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APPROXIMATION OF THE ELASTIC CURVE DIFFERENTIAL EQUATION BY TRIGONOMETRIC SERIES FOR AN ISOTROPIC BEAM WITH A CONSTANT MOMENT OF INERTIA, COMPLEXLY LOADED, ACCORDING TO THE EULER-BERNOULLI THEORY

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Abstract: *The study of the elastic curve has been a constant concern in the field of mechanical engineering and has led to several analytical, graphical, and graph-analytical methods of analysis. In this work, an analytical method of study is presented that can be counted among the energy methods for calculating deformations, since the potential energy of deformation expression is used in the case of beams loaded to simple bending, in the field of elastic deformations, using isotropic materials. The elastic curve expressed by a fourth-order differential equation can also be approximated by an infinite trigonometric series, leading to a good convergence of the results obtained by classical methods. Starting from a series of simple loading cases, which serve to illustrate the method of solving the problem, this study examines two cases with a higher degree of complexity: in the first study, the beam is loaded on the first half unit length by a uniformly distributed load q(x), and in the second study, the beam is loaded by two concentrated forces arranged symmetrically concerning the support points.*

Keywords: *trigonometric series, the potential energy of deformation, isotropic beams*

1. INTRODUCTION

Beams made of isotropic materials are important components of structures and are subjected to various types of loads, such as static and dynamic loads. During service, these beams may exhibit deformations under the action of loads. Understanding and analyzing these deformations is critical for the design and evaluation of mechanical structures, civil structures, etc. Accurate knowledge of the deformation mode of the elastic curve is critical for determining the stress state, load carrying capacity, or load capacity of the beams [1-16].

The nature of the deformation of an isotropic beam (with a constant moment of inertia) loaded (statically) in bending is expressed by the differential equation of the elastic curve. It is a fourth-order function that can be solved by direct integration, using Clebsch's method to evaluate the integration constants (by introducing the condition of smoothness and continuity of the function that determines the elastic curve, but also by introducing boundary conditions at the inflection points).

However, such a function can be solved by various approximation methods, including polynomials (the method of least squares, Cebasev) [17, 18], the polynomial of best approximation (Remez algorithm) [19], Fourier series [20-22], etc.

The use of trigonometric series [23-27] in approximating the elastic curve of isotropic bars is of great importance because of the efficiency and convenience of this method in the analysis and representation of deformations.

Trigonometric series are extremely effective in representing and analyzing periodic functions, and the elastic curve of isotropic bars can be considered a periodic function in many cases. This is because the elastic curve varies periodically along the beam, depending on the distance from one support point to another and the type of loading [35–38]. The trigonometric series allows an accurate representation of these periodic variations and provides a solid basis for

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the study of the elastic curve. The trigonometric series allows for the expression of a periodic function using an infinite number of trigonometric terms. With a sufficient number of terms in the trigonometric series, a very good approximation to the real form of the deformations can be obtained. Trigonometric series have important mathematical properties such as orthogonality and convergence. These properties facilitate the analysis and mathematical calculations of the elastic curve. Orthogonality allows the coefficients of the series to be determined by integration and dot products, and convergence ensures that the partial sum of the terms of the series approaches the original function as more terms are added.

The method of trigonometric series benefits from a high efficiency in the numerical calculation of the coefficients of the series. There are algorithms [28, 29] and well-developed techniques for the fast and accurate calculation of these coefficients, which allow an efficient and convenient analysis of the elastic curve.

This study aims to highlight and analyze the trigonometric series used in studying the elastic curve of isotropic beams with constant moments of inertia. The article presents a detailed approach to the method of applying trigonometric series and demonstrates its advantages and efficiency in the analysis of the elastic curve, starting from two examples with simple loading (case 1: beam supported at both ends and loaded in the middle with a force concentrated – C1; case 2: beam supported at both ends and loaded along its entire length with a uniformly distributed load – C2) and ending with the presentation of two more examples with more complex loading (case 3: beam supported at both ends and loaded on one half of unit length with a uniformly distributed load – C3; case 4: beam supported at both ends symmetrically loaded with two concentrated forces – C4).

In the case of straight beams subjected to bending by various types of loads acting nodally or in a given interval (corresponding to a given unit length), the elastic curve may have a different shape in each interval. Assuming that the elastic curve y(x) is expressed by a single curve, it must satisfy a set of boundary conditions at the inflection points for displacements, slopes, bending moments, and shear forces, regardless of the number of intervals.

For example, if we consider a beam (simply supported) loaded to bending by a nodal force F, which is study case 1, the elastic curve (as shown in Figure 1) is expressed by the following relationship:

$$y(x) = v_1 \cdot sin\frac{\pi \cdot x}{l} \tag{1}$$

The differential equation of the elastic curve is as follows:

$$\frac{d^4y}{dx^4} = \frac{F}{E \cdot I_z} \tag{2}$$

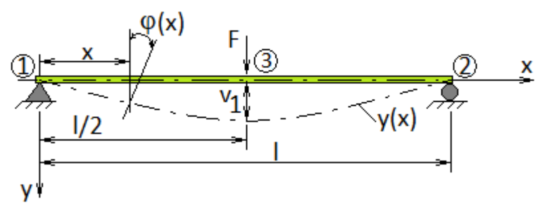

Fig. 1. Beam - simply supported - loaded to bending by a nodal force F.

In Figure 1, the points marked with 1, 2, and 3 indicate the position of the supporting elements or the nodal force F applied to the bar. Equation (1) must satisfy the boundary conditions in the support points (inflection points) as follows: for x=0 (point 1 in Figure 1) and x=l (point 2 in Figure 1) y(x)=0, y'(x)=φ(x)≠0, y''(x)=M_z(x)=0, y'''(x)=T_y(x)≠0; where: y(x) represents the elastic curve, y'(x)= φ(x) is the slope of the cross-section, y''(x)=M_z(x) represents the bending moment and y'''(x)=T_y(x) is the shear force, sin (π·x)/l is a function that depends on the variable x and is a dimensionless quantity, v_1 represents the amplitude of the oscillation y(x) and from the perspective of the cross section of the beam, it is the displacement of the center of gravity in relation to the y axis.

The amplitude v_1 is determined by imposing the condition that external mechanical work (L_e) produced by the nodal force F, applied to the beam, is transformed into potential energy of deformation (U_d) as a result of moving to a new equilibrium position (dv_1). The magnitude dv_1 was produced by a small static increase in the dF of the applied nodal force F (according to Figure 2). In Figure 2, line OA represents the range of elastic deformations of the material from which the beam was made. The external mechanical work dL_e represents the surface area (*abcde*) under the characteristic curve and is expressed as follows:

$$dL_e = A_{abcde} = A_{abcd} + A_{ade} \qquad (3)$$

or

$$dL_e = F \cdot dv_1 + \frac{dF \cdot dv_1}{2} \cong F \cdot dv_1 \qquad (4)$$

The term $\frac{dF \cdot dv_1}{2}$, from relation (4), is neglected considering that it is a small infinite of higher order.

