



TECHNICAL UNIVERSITY OF CLUJ-NAPOCA

ACTA TECHNICA NAPOCENSIS

Series: Applied Mathematics, Mechanics, and Engineering
Vol. 66, Issue Special II, October, 2023

THE CONTACT PROBLEM IN MULTIPARE CONVEX-CONCAVE PRECESSIONAL GEARING

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Abstract: The mathematical model for calculating the contact loads of the multipair convex-concave precessional gear is developed for each pair of gear teeth with the application of the hypothesis of the existence of the functional relationship in the form of powers between the ratio of the contact loads and the widths of the contact spots for each two adjacent pairs of teeth. The calculation program is developed in the MATHCAD environment. The contact forces per unit length of the teeth, the transmitted moments, the maximum pressures on the contact surfaces, the maximum equivalent stresses for each pair of teeth in contact are calculated. Numerical calculations show that the maximum load is taken by the first pair of teeth, and the minimum by the last. At the same time, the maximum stresses are in the last contact, and minimum in the first.

Key words: precessional gearing, multipair contact, convex-concave profile, mathematical model, system of algebraic equations, equivalent stresses.

1. INTRODUCTION

One of the main features of the precessional transmissions elaborated and developed by I. Bostan [1, 2] is the linear and multipair convex-concave contact of the gear.

The precessional gear can be designed with n pairs of teeth in gearing by modifying the shape of the teeth profiles of the central wheel and the satellite wheel [2].

From a solid mechanics point of view, the bearing capacity of the load distribution in contact depends on the geometry of the teeth in the gear. The main parameters are the radii of curvature of the teeth profiles at the contact points and the differences of the curvatures at these points, especially for the convex-concave profiles of the precessional transmissions [2].

In the case of single contact, the problem of calculating the contact stresses is relatively simple, if we use the assumptions of the Hertz problem [3, 5]: isotropic linear elastic material, the contact forces per unit length of the teeth p (N/m) are considered uniformly distributed and normal to the profiles on the contact lines, the

surfaces in contact are ideal and small in relation to the dimensions of the teeth [3].

Initially, the load per unit length of the teeth in contact is determined from the equilibrium equation:

$$b \cdot p \cdot \rho \cdot \cos \alpha = M_t, \quad (1)$$

Therefore

$$p = \frac{M_t}{b \cdot \rho \cdot \cos \alpha}, \quad (2)$$

where M_t – the torque transmitted by the pinion ($N \cdot m$), b – the tooth length (m), ρ – the distance from the middle contact point of the driving wheel to the wheel axis (m), α – the angle between the load p direction and the normal direction at the contact radius (*degrees*).

Afterwards, if the load p is known, the maximum stresses in contact are calculated [1, 4, 7].

In multipair gears with n pairs of teeth in gearing, due to the variation of contacts geometry (fig. 1) and execution errors, the loads p_i ($i=1, n$) are not uniformly distributed over the pairs of teeth, which are simultaneously in gearing.

tenths or even hundreds of millimeters, a very fine mesh of finite elements is required, which leads to a poor convergence of the results, if the model is not well prepared and is not tested.

Therefore, the problem of load distribution in precession gears with multipair convex-concave contact is actual and important.

3. THE CALCULATION MODEL

The reduced dimensions and global rigidity of the precessional transmission allow to admit that the distribution of contact loads depends only on the materials of the teeth and the set of radii of curvature of the driving wheel and the driven wheel, as well as on the normals to the profiles at the contact points. The compatibility equations can be formed in the following way: between the ratios of the contact loads of two adjacent pairs of teeth and the corresponding ratios of the half-widths of the contact surfaces there is a functional relationship in the form of radicals:

$$\frac{p_i}{p_{i+1}} = \sqrt{\frac{a_i}{a_{i+1}}} \quad (i = \overline{1, n-1}). \quad (5)$$

If E_1, E_2 – the longitudinal moduli of elasticity of the pinion and driven wheel materials (Pa), ν_1 and ν_2 – Poisson's coefficients,

$$E_{r_1} = \frac{2}{\pi \cdot E_1} \cdot (1 - \nu_1^2), \quad E_{r_2} = \frac{2}{\pi \cdot E_2} \cdot (1 - \nu_2^2) -$$

reduced moduli of elasticity ($1/Pa$),

$$R_{m_i} = \frac{2}{\frac{1}{r_i} + \frac{1}{R_i}} - \text{the reduced radii of curvature}$$

at the contact points (m),

$$\lambda_i = R_{m_i} \cdot (E_{r_1} + E_{r_2}) - \text{mechanical-geometrical}$$

factors in the contact points (m/Pa), then the half-widths of the contact surfaces are calculated with the relation (Hertzian approach) [4]:

$$a_i = \sqrt{\lambda_i p_i}. \quad (6)$$

If we substitute (6) in (5) and add (3) we obtain n algebraic equations with respect to the loads per unit length of the teeth in the gear.

