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ON THE APPLICATION OF THE THEOREM OF THE ANGULAR MOMENTUM

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Abstract: *The angular momentum theorem for a system of material points has the same classical form if applied relative to a fixed point, or relative to the center of mass (König's angular momentum theorem). This paper aims to determine if there are other points relative to which the angular momentum theorem has the same form as the one obtained when applied relative to a fixed point. The necessary conditions to be fulfilled by these points have been established.*

Key words: *theorem of the angular momentum, center of mass, pole of accelerations, instantaneous center of rotation.*

1. INTRODUCTION

It is known that, in the case of material point systems, the theorem of the angular momentum is applied in the form

$$\frac{d\vec{K}}{dt} = \sum \vec{r}_i \times \vec{F}_i \qquad (1)$$

about a fixed point or about the center of mass [1], [2]. In special cases, in motion of a rigid body parallel to a fixed plane, it can be applied in the form (1) also about the instantaneous center of rotation [3].

Interesting if there are other points about which the theorem of the angular momentum is applied in form (1).

2. THEORETICAL CONSIDERATION

It is considered a material point system $A_i(i=1,...,n)$ (figure 1) of masses respectively $m_i(i=1,...,n)$. Let us consider:

- $O_1x_1y_1z_1$ a fixed cartesian frame;
- $Cx_Cy_Cz_C$ a movable cartesian frame having its origin at the center of mass C and his axes parallel to the axes of the fixed frame;
- $Pxyz$ a movable cartesian frame having its origin at the point P and his axes parallel to the axes of the fixed frame.

Taking into account the notations in the figure 1, one can write the relationship:

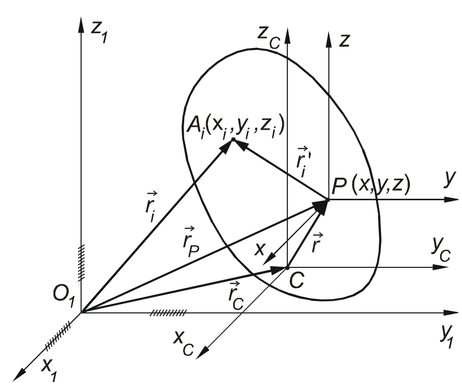

Fig. 1 Theoretical consideration

$$\vec{r}_i = \vec{r}_p + \vec{r}_i' \qquad (2)$$

Differentiating with respect to the time it follows that

$$\vec{v}_i = \vec{v}_p + \vec{v}_i' \qquad (3)$$

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The angular momentum about the fixed point O will be successively:

$$\begin{aligned}\vec{K}_{O_1} &= \sum \vec{r}_i \times m_i \vec{v}_i = \sum[(\vec{r}_P + \vec{r}_i')\times m_i(\vec{v}_P + \vec{v}_i')] = \\ &= \sum \vec{r}_P \times m_i \vec{v}_p + \sum \vec{r}_P \times m_i \vec{v}_i' + \sum \vec{r}_i' \times m_i \vec{v}_P + \\ &+ \sum \vec{r}_i' \times m_i \vec{v}_i' = \vec{r}_P \times \vec{v}_P\left(\sum m_i\right) + \vec{r}_P \times \left(\sum m_i \vec{v}_i'\right) + \\ &+ \left(\sum m_i \vec{r}_i'\right) \times \vec{v}_P + \sum \vec{r}_i' \times m_i \vec{v}_i' = \\ &= \vec{r}_P \times \left[\left(\sum m_i\right)\vec{v}_P + \sum m_i \vec{v}_i'\right] + \left(\sum m_i \vec{r}_i'\right) \times \vec{v}_P + \\ &+ \sum \vec{r}_i' \times m_i \vec{v}_i' = \vec{r}_P \times \sum\left(m_i \vec{v}_P + m_i \vec{v}_i'\right) + \\ &+ \left(\sum m_i \vec{r}_i'\right) \times \vec{v}_P + \sum \vec{r}_i' \times m_i \vec{v}_i' = \\ &= \vec{r}_P \times \sum m_i\left(\vec{v}_P + \vec{v}_i'\right) + \left(\sum m_i \vec{r}_i'\right) \times \vec{v}_P + \sum \vec{r}_i' \times m_i \vec{v}_i'.\end{aligned}$$