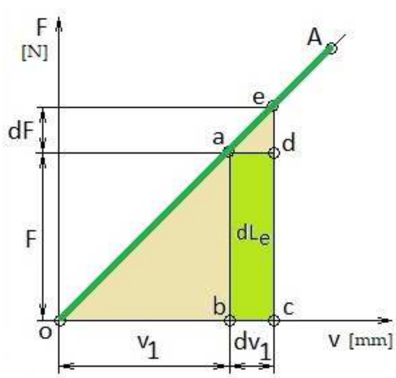

Fig. 2. Graphic representation of the domain of elastic deformations for the evaluation of the variation of the external mechanical work (d*Le*) in the case of a beam (simply supported) loaded to bending, using a nodal force, F.

When the straight beams are loaded for bending, the potential deformation energy U_d (or internal mechanical work, L_i) is expressed through the following relationship:

$$U_d = \int_0^l \frac{M_z^2(x)}{2 \cdot E \cdot I_z} \cdot dx \qquad (5)$$

From Equation (2), we can deduce the expression of the bending moment $M_z(x)$ of the form:

$$M_z(x) = -E \cdot I_z \cdot \frac{d^2 y(x)}{dx^2} = -E \cdot I_z \cdot y''(x) \qquad (6)$$

where the (-) sign is considered because the y axis is in the direction of the convexity of the beam.

Thus, relations (5) and (6) result in:

$$U_d = \frac{E \cdot I_z}{2} \cdot \int_0^l [y''(x)]^2 \cdot dx \qquad (7)$$

where:

$$y(x) = v_1 \cdot sin\frac{\pi \cdot x}{l}$$

By successive derivation and squaring, we obtain:

$$y'(x) = \frac{\pi \cdot v_1}{l} cos\frac{\pi \cdot x}{l}$$

$$y''(x) = -\frac{\pi^2}{l^2} \cdot v_1 \cdot sin\frac{\pi \cdot x}{l}$$

$$[y''(x)]^2 = \frac{\pi^4}{l^4} \cdot v_1^2 \cdot sin^2\frac{\pi \cdot x}{l} \qquad (7a)$$

Introducing relation (7a) into the expression of the potential deformation energy - relation (7), it results:

$$U_d = \frac{E \cdot I_z}{2} \cdot \int_0^l \frac{\pi^4}{l^4} \cdot v_1^2 \cdot sin^2\frac{\pi \cdot x}{l} \cdot dx =$$

$$= \frac{E \cdot I_z}{2} \cdot v_1^2 \cdot \int_0^l \frac{\pi^4}{l^4} \cdot sin^2\frac{\pi \cdot x}{l} \cdot dx \qquad (8)$$

If noted:

$$f''^2(x) = \frac{\pi^4}{l^4} \cdot sin^2\frac{\pi \cdot x}{l}$$

from relation (8) we obtain:

$$U_d = \frac{E \cdot I_z}{2} \cdot v_1^2 \cdot \int_0^l f''^2(x) \cdot dx \qquad (9)$$

In the case of an increase in the amplitude by dv_1, the deformation potential energy is expressed as follows:

$$dU_d = \frac{\partial U_d}{\partial v_1} =$$

$$= v_1 \cdot dv_1 \cdot E \cdot I_z \cdot \int_0^l f''^2(x) \cdot dx \qquad (10)$$

The principle of conservation of energy states that the external mechanical work (L_e) is equal to the internal mechanical work (L_i) also called the potential energy of deformation (U_d). Thus, from Equations (4) and (10), it follows that:

$$dL_e = dU_d$$

or

$$F \cdot dv_1 = v_1 \cdot dv_1 \cdot E \cdot I_z \cdot \int_0^l f''^2(x) \cdot dx \qquad (11)$$

From relation (11) we obtain:

$$v_1 = \frac{F}{E \cdot I_z \cdot \int_0^l f''^2(x) \cdot dx} \qquad (12)$$

where:

$$\int_0^l f''^2(x) \cdot dx = \int_0^l \frac{\pi^4}{l^4} \cdot sin^2\frac{\pi \cdot x}{l} \cdot dx =$$

$$= \frac{\pi^4}{l^4} \cdot \int_0^l sin^2\frac{\pi \cdot x}{l} \cdot dx \qquad (13)$$

Because:

$$sin^2 x = \frac{1}{2} \cdot [1 - cos(2 \cdot x)]$$

result:

$$sin^2 \frac{\pi \cdot x}{l} = \frac{1}{2} \cdot \left[1 - cos\left(2 \cdot \frac{\pi \cdot x}{l}\right)\right]$$

Thus,

$$\int_0^l sin^2 \frac{\pi \cdot x}{l} \cdot dx = \int_0^l \frac{1}{2} \cdot \left[1 - cos\left(2 \cdot \frac{\pi \cdot x}{l}\right)\right]$$

$$= \int_0^l \left[\frac{1}{2} - \frac{1}{2} cos\left(2 \cdot \frac{\pi \cdot x}{l}\right)\right] \cdot dx =$$

$$= \frac{l}{2} - \frac{1}{2} \cdot \int_0^l cos\left(2 \cdot \frac{\pi \cdot x}{l}\right) \cdot dx \quad (14)$$

Because:

$$\int cos(a \cdot x) \cdot dx = \frac{sin(a \cdot x)}{a}$$

result:

$$\int_0^l cos\left(2 \cdot \frac{\pi \cdot x}{l}\right) \cdot dx = \frac{sin\left(\frac{2 \cdot \pi}{l} \cdot x\right)}{\frac{2 \cdot \pi}{l}} \Big|_0^l = 0$$

Thus, from relation (14) we obtain:

$$\int_0^l sin^2 \frac{\pi \cdot x}{l} \cdot dx = \frac{l}{2} \quad (15)$$

The l/2 term in equation (15) represents the integral in relation (13). By substitution we get:

$$\int_0^l f''^2(x) \cdot dx = \frac{\pi^4}{l^4} \cdot \frac{l}{2} = \frac{\pi^4}{2 \cdot l^3} \quad (16)$$

Substituting Equation (16) into Equation (12) results in:

$$v_1 = \frac{F}{E \cdot I_z \cdot \frac{\pi^4}{2 \cdot l^3}} = 0.02053 \cdot \frac{F \cdot l^3}{E \cdot I_z} \quad (17)$$

The exact solution for this case is expressed through the following relation:

$$v_1 = \frac{F \cdot l^3}{48 \cdot E \cdot I_z} = 0.02083 \cdot \frac{F \cdot l^3}{E \cdot I_z} \quad (18)$$

Comparing the results obtained through Eqs. (17) and (18) results in a relative deviation of 1.445%.

In the following case study, the beam is loaded through a uniformly distributed force q(x), also applied in the transverse plane (according to Figure 3), the amplitude v1 is also determined, with relationship (1) as the starting point. In Figure 3, the points marked with 1 and 2 indicate the position of the supporting elements.

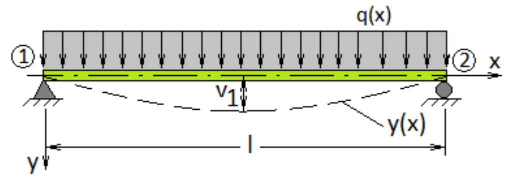

Fig. 3. Straight beam (simply supported) loaded to bending by a uniformly distributed force, q(x).

The amplitude v_1 is calculated by imposing the condition that the external mechanical work (L_e), produced by the uniformly distributed force, q(x), applied to the beam, is transformed into potential energy of deformation (U_d) as a result of moving to a new equilibrium position (dv_1). The quantity dv_1 is produced by a small static increase in dF of the uniformly distributed load q(x) applied (according to Figure 4).