For $p_i > 0$ this system has only one analytical solution:

$$p_1 = \frac{2M_t}{b \cdot d_m \cdot \left(n_1 + \sqrt[3]{\frac{\lambda_2}{\lambda_1}} \cdot n_2 + \dots + \sqrt[3]{\frac{\lambda_n}{\lambda_1}} \cdot n_n \right)}, \quad (7)$$

$$p_2 = \sqrt[3]{\frac{\lambda_2}{\lambda_1}} \cdot p_1, \quad p_3 = \sqrt[3]{\frac{\lambda_3}{\lambda_1}} \cdot p_1, \quad \dots, \quad p_n = \sqrt[3]{\frac{\lambda_n}{\lambda_1}} \cdot p_1,$$

where $n_i = \cos \alpha_i$ ($i = \overline{1, n}$), d_m – the median diameter of the pinion (m).

We observe that the contact loads are proportional to the torque and inversely proportional to the length of the teeth in contact, and to the median diameter of the driving wheel.

Afterwards, if the contact loads for each pair of teeth in gear are known, by using the known relationships [4], the widths of the contact surfaces, the maximum pressures, the main stresses, the maximum equivalent stresses according to the von Mises criterion, the kinematic proximity of the teeth pairs. the torques taken by each pair of teeth in the gear can be calculated.

4. THE CALCULATION EXAMPLE

For the precessional transmission with four-pair of teeth in gearing and with the parameters provided by the author of the monograph [1, 2]: torsional moment $M_t = 3 \text{ N}\cdot\text{m}$, the moduli of elasticity of the pinion teeth $E_1 = 2 \cdot 10^5 \text{ MPa}$ and of the driven wheel $E_2 = 2 \cdot 10^5 \text{ MPa}$, the transverse contraction coefficients $\nu_1 = 0.3$ and $\nu_2 = 0.3$, the median diameter of the pinion $d_m = 80 \text{ mm}$, the length of the teeth $b = 11 \text{ mm}$, the radii of curvature in the contact poles located in the centers of the rectangular contact surfaces, for the pinion with the semicircular teeth profile (fig. 1) are $r_i = 6 \text{ mm}$ ($i = \overline{1, 4}$), and for the driven wheel, for the first pair of teeth the radius of curvature is $R_1 = -6.022 \text{ mm}$, for the second pair $R_2 = -6.216 \text{ mm}$, for the third pair $R_3 = -7.4 \text{ mm}$, and for the fourth pair $R_4 = -12.5 \text{ mm}$ (the sign " - " is applied due to the convex-concave nature of the gearing), the angles between the directions of the loads and the normals at the median radii of the pinion for the first pair $\alpha_1 = 37.5^\circ$, for the second pair $\alpha_2 = 19.0^\circ$, for the third pair $\alpha_3 = 17.0^\circ$

and for the fourth pair $\alpha_4=17.5^\circ$, there were calculated:

Reduced modulus of elasticity:

$$E_{r_1} = \frac{2}{\pi \cdot E_1} \cdot (1 - \nu_1^2) = 2,897 \cdot 10^{-12} \frac{1}{Pa}, \quad (8)$$

$$E_{r_2} = E_{r_1}.$$

Reduced contact radii for each pair of teeth in gearing:

$$\begin{aligned} R_{m_1} &= \frac{2}{\frac{1}{r_1} + \frac{1}{R_1}} = 3285 \text{ mm}, \\ R_{m_2} &= \frac{2}{\frac{1}{r_2} + \frac{1}{R_2}} = 345,333 \text{ mm}, \\ R_{m_3} &= \frac{2}{\frac{1}{r_3} + \frac{1}{R_3}} = 63,429 \text{ mm}, \\ R_{m_4} &= \frac{2}{\frac{1}{r_4} + \frac{1}{R_4}} = 23,077 \text{ mm}. \end{aligned} \quad (9)$$

Mechanical-geometrical factors

$$\begin{aligned} \lambda_1 &= R_{m_1} \cdot (E_{r_1} + E_{r_2}) = 1,903 \cdot 10^{-8} \left(\frac{Pa}{mm} \right)^{-1}; \\ \lambda_2 &= R_{m_2} \cdot (E_{r_1} + E_{r_2}) = 2,001 \cdot 10^{-9} \left(\frac{Pa}{mm} \right)^{-1}; \\ \lambda_3 &= R_{m_3} \cdot (E_{r_1} + E_{r_2}) = 3,675 \cdot 10^{-10} \left(\frac{Pa}{mm} \right)^{-1}; \\ \lambda_4 &= R_{m_4} \cdot (E_{r_1} + E_{r_2}) = 1,337 \cdot 10^{-10} \left(\frac{Pa}{mm} \right)^{-1}. \end{aligned} \quad (10)$$