Taking into account the relation (3), the expression of the angular momentum about the fixed point O_1 becomes:

$$\begin{aligned}\vec{K}_{O_1} &= \vec{r}_p \times \sum m_i \vec{v}_i + \\ &+ \left(\sum m_i \vec{r}_i'\right) \times \vec{v}_P + \sum \vec{r}_i' \times m_i \vec{v}_i'\end{aligned} \tag{4}$$

But:

$\sum m_i \vec{v}_i = m\vec{v}_C$ is the linear momentum for a system of material points where

$$m = \sum m_i \tag{5}$$

is the mass of the system and $\vec{v}_C$ is the velocity of the center of mass;

$\overrightarrow{PC} = \dfrac{\sum m_i \vec{r}_i'}{\sum m_i}$ is the position vector of the center of mass, so

$$\sum m_i \vec{r}_i' = m\overrightarrow{PC} \tag{6}$$

$$\vec{K}_P = \sum \vec{r}_i' \times m_i \vec{v}_i' \tag{7}$$

is the angular momentum in relative motion about the point P.

Replacing in (4) it results

$$\vec{K}_{O_1} = \vec{r}_p \times m\vec{v}_C + m\overrightarrow{PC} \times \vec{v}_P + \vec{K}_P' \tag{8}$$

Taking into account the relations (2) and (8), the theorem of the angular momentum about the fixed point O_1,

$$\frac{d\vec{K}_{O_1}}{dt} = \sum \vec{r}_i \times \vec{F}_i\,, \tag{9}$$

becomes

$$\begin{aligned}&\frac{d}{dt}\left(\vec{r}_P \times m\vec{v}_C + m\overrightarrow{PC} \times \vec{v}_P + \vec{K}'_P\right) = \\ &= \sum\left(\vec{r}_P + \vec{r}_i'\right) \times \vec{F}_i\end{aligned} \tag{10}$$

But $\overrightarrow{PC} = \vec{r}_C - \vec{r}_P$ from where, by differentiating, it results

$$\frac{d\overrightarrow{PC}}{dt} = \vec{v}_C - \vec{v}_P \tag{11}$$

By replacing (11) in (10) it is obtained:

$$\begin{aligned}&\frac{d}{dt}\left(\vec{r}_P \times m\vec{v}_C + m(\vec{r}_C - \vec{r}_P) \times m\vec{v}_P + \vec{K}'_P\right) = \\ &= \sum\left(\vec{r}_P + \vec{r}_i'\right) \times \vec{F}_i\end{aligned} \tag{12}$$

or

$$\begin{aligned}&\frac{d}{dt}\left(\vec{r}_P \times m\vec{v}_C + \vec{r}_C \times m\vec{v}_P - \vec{r}_P \times m\vec{v}_P + \vec{K}'_P\right) = \\ &= \sum \vec{r}_P \times \vec{F}_i + \sum \vec{r}_i \times \vec{F}_i\end{aligned} \tag{13}$$

By making the calculations is obtained

$$\begin{aligned}&\vec{v}_P \times m\vec{v}_C + \vec{r}_P \times m\vec{a}_C + \vec{v}_C \times m\vec{v}_P + \\ &+ \vec{r}_C \times m\vec{a}_P - \vec{v}_P \times m\vec{v}_P - \vec{r}_P \times m\vec{a}_P + \\ &+ \frac{d}{dt}\left(\vec{K}'_P\right) = \vec{r}_P \times \sum \vec{F}_i + \sum \vec{r}_i \times \vec{F}_i\end{aligned} \tag{14}$$

But

$$m\vec{a}_C = \sum \vec{F}_i \tag{15}$$

the theorem of the motion of the center of mass and

$$\vec{v}_P \times m\vec{v}_P = 0 \tag{16}$$

because the vectors are parallel.