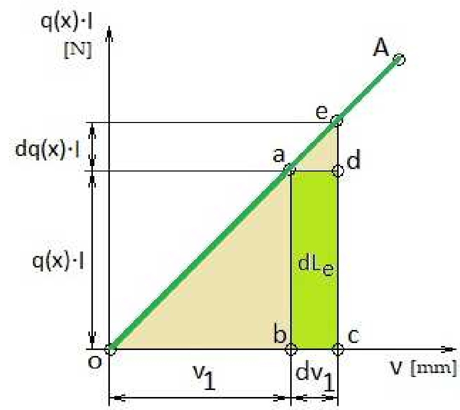

Fig. 4. Evaluation of the variation of the external mechanical work (dL_e) in the case of a beam simply supported and subjected to bending using a uniformly distributed load, q(x).

In Figure 4, segment OA represents the range of elastic deformations corresponding to the material from which the beam was made. The external mechanical work dL_e represents the area of the surface *abcde* under the characteristic curve and can be evaluated using the following formula:

$$dL_e = A_{abcde} = A_{abcd} + A_{ade} =$$

$$= q(x) \cdot l \cdot dv_1 + \frac{dq(x) \cdot dv_1}{2} \quad (19)$$

The term $\frac{dq(x) \cdot dv_1}{2}$, from relation (19), is neglected considering that it is a small infinity of higher order.

From equation (19), it follows that:

$$dL_e = q(x) \cdot l \cdot dv_1 \quad (20)$$

From relations (20) and (11), we obtain:

$$q(x) \cdot l \cdot dv_1 =$$

$$= v_1 \cdot dv_1 \cdot E \cdot I_z \cdot \int_0^l f''^2(x) \cdot dx \qquad (21)$$

By dividing equation (21) by dv_1, it follows that:

$$v_1 = \frac{q(x) \cdot l}{E \cdot I_z \cdot \int_0^l f''^2(x) \cdot dx} =$$

$$= \frac{q(x) \cdot l}{E \cdot I_z \cdot \frac{\pi^4}{l^4} \cdot \int_0^l sin^2 \frac{\pi \cdot x}{l} \cdot dx} \qquad (22)$$

For x=l/2 from equation (22), we obtain:

$$v_1 = \frac{q(x) \cdot l^4}{\pi^4 \cdot E \cdot I_z} = 0.01026 \cdot \frac{q(x) \cdot l^4}{E \cdot I_z} \qquad (23)$$

The exact solution for this support and loading case is expressed as follows:

$$v_1 = \frac{5 \cdot q(x) \cdot l^4}{384 \cdot E \cdot I_z} = 0.01302 \cdot \frac{q(x) \cdot l^4}{E \cdot I_z} \qquad (24)$$

Comparing the results obtained through relations (23) and (24) results in a relative deviation of 21.19%.

By analyzing the relative deviations obtained in the two cases of support and loading (in case 1 it is 1.445% and in the second case 21.19%), it can be seen that the method of replacing the elastic curve with a single sinusoid does not provide a satisfactory result. Therefore, the elastic curve is expressed as follows, representing an infinite trigonometric series.

2. DEVELOPMENT OF THE ELASTIC CURVE DIFFERENTIAL EQUATION IN A TRIGONOMETRIC SERIES

For the first load case previously analyzed (beam - simply supported - loaded in the transverse plane with a nodal force, F), the trigonometric series expressing the equation of the elastic curve is of the form:

$$y(x) = v_1 \cdot sin \frac{\pi \cdot x}{l} + v_2 \cdot sin \frac{2 \cdot \pi \cdot x}{l} + \cdots$$

$$+ v_n \cdot sin \frac{n \cdot \pi \cdot x}{l} \qquad (25)$$

In Figure 5, successive sinusoids are represented, and in Figure 6, the same sinusoids are highlighted by their overlap.

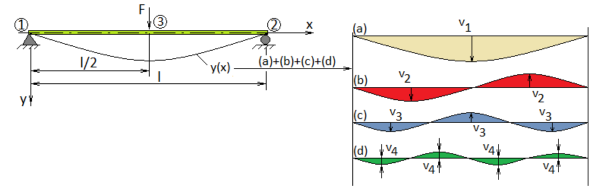

Fig. 5. Study case 1 (C1) – the elastic curve y(x) expressed by a sinusoidal series.

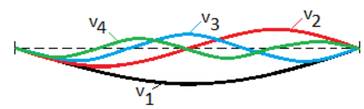

Fig. 6. Superposition of the sinusoidal series.

It should be specified that each component term of the sinusoidal series, as well as its derivatives, satisfies the boundary conditions in the supports (points 1 and 2 in Figure 5).

When choosing the form of the trigonometric series, it must be considered that, because in relation (7), the potential deformation energy dU_d depends on the square of the second derivative of the function y(x), it is very important that all the conditions for this derivative are satisfied to get relevant results. The condition for the third derivative of the function y(x), which represents the size of the shear force, may not be satisfied (Timoshenko-Ehrenfest's theory of beams), without having a particular influence on the accuracy of the results [30-34].

To determine the amplitude of each sinusoid (v_1, v_2, ..., v_n), the principle of energy conservation was applied.

From relation (7), it follows that:

$$U_d = \frac{E \cdot I_z}{2} \cdot \int_0^l [y''(x)]^2 \cdot dx =$$

$$= \frac{E \cdot I_z}{2} \cdot \int_0^l \left[-v_1 \cdot \frac{\pi^2}{l^2} \cdot sin \frac{\pi \cdot x}{l} - v_2 \cdot \frac{2^2 \cdot \pi^2}{l^2} \cdot sin \frac{2 \cdot \pi \cdot x}{l} - v_n \cdot \frac{n^2 \cdot \pi^2}{l^2} \cdot sin \frac{n \cdot \pi \cdot x}{l} \right]^2 \cdot dx \qquad (26)$$

By squaring under the integral, two types of terms are obtained:

$$v_n^2 \cdot \frac{n^4 \cdot \pi^4}{l^4} \cdot sin^2 \frac{n \cdot \pi \cdot x}{l}$$

and

$$2 \cdot v_n \cdot v_m \cdot \frac{n^2 \cdot m^2 \cdot \pi^4}{l^4} \cdot sin\frac{n \cdot \pi \cdot x}{l} \cdot sin\frac{m \cdot \pi \cdot x}{l}$$

By integrating these two terms, we obtain:

$$\int_0^l v_n^2 \cdot \frac{n^4 \cdot \pi^4}{l^4} \cdot sin^2\frac{n \cdot \pi \cdot x}{l} \cdot dx = v_n^2 \cdot \frac{n^4 \cdot \pi^4}{l^4} \cdot \frac{l}{2} = v_n^2 \cdot \frac{n^4 \cdot \pi^4}{2 \cdot l^3}$$

and

$$\int_0^l 2 \cdot v_n \cdot v_m \cdot \frac{n^2 \cdot m^2 \cdot \pi^4}{l^4} \cdot sin\frac{n \cdot \pi \cdot x}{l} \cdot sin\frac{m \cdot \pi \cdot x}{l} \cdot dx = 0$$

Thus, the expression of the potential energy will be of the form:

$$U_d = \frac{\pi^4 \cdot E \cdot I_z}{4 \cdot l^3} \cdot (v_1^2 + 2^4 \cdot v_2^2 + 3^4 \cdot v_3^2 + \dots + n^4 \cdot v_n^2) = \frac{\pi^4 \cdot E \cdot I_z}{4 \cdot l^3} \cdot \sum_{n=1}^{\infty} n^4 \cdot v_n^2 \quad (27)$$

Equation (27) represents the sum of potential deformation energies that accumulate in the material of the beam during deformation in the elastic domain, corresponding to each sinusoid.