The direction cosines of the normals to the teeth profiles at the contact points:

$$n_1 = \cos \alpha_1 = 0,793; \quad n_2 = \cos \alpha_2 = 0,946; \quad (11)$$

$$n_3 = \cos \alpha_3 = 0,956; \quad n_4 = \cos \alpha_4 = 0,954.$$

The dimensionless factor ω that takes into account all the geometric and mechanical parameters that characterize the vicinity of the contact surfaces is

$$\begin{aligned} \omega &= n_1 + \sqrt[3]{\frac{\lambda_2}{\lambda_1}} \cdot n_2 + \sqrt[3]{\frac{\lambda_3}{\lambda_1}} \cdot n_3 \\ &+ \sqrt[3]{\frac{\lambda_4}{\lambda_1}} \cdot n_4 = 1,679. \end{aligned} \quad (12)$$

Then, the contact loads for each pair of teeth in contact according to (7):

$$\begin{aligned} p_1 &= \frac{2Mt}{b \cdot d_m \cdot \omega} = 4,061 \frac{N}{mm}; \\ p_2 &= \sqrt[3]{\frac{\lambda_2}{\lambda_1}} \cdot p_1 = 1,917 \frac{N}{mm}; \\ p_3 &= \sqrt[3]{\frac{\lambda_3}{\lambda_1}} \cdot p_1 = 1,09 \frac{N}{mm}; \\ p_4 &= \sqrt[3]{\frac{\lambda_4}{\lambda_1}} \cdot p_1 = 0,778 \frac{N}{mm}. \end{aligned} \quad (13)$$

The maximum load is carried out in the first pair of teeth, and the minimum is in the last. The maximum load is about 5 times higher than the minimum load.

The half-widths of the contact surfaces:

$$\begin{aligned} a_1 &= \sqrt{\lambda_1 \cdot p_1} = 0,278 \text{ mm}; \\ a_2 &= \sqrt{\lambda_2 \cdot p_2} = 0,062 \text{ mm}; \\ a_3 &= \sqrt{\lambda_3 \cdot p_3} = 0,02 \text{ mm}; \\ a_4 &= \sqrt{\lambda_4 \cdot p_4} = 0,01 \text{ mm}. \end{aligned} \quad (14)$$

The maximum pressures on the contact surfaces [4]:

$$\begin{aligned} q1_{\max} &= 2 \cdot \frac{p_1}{\pi \cdot a_1} = 9,3 \text{ MPa}; \\ q2_{\max} &= 2 \cdot \frac{p_2}{\pi \cdot a_2} = 19,71 \text{ MPa}; \\ q3_{\max} &= 2 \cdot \frac{p_3}{\pi \cdot a_3} = 34,67 \text{ MPa}; \\ q4_{\max} &= 2 \cdot \frac{p_4}{\pi \cdot a_4} = 48,56 \text{ MPa}. \end{aligned} \quad (15)$$

The maximum pressure is in the last contact, and the minimum pressure is in the first.

It is known that in the Flamant problem of action of a concentrated force normal to the half-plane [4], the maximum stresses are on the line of action of the force. We place the origins of the orthogonal Cartesian coordinate systems in the centers of the contact surfaces, we orient the z axes in the directions parallel to the large sides of the rectangular contact surfaces, and the y axes in the directions normal to these surfaces.

With these notations, the laws of pressure variation on the widths x of the contact surfaces will be [4]:

$$\begin{aligned} q1(x) &= \frac{q1_{\max}}{a_1} \cdot \sqrt{a_1^2 - x^2}; \\ q2(x) &= \frac{q2_{\max}}{a_2} \cdot \sqrt{a_2^2 - x^2}; \\ q3(x) &= \frac{q3_{\max}}{a_3} \cdot \sqrt{a_3^2 - x^2}; \\ q4(x) &= \frac{q4_{\max}}{a_4} \cdot \sqrt{a_4^2 - x^2}. \end{aligned} \quad (16)$$

The results of the calculations are presented in figures 2-5 .

