale de definire a formei difuzoarelor curbate. Se argumentează că aceste metode au deficiențe, cum ar fi discontinuitatea în curbura canalului sau schimbarea ratelor de expansiune. Este prezentat un algoritm iterativ pentru a depăși aceste probleme prin definirea formelor difuzorului folosind spirale logaritmice. Această abordare permite controlul precis asupra unghiului de expansiune a difuzorului, unghiului de rotire, precum și a curburii. În plus, se arată cum geometria unui difuzor logaritmic poate fi adaptată și integrată într-o cascadă de palete de expansiune.

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