For small deformations of an elastic system, starting from the equilibrium position, the deformation potential energy U_d is equal to the mechanical work L_e produced by the external forces (nodal force F) during this deformation. According to relation (25), a small deformation is equivalent to a small change in the coefficients v_1, v_2, and v_3, which, in reality, represent the displacements of the center of gravity of the transversal section of the beam in different nodes. Thus, dv_n represents to be the variation of coefficient v_n.

With this variation the term

$$v_n \cdot sin\frac{n \cdot \pi \cdot x}{l} \quad (28)$$

turns into

$$(v_n + dv_n) \cdot sin\frac{n \cdot \pi \cdot x}{l} \quad (29)$$

This increase is similar to the insertion of an additional sinusoidal function in the trigonometric series (25), which can be expressed as follows:

$$dv_n \cdot sin\frac{n \cdot \pi \cdot x}{l} \quad (30)$$

Through this additional deformation, the point of application of force F (node 3 in Figure 1), positioned at a distance l/2 from the origin of the orthogonal system of reference axes (node 1 in Figure 1), moves in the vertical plane with

$$dv_n \cdot sin\frac{n \cdot \pi \cdot x}{l}$$

The external mechanical work dL_{ext}, generated by the nodal force F, will be of the following form:

$$dL_{ext} = F \cdot dv_n \cdot sin\frac{n \cdot \pi \cdot a}{l} \quad (31)$$

where a=l/2 and represents the elevation corresponding to the point of application of nodal force F to node 1 in Figure 5.

The internal mechanical work dL_{int} expressed according to the amplitudes v_1, v_2, v_3, ..., v_n by the variation of a single coefficient, will present an increase of it and can be represented by the relation:

$$dL_{int} = \frac{\partial U_d}{\partial v_n} \cdot dv_n = \frac{\partial}{\partial v_n} \cdot \left(\frac{E \cdot I_z \cdot \pi^4}{4 \cdot l^3} \cdot \sum_{n=1}^{\infty} n^4 \cdot v_n^4\right) \cdot dv_n \Rightarrow$$

$$dL_{int} = \frac{E \cdot I_z \cdot \pi^4}{2 \cdot l^3} \cdot n^4 \cdot v_n \cdot dv_n \quad (32)$$

According to the principle of energy conservation, the internal mechanical work dL_{int}, expressed in relation (32), is equalized with the external mechanical work dL_{ext}, expressed with relation (31), and it follows that:

$$F \cdot dv_n \cdot sin\frac{n \cdot \pi \cdot a}{l} = \frac{E \cdot I_z \cdot \pi^4}{2 \cdot l^3} \cdot n^4 \cdot v_n \cdot dv_n \quad (33)$$

Where to get:

$$v_n = \frac{2 \cdot F \cdot l^3 \cdot sin\frac{n \cdot \pi \cdot a}{l}}{E \cdot I_z \cdot \pi^4 \cdot n^4} \quad (34)$$

By substituting the coefficients v_n in equation (25), we obtain:

$$y(x) = \frac{2 \cdot F \cdot l^3}{E \cdot I_z \cdot \pi^4} \cdot \left(\frac{1}{1^4} \cdot sin\frac{\pi \cdot a}{l} \cdot sin\frac{\pi \cdot x}{l} + \frac{1}{2^4} \cdot sin\frac{2 \cdot \pi \cdot a}{l} \cdot sin\frac{2 \cdot \pi \cdot x}{l} + \frac{1}{3^4} \cdot sin\frac{3 \cdot \pi \cdot a}{l} \cdot sin\frac{3 \cdot \pi \cdot x}{l}\right) \quad (35)$$

or

$$y(x) = \frac{2 \cdot F \cdot l^3}{E \cdot I_z \cdot \pi^4} \cdot \sum_{n=1}^{\infty} \frac{1}{n^4} \cdot sin \frac{n \cdot \pi \cdot a}{l} \cdot sin \frac{n \cdot \pi \cdot x}{l} \quad (36)$$

If it is considered that the beam has a total length of l=200[mm] and is loaded with a force F=100[N], having the rectangular cross-section defined using the dimensions h=12[mm] and b=4[mm], the variation of the elastic curve is evaluated, in the following, using relation (36), of the finite element method (FEA) but also using relations (36a) and (36b).

$$v_{13} = \frac{1}{E \cdot I_z} \cdot \left(-\frac{F \cdot x^3}{12} + C \cdot x + D \right) \quad (36a)$$

$$v_{32} = \frac{-\frac{F \cdot x^3}{12} + \frac{F \cdot \left(x - \frac{l}{2}\right)^3}{6} + C \cdot x + D}{E \cdot I_z} \quad (36b)$$

Relations (36a) and (36b) are a reformulation of relation (18) that allows the evaluation of the deformation of the elastic curve in different cross-sections of the beam, whose position is specified by quantity x. It should be specified that these two relations are obtained by integrating the differential equation of the elastic curve and the integration constants C and D are determined by introducing the boundary conditions at the inflection points.

The magnitude of these two integration constants are:

$$D = 0; C = \frac{3 \cdot F \cdot l^8}{48} \quad (36c)$$

In relation to (36a) and (36b), Young's modulus is 2.1e5 [MPa], and the axial moment of inertia about the main axis of inertia z is 576[mm^4].

Finite Element Analysis (FEA) was performed using RDM v.7.04 software, Ossatures module, using bar-type finite elements with three degrees of freedom.

21 distinct sections located 10 mm from each other were considered. Figure 7 shows the variation diagrams of the elastic curve y(x).

How should the eigenfunctions of the Euler–Bernoulli model compare with finite element modeling and the trigonometric series? Considering the results given by relations (36a) and (36b) - the Euler-Bernoulli model - as reference values, the relative deviation (in percentage) is calculated based on the results obtained using finite element analysis (FEA) and using the relation (36) for n equal to 1, 3, and 5.

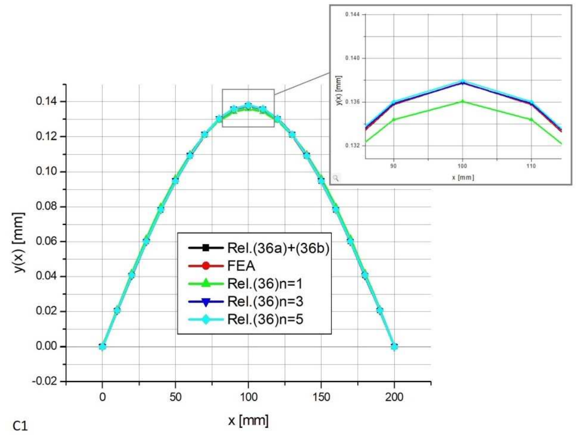

Fig. 7. The variation of the elastic curve was evaluated utilizing relations (36a), (36b), (36) and FEA for the isotropic beam simply supported and loaded with a concentrated force F at the middle.