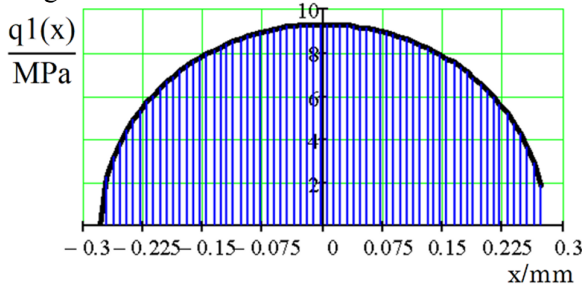


Fig.2. Pressure variation across 1st teeth contact width

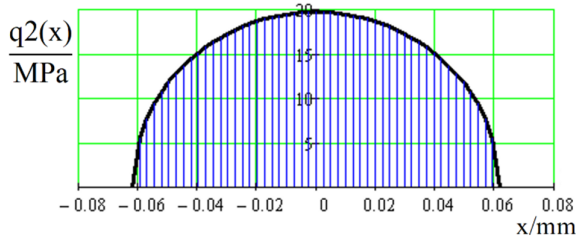


Fig.3. Pressure variation across 2nd teeth contact width

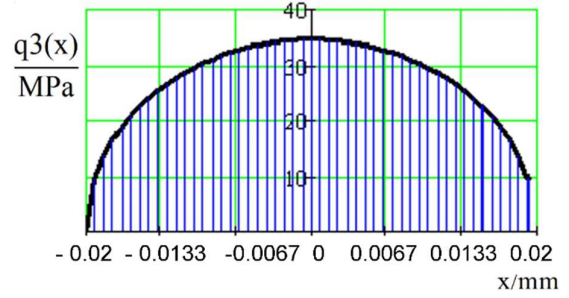


Fig.4. Pressure variation across 3rd teeth contact width

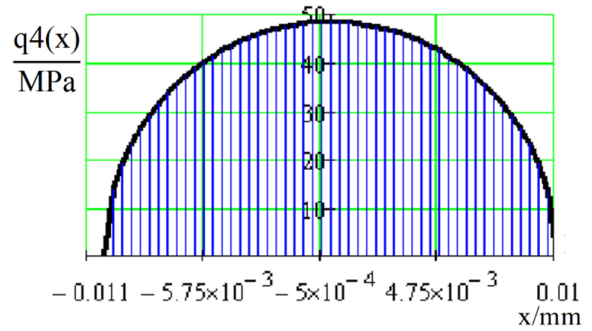


Fig.5. Pressure variation across 4th teeth contact width

The states of stress in the vicinity of the contact surfaces are spatial.

For the y axis points of the chosen coordinate system, the tangential stresses $\tau_{xy} = \tau_{yx} = 0$ and $\tau_{zx} = \tau_{xz} = 0$. So, the normal stresses $\sigma_{xx}, \sigma_{yy}, \sigma_{zz}$ are the main stresses notated $\sigma_1, \sigma_2, \sigma_3$ and are calculated with the relations [4]:

$$\begin{aligned} \sigma_1(y) &= -q_{i_{\max}} \cdot \left[\frac{1 + 2 \left(\frac{y}{a_i} \right)^2}{\sqrt{1 + \left(\frac{y}{a_i} \right)^2}} - 2 \cdot \frac{y}{a_i} \right]; \\ \sigma_2(y) &= -q_{i_{\max}} \cdot \frac{1}{\sqrt{1 + \left(\frac{y}{a_i} \right)^2}}, \quad i = \overline{1, 4}; \\ \sigma_3(y) &= -2 \cdot \nu \cdot q_{i_{\max}} \cdot \left[\sqrt{1 + \left(\frac{y}{a_i} \right)^2} - \frac{y}{a_i} \right]. \end{aligned} \quad (17)$$

The results are presented in Figures 6 – 21.

For the first pair of teeth in gearing (fig. 6 – 9):

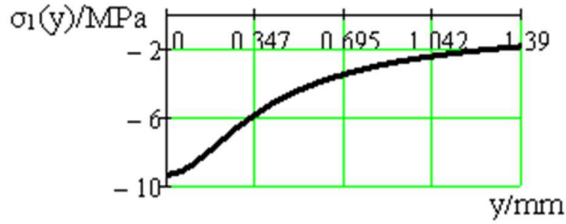


Fig.6. Variation of the main stress σ_1 (pair 1)

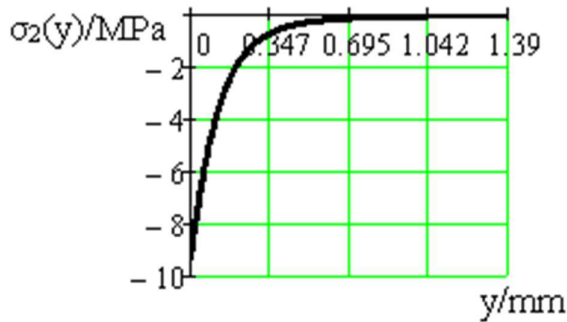


Fig.7. Variation of the main stress σ_2 (pair 1)

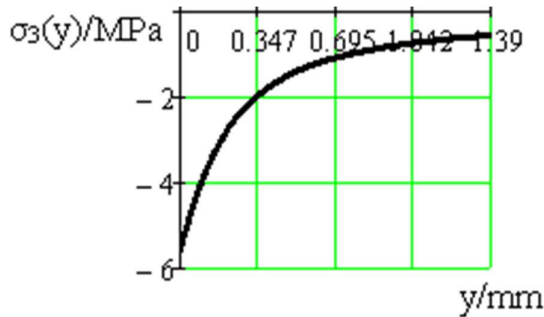


Fig.8. Variation of main voltage σ_3 (pair 1)