By replacing (15) and (16) in (14) it is obtained

$$\begin{aligned}&\vec{v}_P \times m\vec{v}_C + \vec{v}_C \times m\vec{v}_P + \vec{r}_C \times m\vec{a}_P - \\ &- \vec{r}_P \times m\vec{a}_P + \frac{d}{dt}\left(\vec{K}'_P\right) = \sum \vec{r}_i \times \vec{F}_i\end{aligned} \tag{17}$$

But

$$\vec{v}_P \times m\vec{v}_C = -\vec{v}_C \times m\vec{v}_P\,. \tag{18}$$

Substituting (18) into (17) it results

$$\vec{r}_C \times m\vec{a}_P - \vec{r}_P \times m\vec{a}_P + \frac{d}{dt}\left(\vec{K}'_P\right) = \sum \vec{r}_i \times \vec{F}_i$$

or

$$\left(\vec{r}_C - \vec{r}_P\right) \times m\vec{a}_P + \frac{d}{dt}\left(\vec{K}'_P\right) = \sum \vec{r}_i \times \vec{F}_i \tag{19}$$

Because $\vec{r}_C - \vec{r}_P = \overrightarrow{PC}$, it results:

$$\overrightarrow{PC} \times m\vec{a}_P + \frac{d}{dt}\left(\vec{K}'_P\right) = \sum \vec{r}_i \times \vec{F}_i \tag{20}$$

We are interested in the conditions under which the theorem of the angular momentum can be applied about the point P in the form:

$$\frac{d\vec{K}'_P}{dt} = \sum \vec{r}'_i \times \vec{F}_i . \qquad (21)$$

For this it is necessary that:

$$\overrightarrow{PC} \times m\vec{a}_P = 0 . \qquad (22)$$

The following situations arise:

a) $\overrightarrow{PC} = 0$, so $P \equiv C$. The P point coincides with the center of mass. This is already known: $\dfrac{d\vec{K}_C}{dt} = \sum \vec{M}_C\left(\vec{F}_i\right)$

b) $\vec{a}_P = 0$, so $P \equiv J$. Point P coincides with the pole of accelerations J. Also this is already known. But the determination of the position of the pole of accelerations is more difficult. Sometimes, in some problems, it is known that the velocity vector of a point is constant during the motion.

c) $\overrightarrow{PC} \parallel \vec{a}_P$, that is, the vector $\overrightarrow{CP}$ is collinear with $\vec{a}_P$.

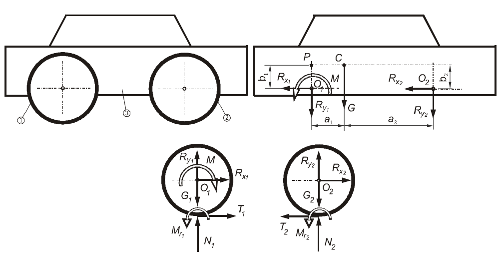

Fig. 2. Example 1

For example, for the system in Figure 2, if for the rigid body 3 the theorem of the angular momentum about the point P (which satisfies condition c)) is applied, we obtain the equation

$$R_{x1} \cdot b_1 + G \cdot a_1 + R_{x2} \cdot b_2 + R_{y2} \cdot (a_1 + a_2) - M = 0$$

containing only 3 unknowns compared to equation

$$R_{x1} \cdot b_1 - R_{y1} \cdot a_1 + R_{x2} \cdot b_2 + R_{y2} \cdot a_2 - M = 0$$

which contains 4 unknowns and is obtained by applying the theorem of the angular momentum about the center of mass C.

And for the system in Figure 3, if the theorem of the angular momentum about the point P (which fulfills the condition c) is applied for the rigid body 2, an equation is obtained with fewer unknowns than if we applied the theorem of the angular momentum about the center of mass C.

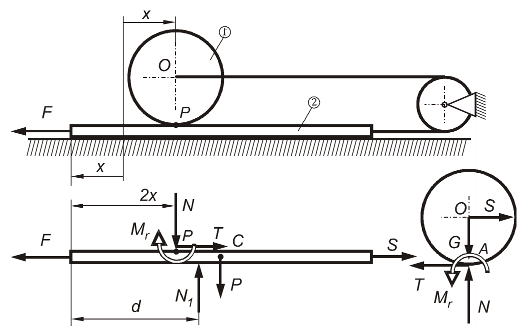

Fig.3. Example 2

Condition (22) is more general. It also includes the condition for applying the theorem of the angular momentum about the instantaneous center of rotation, I, in the case of motion of a rigid body parallel to a fixed plane. The demonstration is immediate.