Figure 8 shows the results obtained analytically and numerically (FEA).

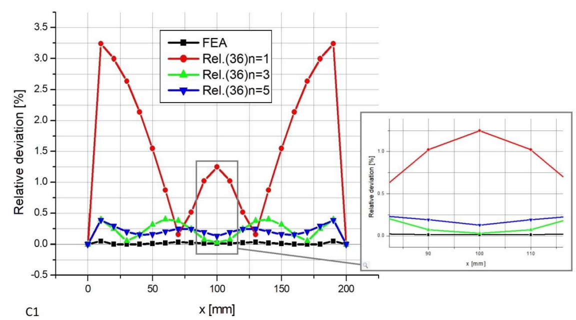

Fig. 8. Graphical representation of the relative deviation.

Before studying the second case loading (C2), according to Figure 3, it is noted that for the calculation of the increase in internal mechanical work dL_{int} [relation (32)], the type of applied load (nodal force, nodal moment, or load distributed on a certain unit of length) does not influence the variation of the displacement v_n, provided that this parameter is integrated in the field of elastic deformations.

Thus, for the second case of support and loading (C2), the variation of the external mechanical work dL_{ext} becomes:

$$\frac{dL_{ext}}{dv_n} = \int_0^l q(x) \cdot dx \cdot sin \frac{n \cdot \pi \cdot x}{l} =$$

$$= \left[q(x)\cdot\left(-\frac{l}{n\cdot\pi}\right)\cdot cos\frac{n\cdot\pi\cdot x}{l}\right]_0^l =$$

$$= -\frac{q(x)\cdot l}{n\cdot\pi}\cdot(cosn\cdot\pi-1) \qquad (37)$$

Equating this relationship mathematically with the increase of the internal mechanical work dL_{int}, expressed through equation (32), results:

$$\frac{E\cdot I_z\cdot\pi^4}{2\cdot l^3}\cdot n^4\cdot v_n = -\frac{q(x)\cdot l}{n\cdot\pi}\cdot(cosn\cdot\pi-1)$$

$$v_n = \frac{-2\cdot q(x)\cdot l^4}{E\cdot I_z\cdot\pi^5\cdot n^5}\cdot(cosn\cdot\pi-1) \qquad (38)$$

In this case, the equation of the elastic curve is:

$$y(x) = -\sum_{n=1}^{\infty}\frac{2\cdot q(x)\cdot l^4}{E\cdot I_z\cdot\pi^5\cdot n^5}\cdot(cosn\cdot\pi-1)\cdot \cdot sin\frac{n\cdot\pi\cdot x}{l} \qquad (39)$$

For n=1, 3, 5, ...

$$v_n = \frac{4\cdot q(x)\cdot l^4}{E\cdot I_z\cdot\pi^5\cdot n^5} \qquad (40)$$

and for n=2, 4, 6, ...

$$v_n = 0 \qquad (41)$$

The equation of the elastic curve is expressed in the following form:

$$y(x) = \frac{4\cdot q(x)\cdot l^4}{E\cdot I_z\cdot\pi^5}\cdot \cdot\left(\frac{1}{1^5}\cdot sin\frac{1\cdot\pi\cdot x}{l}+\frac{1}{3^5}\cdot sin\frac{3\cdot\pi\cdot x}{l}+\frac{1}{5^5} \cdot sin\frac{5\cdot\pi\cdot x}{l}+\cdots\right) \qquad (42)$$

In this case, the maximum displacement of the center of gravity of the cross-section will occur at the point located in the middle of the distance between the two supports, for which the variable x has the value of l/2:

$$v = \frac{4\cdot q(x)\cdot l^4}{E\cdot I_z\cdot\pi^5}\cdot \cdot\left(\frac{1}{1^5}-\frac{1}{3^5}+\frac{1}{5^5}-\frac{1}{7^5}+\cdots\right) \qquad (43)$$

Next, in a sequential form, the first term is taken into account, then the first two terms and, finally, the first three terms in the parenthesis, the following results are obtained:

$$y(x) = \frac{4\cdot q(x)\cdot l^4}{306.019\cdot E\cdot I_z} =$$

$$= 0.013071\cdot\frac{q(x)\cdot l^4}{E\cdot I_z} \qquad (44)$$

$$y(x) = 0.0130172\cdot\frac{q(x)\cdot l^4}{E\cdot I_z} \qquad (45)$$

$$y(x) = 0.0130214\cdot\frac{q(x)\cdot l^4}{E\cdot I_z} \qquad (46)$$

To determine the size y(x) in a cross-section, apply the relation (46a) obtained based on the model of the Euler-Bernoulli theory.

$$y(x) = \frac{q(x)\cdot l^4}{24\cdot E\cdot I_z}\cdot\left(\frac{x}{l}-2\cdot\frac{x^3}{l^3}+\frac{x^4}{l^4}\right) \qquad (46a)$$

21 distinct sections located at a distance of 10 millimeters from one another are considered. The same data is used as in the previous study case, specifying that the uniformly distributed load q(x) has a value of 1 [N/mm]. Figure 9 shows the variation diagrams of the elastic curve y(x).

Considering the results given by relation (43a) as reference values, the relative deviation (in percent) is calculated from the results obtained using finite element analysis (FEA) and using relation (42) for n equal to 1, 3, and 5. In Figure 10 the results are graphically represented.

If a uniformly distributed load q(x) is applied to the beam that acts on the unit length l/2 from the origin of the coordinate axes (according to Figure 11), the amplitude y(x) is determined considering that the uniformly distributed load q(x) represents a sum of elementary concentrated charges of the form q(x)·da. In Figure 11, the nodes where either the support elements are defined, or the length unit on which the uniformly distributed force q(x) is applied to the beam, are marked with 1 - 3.

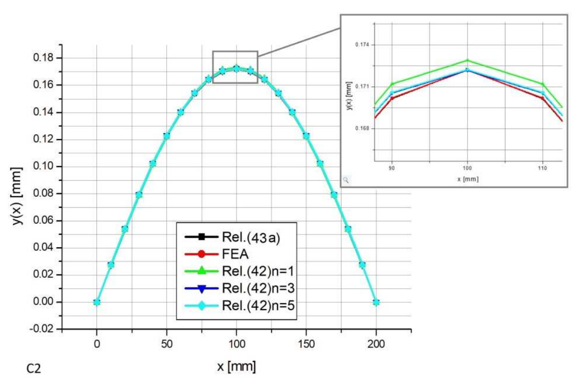

Fig. 9. The variation of the elastic curve was evaluated using relations (43a), (42), and FEA for the isotropic beam simply supported and loaded with a uniformly distributed force, q(x), over the entire unit length.

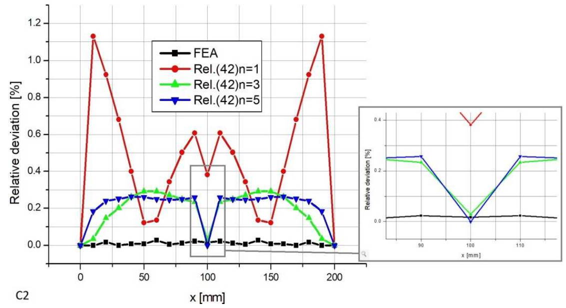

Fig. 10. Graphical representation of the relative deviation.