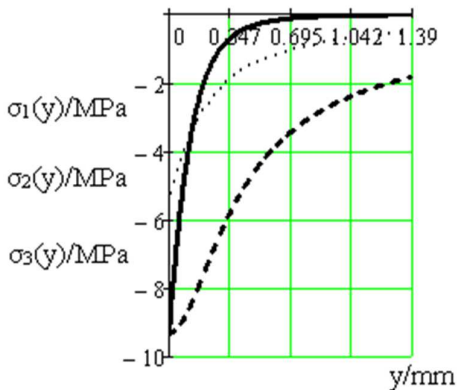


Fig.9. Variations of the main stresses $\sigma_1, \sigma_2, \sigma_3$ (pair 1)

For the second pair of teeth in gearing (fig. 10 – 13):

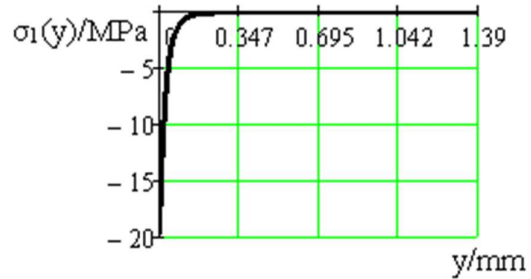


Fig.10. Variation of the main stress σ_1 (pair 2)

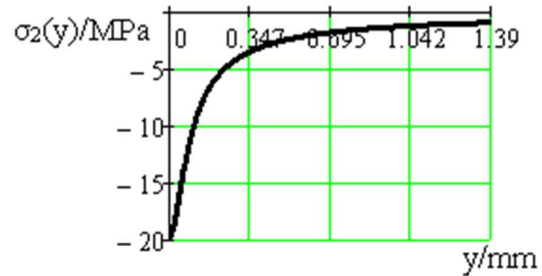


Fig.11. Variation of the main stress σ_2 (pair 2)

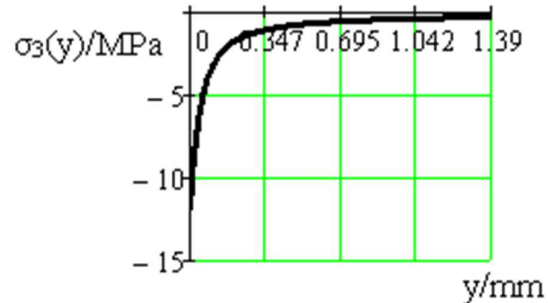


Fig.12. Variation of the main stress σ_3 (pair 2)

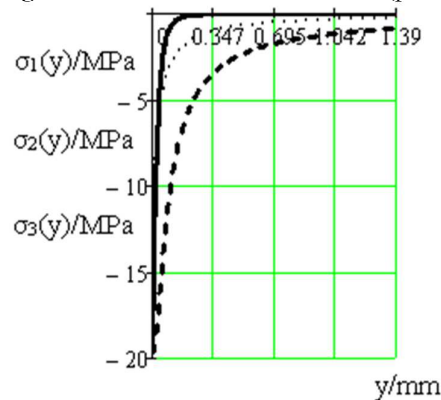


Fig.13. Variations of the main stresses $\sigma_1, \sigma_2, \sigma_3$ (pair 2)

Fig.17. Variations of the main stresses $\sigma_1, \sigma_2, \sigma_3$ (pair 3)

For the third pair of teeth in gearing (fig. 14 – 17):

For the fourth pair of teeth in gearing (fig. 18 – 21):

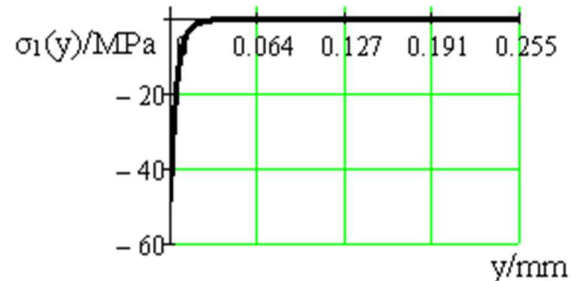
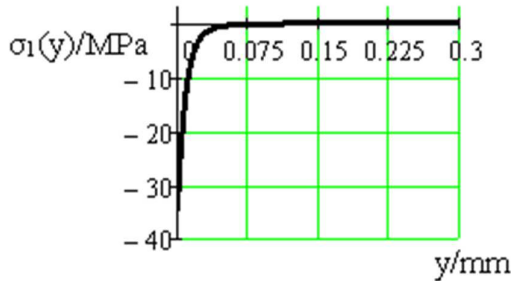


Fig.14. Variation of the main stress σ_1 (pair 3)

Fig.18. Variation of the main stress σ_1 (pair 4)