Let us consider that the origin of the movable cartesian frame is in the center of mass C and the axis Cz is perpendicular to the fixed plane. Under these circumstances, with the notations below, one can write:

$$\overrightarrow{CI} = \xi\vec{i} + \eta\vec{j}\,;\ \vec{\omega} = \omega\vec{k}\,;\ \vec{v}_I = 0 = \vec{v}_C + \vec{\omega} \times \overrightarrow{CI}\,;$$

$$\vec{\omega} \times \overrightarrow{CI} = -\vec{v}_C\,;\ \vec{v}_C = v_{Cx}\vec{i} + v_{Cy}\vec{j}\,;$$

$$\vec{a}_C = a_{Cx}\vec{i} + a_{Cy}\vec{j}\,;\ \vec{a}_I = \vec{a}_C + \vec{\varepsilon} \times \overrightarrow{CI} + \vec{\omega} \times \vec{\omega} \times \overrightarrow{CI}$$

$$\text{or} \quad \vec{a}_I = \vec{a}_C + \vec{\varepsilon} \times \overrightarrow{CI} - \vec{\omega} \times \vec{v}_C .$$

The condition (22), concerning the instantaneous center of rotation, I, can also be written:

$$\overrightarrow{IC} \times \vec{a}_I = 0 \ \text{or} \ \overrightarrow{CI} \times \vec{a}_I = 0 ,$$

which leads to

$$\overrightarrow{CI} \times \vec{a}_I = \overrightarrow{CI} \times \left(\vec{a}_C + \vec{\varepsilon} \times \overrightarrow{CI}\right) - \overrightarrow{CI} \times (\vec{\omega} \times \vec{v}_C) = 0$$

from which the condition results

$$\overrightarrow{CI}\times\left(\overrightarrow{a_C}+\overrightarrow{\varepsilon}\times\overrightarrow{CI}\right)=0$$

This relationship, through successive processing, leads to:

$$\overrightarrow{CI}\times\left(\overrightarrow{a_C}+\vec{\varepsilon}\times\overrightarrow{CI}\right)=\left(\xi\, a_{Cy}+\varepsilon\xi^2-\eta a_{Cx}+\varepsilon\eta^2\right)\vec{k}=$$
$$=\left[\xi\left(a_{Cy}+\varepsilon\xi\right)-\eta\left(a_{Cx}-\varepsilon\eta\right)\right]\vec{k}=$$
$$=\left[\xi\left(\frac{\omega a_{Cy}-\varepsilon v_{Cy}}{\omega}\right)-\eta\left(\frac{\omega a_{Cx}-\varepsilon v_{Cx}}{\omega}\right)\right]\vec{k}=$$
$$=-\omega\left[\xi\left(-\frac{\omega a_{Cy}-\varepsilon v_{Cy}}{\omega^2}\right)+\eta\left(\frac{\omega a_{Cx}-\varepsilon v_{Cx}}{\omega^2}\right)\right]\vec{k}=$$
$$=-\omega\left[\xi\frac{d}{dt}\left(-\frac{v_{Cy}}{\omega}\right)+\eta\frac{d}{dt}\left(\frac{v_{Cx}}{\omega}\right)\right]\vec{k}=$$
$$=-\frac{\omega}{2}\left(2\xi\dot{\xi}+2\eta\dot{\eta}\right)\vec{k}=-\frac{\omega}{2}\frac{d}{dt}\left(\xi^2+\eta^2\right)\vec{k}=$$
$$=-\frac{\omega}{2}\frac{d}{dt}\left(CI^2\right)\vec{k}=0$$

from which the condition results *CI=constant* [3].

3. CONCLUSION

The theorem of the angular momentum is applied in known form (1) about a fixed point or about the center of mass. It has been shown in this paper that there are still other points about which this theorem can be applied in the same form. These points must, however, meet certain conditions (22). Applying the theorem of the angular momentum about these particular points may be useful in problems because the obtained equations may be simpler.

8. REFERENCES

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Asupra aplicării teoremei momentului cinetic

Rezumat: Teorema momentului cinetic pentru un sistem de puncte materiale are aceeași formă clasică dacă se aplică în raport cu un punct fix, sau în raport cu centrul de masă (teorema lui König pentru momentul cinetic). Această lucrare își propune să determine dacă există și alte puncte in raport cu care teorema momentului cinetic are aceeași formă ca cea obținută la aplicarea în raport cu un punct fix. S-au stabilit condițiile necesare a fi îndeplinite de aceste puncte.

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