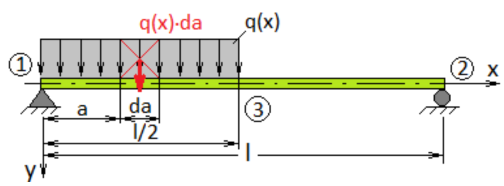

Fig. 11. Simply supported beam loaded with uniformly distributed load, q(x), acting per unit length l/2.

Thus, the elastic curve y(x) can be expressed using the relation (36) in which the term F is replaced by the expression q(x)·da as follows:

$$y(x)=\frac{2\cdot q(x)\cdot da\cdot l^3}{E\cdot I_z\cdot \pi^4}\cdot \sum_{n=1}^{\infty}\frac{1}{n^4}\cdot sin\frac{n\cdot\pi\cdot a}{l}\cdot sin\frac{n\cdot\pi\cdot x}{l} \quad (47)$$

The maximum displacement y(x) is determined by integrating the series (47) about the unit length a from 0 to l/2:

$$\int_0^{\frac{l}{2}}\frac{2\cdot q(x)\cdot da\cdot l^3}{E\cdot I_z\cdot\pi^4}\cdot sin\frac{n\cdot\pi\cdot a}{l}\cdot sin\frac{n\cdot\pi\cdot x}{l}= =\frac{2\cdot q(x)\cdot l^3}{E\cdot I_z\cdot\pi^4}\cdot\frac{1}{n^4}\cdot sin\frac{n\cdot\pi\cdot x}{l}\cdot \int_0^{l/2}sin\frac{n\cdot\pi\cdot a}{l}\cdot da \quad (48)$$

where

$$\int_0^{l/2}sin\frac{n\cdot\pi\cdot a}{l}\cdot da= =-\frac{l}{n\cdot\pi}\cdot cos\frac{n\cdot\pi\cdot c}{l}\Big|_0^{l/2}= =\frac{l}{n\cdot\pi}\cdot\left(1-cos\frac{n\cdot\pi}{2}\right) \quad (49)$$

It is noted:

$$\left(1-cos\frac{n\cdot\pi}{2}\right)=c$$

and has the following values:

$$\boldsymbol{n=1}\rightarrow c=1;\ \boldsymbol{n=2}\rightarrow c=2;\ \boldsymbol{n=3}\rightarrow c=1;\ \boldsymbol{n=4}\rightarrow c=0$$

Thus, the amplitude y(x), for x=l/2, will be expressed through the following relation:

$$y(x)_{l/2}=\frac{2\cdot q(x)\cdot l^4}{E\cdot I_z\cdot n^5\cdot\pi^5}\cdot sin\frac{n\cdot\pi}{2}\cdot\left(1-cos\frac{n\cdot\pi}{2}\right) \quad (50)$$

or

$$y(x)_{l/2}=\frac{2\cdot q(x)\cdot l^4}{E\cdot I_z\cdot\pi^5}\cdot\left[\sum_{n=1,3,5,\ldots}\frac{1}{n^5}\cdot sin\frac{n\cdot\pi\cdot x}{l}+\sum_{n=2,6,10,\ldots}\frac{2}{n^5}\cdot sin\frac{n\cdot\pi\cdot x}{l}\right] \quad (51)$$

For n=1 it follows:

$$y(x)_{l/2}=\frac{q(x)\cdot l^4}{153.009\cdot E\cdot I_z} \quad (52)$$

The final relation for calculating the displacement of the center of gravity of the cross section of the bar for this case of loading and support, according to the Euler-Bernoulli model, is of the form:

$$y(x)_{l/2}=\frac{q(x)\cdot l^4}{153.6\cdot E\cdot I_z} \quad (53)$$

To determine the quantity y(x) in a certain cross-section, apply relations (53a) and (53b) – the Euler-Bernoulli model.

$$y(x)_{13}=\frac{1}{384\cdot E\cdot I_z}\cdot[16\cdot q(x)\cdot x^4-24\cdot q(x)\cdot l\cdot x^3+9\cdot q(x)\cdot l^3\cdot x]\ (53a)$$

$$y(x)_{32}=\frac{1}{384\cdot E\cdot I_z}\cdot\left[16\cdot q(x)\cdot x^4-16\cdot q(x)\cdot\left(x-\frac{l}{2}\right)^4-24\cdot q(x)\cdot l\cdot x^3+9\cdot q(x)\cdot l^3\cdot x\right]\ (53b)$$

21 distinct sections located at a distance of 10 millimeters from one another are considered. The same data are used as in the previous case study. Figure 12 shows the variation diagrams of the elastic curve y(x).

Considering the results given by relations (53a) and (53b) as reference values, the relative deviation (in percent) is calculated about the results obtained using finite element analysis (FEA) and using relation (51) for n equal to 1, 3, and 5.

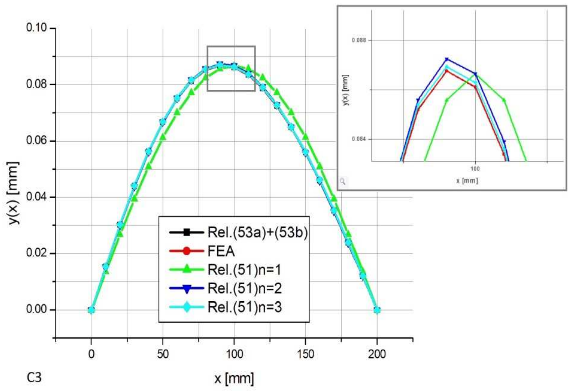

Fig. 12. The variation of the elastic curve was evaluated using relations (53a), (53b), (51), and FEA for the isotropic beam simply supported and loaded with a uniformly distributed load, q(x), over the first half unit length.

Figure 13 shows the results obtained analytically and numerically (FEA).

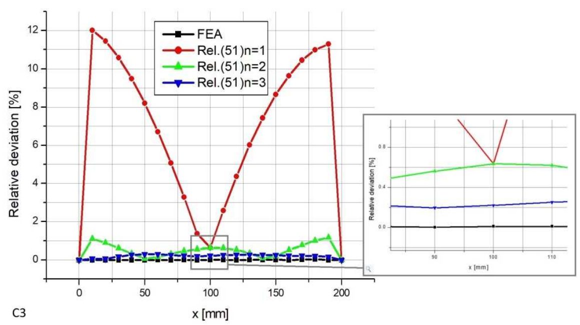

Fig. 13. Graphical representation of the relative deviation.

Figure 14 defines a beam loaded to bending using two nodal forces, F1 and F2. The beam is also simply supported and represents study case 4 (C4).

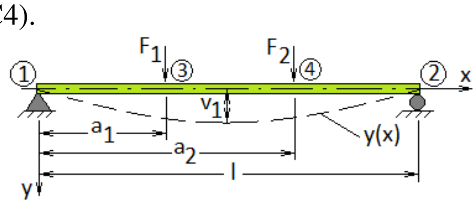

Fig. 14. A simply supported beam is subjected to bending using two concentrated forces, F_1 and F_2.