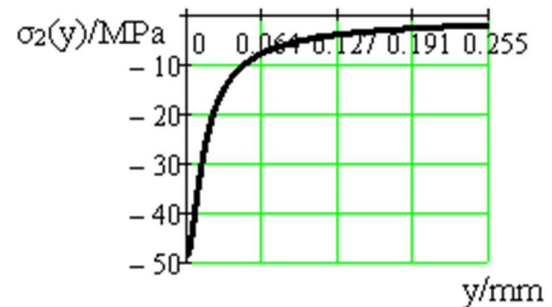
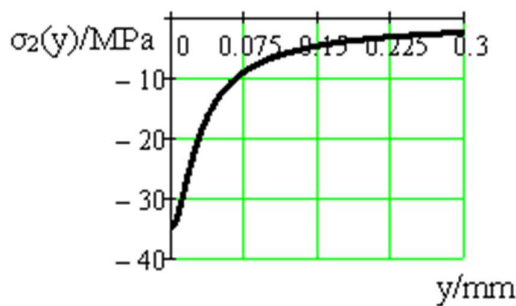


Fig.15. Variation of the main stress σ_2 (pair 3)

Fig.19. Variation of main stress σ_2 (pair 4)

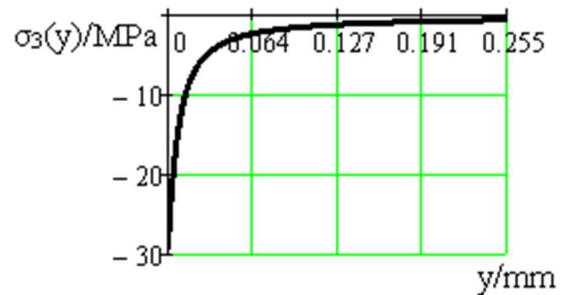
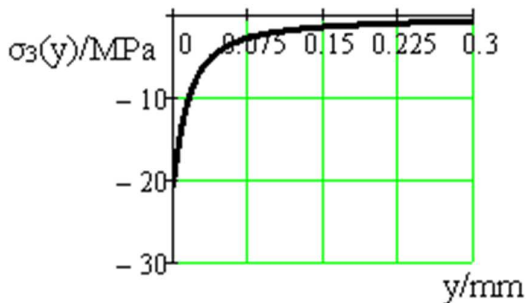
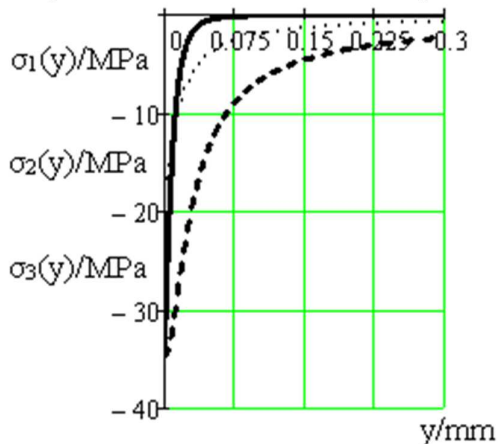


Fig.16. Variation of the main stress σ_3 (pair 3)

Fig.20. Variation of the main stress σ_3 (pair 4)



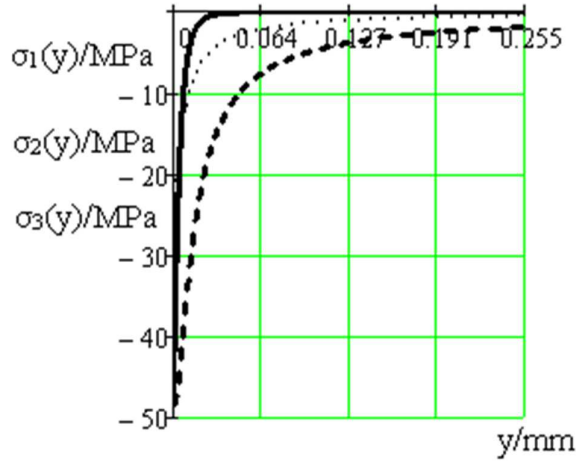


Fig.21. Variations of the main stresses σ_1 , σ_2 , σ_3 (pair 4)

All main stresses are negative and rapidly decreasing. So, the states of stress at the points of the normals to the profiles of the gearing teeth that pass through the centers of the contact surfaces are of spatial compression.

According to the von Mises criterion the equivalent stresses are calculated with the relation:

$$\sigma_e = \frac{1}{\sqrt{2}} \cdot \sqrt{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2}, \quad (18)$$

the variation of which in the direction of the mentioned normals is presented in figures 22 – 26.

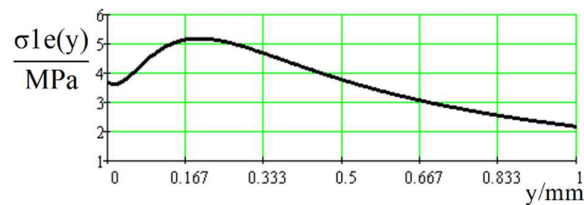


Fig.22. The equivalent stresses for the first pair of teeth in the gear

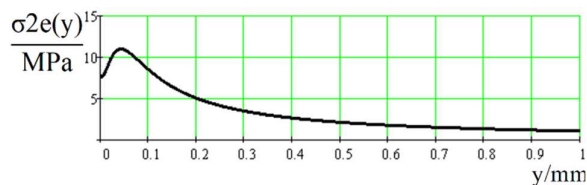


Fig.23. The equivalent stresses for the second pair of teeth in the gear

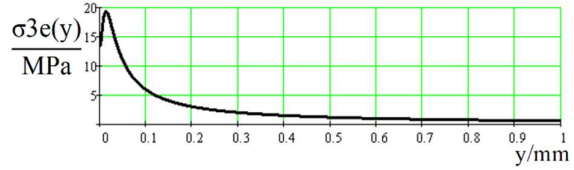


Fig.24. The equivalent stresses for the third pair of teeth in the gear

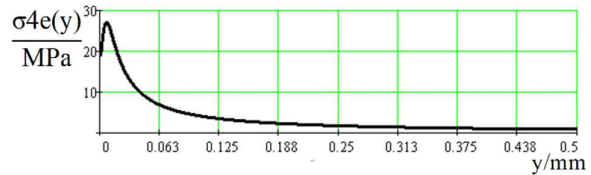


Fig.25. The equivalent stresses for the fourth pair of teeth in the gear

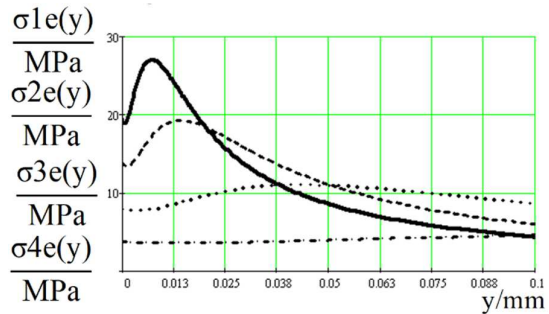


Fig.26. The equivalent stresses at the contact points of the teeth in the gear

The forces taken by the teeth of the pinion in gearing:

$$F_1 = p_1 \cdot b = 44,7 N; \quad F_2 = p_2 \cdot b = 21,1 N; \quad (19)$$

$$F_3 = p_3 \cdot b = 12,0 N; \quad F_4 = p_4 \cdot b = 8,6 N.$$

The moments taken by the pinion teeth in report to the pinion axis:

$$\begin{aligned} M_{t_1} &= p_1 \cdot \frac{d_m}{2} \cdot b \cdot n_1 = 1,42 N \cdot m; \\ M_{t_2} &= p_2 \cdot \frac{d_m}{2} \cdot b \cdot n_2 = 0,80 N \cdot m; \\ M_{t_3} &= p_3 \cdot \frac{d_m}{2} \cdot b \cdot n_3 = 0,46 N \cdot m; \\ M_{t_4} &= p_4 \cdot \frac{d_m}{2} \cdot b \cdot n_4 = 0,33 N \cdot m. \end{aligned} \quad (20)$$

The weights of the moments taken by the pinion teeth in relation to the torsional moment on its axis.

$$m_1 = \frac{M_{t_1}}{M_t} = 47,3\%; \quad m_2 = \frac{M_{t_2}}{M_t} = 26,6\%;$$

$$m_3 = \frac{M_{t_3}}{M_t} = 15,3\%; \quad m_4 = \frac{M_{t_4}}{M_t} = 10,9\%. \quad (21)$$

The distribution diagrams of the main characteristics of the investigated precessional transmission are presented in Figures 27 – 30.