If it is considered that $a_2=2\cdot a_1$, $l=3\cdot a_1$, and $F_1=F_2=F$ the evaluation of the variation of the external mechanical work (dL_e) has as its starting point the determination of the external mechanical work L_e performed by the two concentrated forces (according to Figure 15) as follows:

-in the first stage, the beam is loaded with force F_1 at point 3, generating at point 4 displacement v_{41} and at point 3 displacement v_{31}, in which case the external mechanical work is of the form:

$$L_{e1} = \frac{F_1 \cdot v_{31}}{2} \quad (54)$$

- the next stage is the one in which the beam is additionally loaded, in point 4, with the force F_2 which generates in point 3 the displacement v_{32} and in point 4 the displacement v_{42} in which case the external mechanical work is of the form:

$$L_{e2} = \frac{F_2 \cdot v_{42}}{2} + F_1 \cdot v_{32} \quad (55)$$

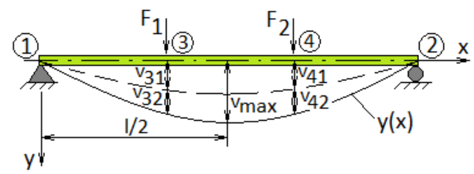

Fig. 15. Evaluation of the variation of the external mechanical work (dL_e) in the case of a beam simply supported and subjected to bending using two concentrated forces, F_1 and F_2.

Thus, the total external mechanical work L_e is of the form:

$$L_e = L_{e1} + L_{e2} = \\ = \frac{F_1 \cdot v_{31}}{2} + \frac{F_2 \cdot v_{42}}{2} + F_1 \cdot v_{32} \quad (56)$$

The initially adopted boundary conditions lead to:

$$v_3 = v_{31} + v_{32}, v_4 = v_{41} + v_{42}, v_3 = v_4$$
$$v_{32} = v_3 - v_{31} = v_{41}, v_{42} = v_3 - v_{41} \Rightarrow$$
$$v_{42} = v_3 - (v_3 - v_{31}) = v_{31} \quad (57)$$

From relations (56) and (57) it follows:

$$L_e = \frac{F \cdot v_{31}}{2} + \frac{F \cdot v_{31}}{2} + F \cdot v_{32} = \\ = 2 \cdot F \cdot v_3 \quad (58)$$

The variation of the external mechanical work, dL_{ext}, produced by an elementary increase of the forces F_1 and F_2, based on the initially adopted boundary conditions, is of the form:

$$dL_{ext} = 2 \cdot F \cdot dv_n \cdot sin\frac{n \cdot \pi \cdot a}{l} \quad (59)$$

Equating relation (59) with relation (32) results:

$$2 \cdot F \cdot dv_n \cdot sin\frac{n \cdot \pi \cdot a}{l} = \\ = \frac{E \cdot I_z \cdot \pi^4}{2 \cdot l^3} \cdot n^4 \cdot v_n \cdot dv_n \quad (60)$$

Where to get:

$$v_n = \frac{4 \cdot F \cdot l^3 \cdot sin\frac{n \cdot \pi \cdot a}{l}}{E \cdot I_z \cdot \pi^4 \cdot n^4} \quad (61)$$

Next, the v_n coefficients are replaced in relation (25), resulting:

$$y(x) = \frac{4 \cdot F \cdot l^3}{E \cdot I_z \cdot \pi^4} \cdot \cdot \left(\frac{1}{1^4} \cdot sin\frac{\pi \cdot a}{l} \cdot sin\frac{\pi \cdot x}{l} + \frac{1}{2^4} \cdot sin\frac{2 \cdot \pi \cdot a}{l} \cdot sin\frac{2 \cdot \pi \cdot x}{l} + \frac{1}{3^4} \cdot sin\frac{3 \cdot \pi \cdot a}{l} \cdot sin\frac{3 \cdot \pi \cdot x}{l}\right) \quad (62)$$

or

$$y(x) = \frac{4 \cdot F \cdot l^3}{E \cdot I_z \cdot \pi^4} \cdot \cdot \sum_{n=1}^{\infty} \frac{1}{n^4} \cdot sin\frac{n \cdot \pi \cdot a}{l} \cdot sin\frac{n \cdot \pi \cdot x}{l} \quad (63)$$

where: a=l/3 and x=l/2.

The exact solution for this case of loading and bearing, based on the Euler-Bernoulli model, is expressed using the following calculation relations:- for the first interval (bounded using points 1-3, according to Figure 15):

$$y(x)_{13} = \frac{1}{E \cdot I_z} \cdot \left(-\frac{F \cdot x^3}{6} + C \cdot x\right) (64a)$$

- for the second interval (bounded using points 3-4, according to Figure 15):

$$y(x)_{34} = \frac{1}{E \cdot I_z} \cdot \left[-\frac{F \cdot x^3}{6} + \frac{F(x-a)^3}{6} + C \cdot x\right] \quad (64b)$$

It is specified that the mathematical formulas (64a) and (64b) were derived by integrating the differential equation of the elastic curve and to establish the integration constant C the Clebsch method was used based on the condition of smoothness and continuity of the differential equation of the elastic curve but also by introducing boundary conditions at the points where the isotropic beam has canceled one degree of freedom - y(x)=0. The calculation relation for the integration constant C is of the form:

$$C = \frac{F}{6 \cdot l} \cdot [l^3 - (l-a)^3 - a^3] \quad (65)$$

21 distinct sections located at a distance of 10 millimeters from one another are considered. The same data are used as in case 1 of the study.

Figure 16 shows the variation diagrams of the elastic curve y(x).

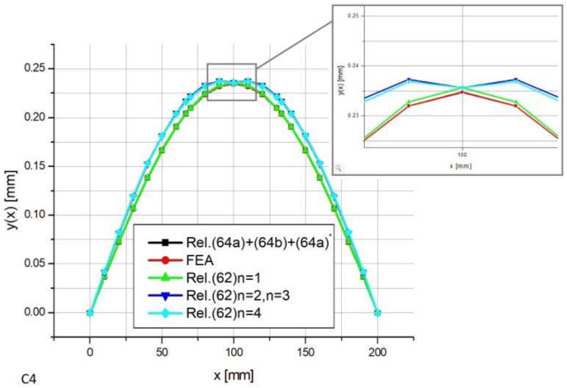

Fig. 16. The variation of the elastic curve was evaluated using relations (64a), (64b), (64a)*, (62), and FEA for the isotropic beam simply supported and subjected to bending using two concentrated forces, F_1 and F_2.

The relation (64a) can also be applied for the interval 4-2 if the origin of the coordinate axis system is at point 2. This aspect is highlighted by noting the relation (64a) with (64a)*. It is also stated that in relation (62) for n=2 and n=3, the same results are obtained because

$$sin\frac{3 \cdot \pi \cdot a}{l} = 0 \; for \; a = \frac{l}{3}$$

Considering the results given by relations (64a), (64a)*, and (64b) as reference values, the relative deviation (in percent) is calculated about the results obtained using finite element analysis (FEA) and using relation (62) for n equal to 1, 2, 3 and 4 . Figure 17 shows the results.

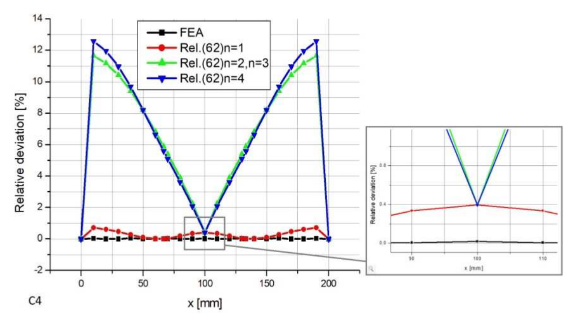

Fig. 17. Graphical representation of the relative deviation.