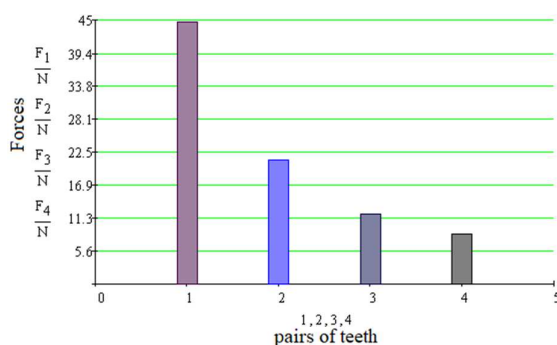


Fig.27. Distribution of the forces taken by the pairs of teeth in gearing

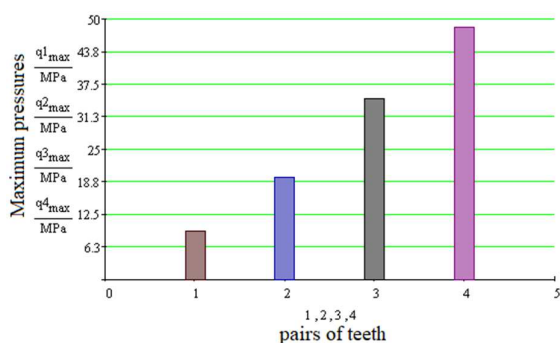


Fig.28. Distribution of maximum pressures on pairs of teeth in gearing

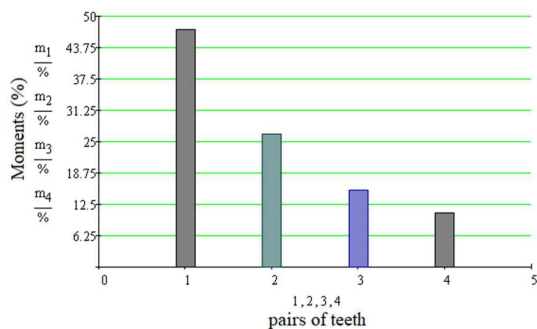


Fig.29. Distribution of the moments taken by the pairs of teeth in gearing (%)

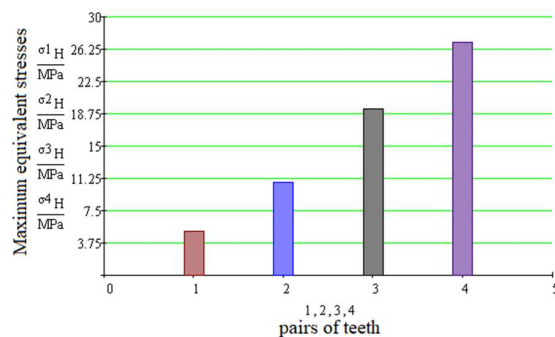


Fig.30. The maximum equivalent stresses in pairs of teeth in gearing

5. CONCLUSION

1. The mathematical model for calculating the distribution of contact loads in precessional transmissions with multipair convex-concave gearing has been developed.
2. The analytical solution of the resulting system of equations has been obtained.
3. The analytical solution obtained for n pairs of teeth in contact, in the case of a single pair of teeth, coincides with the classical solution in the Hertzian approach, which confirms the applicability of the proposed mathematical model.
4. The calculations for the transmission with four pairs of teeth in gearing have been performed and the obtained results have been analyzed.
5. The maximum equivalent stress 27.06 MPa is achieved in the last pair of teeth in the gear (fig. 25) and is at a depth of 0.0074 mm . For steels with an elastic limit of about 250 MPa , the safety factor will be 9.23 units, which ensures good operation of the transmission.
6. The maximum pressure 48.56 MPa is achieved in the last pair of teeth (fig. 28) and is higher than the corresponding maximum equivalent stress (fig. 25). This fact is explained by the almost spherical state of stress at the points of the contact surface.

7. The force taken by the first pair of teeth is maximum (fig. 27) having value equals to 44.7 N, and the force taken by the last pair of teeth is minimum and is 8.6 N.
8. The first pair of teeth takes about 50% of the torsional moment transmitted by the pinion (fig. 29).
9. The widths of the contact surfaces vary between 0.556 mm for the first contact and 0.02 mm for the last contact (fig. 2 – 5). This explains the fact that the maximum loads are in the first contact, and the maximum stresses are in the last contact.
10. The compatibility equations proposed for solving the multipair convex-concave contact problem in precessional transmissions can be modified to satisfy more adequately the experimental results.

Acknowledgements: The article was published by the authors by conducting scientific research under the State Research-Innovation Project, No. 160-PS of 31.01.2020 "Increasing the competitiveness of precessional transmissions by developing and capitalizing on the gear with conform contact of the teeth and expanding their application area". Project number 20.80009.700.24, dated 31.01.2020. Project leader - Acad. Ion Bostan.

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Problema contactului în angrenarea precesională convex-concavă multipară

Este elaborat modelul matematic de calcul al sarcinilor de contact ale angrenării precesionale convex-concave multipare pentru fiecare pereche de dinți angrenați cu aplicarea ipotezei de existență a relației funcționale în formă de puteri între raportul sarcinilor de contact și al lățimilor petelor de contact pentru fiecare două perechi adiacente de dinți. Este elaborat programul de calcul în mediul MATHCAD. Sunt calculate forțele de contact pe unitatea de lungime a dinților, momentele transmise, presiunile maxime pe suprafețele de contact, tensiunile echivalente maxime pentru fiecare pereche de dinți în contact. Calculele numerice arată că sarcina maximală este preluată de prima pereche de dinți, iar minimală de ultima. În același timp, tensiunile maxime sunt în ultimul contact, iar minime în primul.

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