Based on these studies, the relevance of the approximation of the differential equation of the elastic fiber using trigonometric series can be emphasized, highlighting the following conclusions.

3. CONCLUSIONS

This study aims to highlight and analyze the use of trigonometric series in the study of the elastic curve of isotropic beams with constant moments of inertia. The article aims to present a detailed approach to the method of using trigonometric series in the approximation of the elastic curve, starting from two examples of simple loading (case 1: a beam supported at both ends and loaded in the middle with a concentrated force; case 2: a beam supported at both ends and loaded along its entire length with a uniformly distributed load) and continuing with the presentation of two more examples of more complex loading (case 3: a beam supported at both ends and loaded over half a unit length with a uniformly distributed load; case 4: a beam supported at both ends symmetrically loaded with two concentrated forces) by the Euler - Bernoulli theory.

The following can be concluded from the examples presented in this study:

The analysis of the relative deviations obtained in the two models with simple support and loading (in case 1 it is 1.445% and in the second case 21.19%) shows that the method of replacing the elastic curve with a single sinusoid does not give a satisfactory result. For this reason, the elastic curve for the four support and load cases was additionally expressed by trigonometric series.

To understand the influence of the type of load applied to the bar on the approximation of the differential equation of the elastic curve - relation (2) – using trigonometric series - relation (25) - (over the whole unit length, but also for l/2), the 4 cases studied are divided into two categories: group 1 formed by case 1 and case 4 in which the simply supported beam is loaded using concentrated force/forces (according to Figure 18) and group 2 formed by case 2 and case 3 in which the simply supported beam is loaded using a uniformly distributed load, q(x) (according to Figure 19).

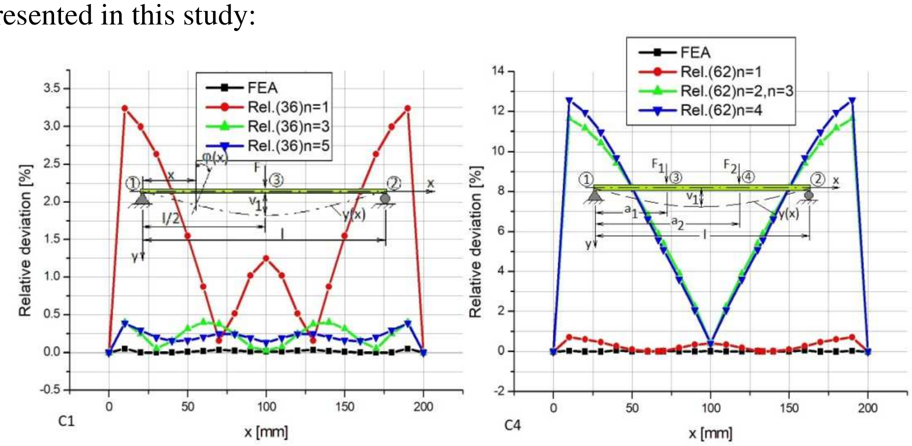

Fig. 18. A comparative representation of the variation diagrams of the relative deviations for the cases of support and loading from group 1.

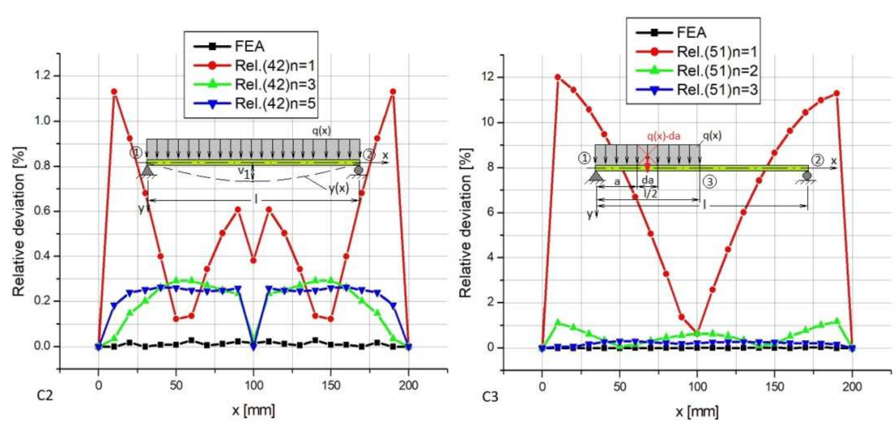

Fig. 19. A comparative representation of the variation diagrams of the relative deviations

for the leaning and loading cases from group 2.

Thus, for study group 1 (Figure 18), it can be seen that in case 1 (C1), the approximation is relevant for n=3 and n=5 for the total unit length and for l/2 in the case of n=3 (0.029 %). When more concentrated forces are applied, as in load case 4 (C4), the best approximation is for n=1 for the entire unit length and for l/2, n=1, 2, 3 and 4 give the same result, with an allowable convergence of 0.394%.

In the situation of study group 2 (Figure 19), it can be observed that in case 2 (C2), the approximation is relevant for n=3 and n=5 for the whole length unit and for l/2 in the case of n=5, for which there is a perfect convergence. In case 3 (C3), a good convergence of the results for n=3 is allowed for both the total length unit and for l/2 with a relative deviation of 0.22%.

Trigonometric series can quickly converge to the original function, meaning that even a truncated approximation with a finite number of terms can provide an accurate representation of the elastic curve under certain conditions.

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APROXIMAREA ECUAŢIEI DIFERENŢIALE A FIBREI MEDII DEFORMATE PRIN SERII TRIGONOMETRICE PENTRU BARE IZOTROPE CU MOMENT DE INERŢIE CONSTANT, ÎNCĂRCATE COMPLEX, ÎN ACORD CU TEORIA EULER – BERNOULLI

Rezumat: Studiul fibrei medii deformate a reprezentat o preocupare continuă în domeniul ingineriei mecanice obţinându-se o serie de metode analitice, grafice şi grafo-analitice de analiză. În această lucrare se prezintă o metodă analitică de studiu care poate fi clasificată ca făcând parte din metodele energetice de calcul a săgeţii prin prisma faptului că se utilizează expresia energiei potenţiale de deformare în cazul barelor solicitate la încovoiere simplă, în domeniul deformaţiilor elastice, utilizând materiale izotrope. Fibra medie deformată, care este exprimată printr-o ecuaţie diferenţială de ordinul patru, poate fi reprezentată şi prin intermediul unei serii trigonometirice infinite care conduce spre o convergenţă bună în raport cu rezultatele obţinute cu ajutorul metodelor clasice. Pornind de la o serie de cazuri simple de încărcare cu ajutorul cărora se exemplifică modul de rezolvare a problemei, în această lucrare sunt studiate două cazuri cu un grad de complexitate mai ridicat: în primul caz bara este încărcată pe prima jumătate de unitate de lungime cu o sarcină uniform distribuită q(x) iar în al doilea caz bara este încărcată prin intermediul a două forţe concentrate dispuse simetric în raport cu punctele de sprijin.

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