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ABOUT LOCAL AND VOLUME DISPLACEMENTS OF HIGHLY COMPLIANT SPHERES

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Abstract: *The paper presents an evaluation of displacements for rubber spheres in compression experiments for: two spheres compressed between two planes and parallel surfaces, two spheres compressed between two bodies with hemispherical cavities and one sphere compressed between two planes and parallel surfaces of rigid bodies.*
Theoretical and numerical considerations on the experimental displacement of the rubber spheres are presented, and the difference between the local displacements and the volume displacements is assessed.
During these experiments, three parameters were recorded: the normal load, the normal approach of the contacting spheres, and the relative contact area between spheres or between one sphere and the contact plane.
From these results it can be concluded that: the rubber spheres exhibit large local and volume displacements, which increase with the load level; the local displacements increase with load faster than volume displacements, but both types of displacements have rates that diminish with the increasing load.

Key words: *rubber sphere, contact displacements, volume displacements.*

1. INTRODUCTION

The loading of a mechanical part has the effect of generating states of stresses and displacements in the volume of material. These two states are correlated by the law of response of the material to external stress.

From a practical point of view, the presence of stresses can lead to reaching the resistance capacity of the material and the presence of displacements can particularly affect the functioning of machine elements by changing their shape and/or relative position. For example, contact displacements of balls and bearing rings can alter the value of the gap in electric motors and their efficiency. Another example, where the presence of displacements is very important to manage, are the technological devices, where the displacements are reflected in positioning and basing errors.

2. THEORETICAL CONSIDERATIONS

In the case of linear elastic behavior, described by Hooke's law, one can write:

$$\sigma = E \cdot \varepsilon \tag{1}$$

where,
$\sigma = \frac{F}{A}$, E– Young's modulus, F – force, A – sectional area, $\varepsilon = \frac{\Delta l}{l_o}$, Δl - displacement, l_0 – initial length.

If the load is not uniform throughout the volume of the part, varying from point to point, Hooke's law can be written in the generalized form, namely:

$$\begin{aligned} \sigma_x &= 2G\varepsilon_x + \lambda\varepsilon_v \quad & \tau_{xy} &= G\gamma_{xy} \\ \sigma_y &= 2G\varepsilon_y + \lambda\varepsilon_v \quad & \tau_{yz} &= G\gamma_{yz} \\ \sigma_z &= 2G\varepsilon_z + \lambda\varepsilon_v \quad & \tau_{zx} &= G\gamma_{zx} \end{aligned} \tag{2}$$

with $\varepsilon_v = \varepsilon_x + \varepsilon_y + \varepsilon_z = \frac{1-2\nu}{E}(\sigma_x + \sigma_y + \sigma_z)$, $G = \frac{E}{2(1+\nu)}$, $\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)}$, ν -Poisson coefficient.

These relationships show that local displacements of the material differ from point to point, depending on the stress state and Young's modulus values.

In the case of rubbers and polymers Young's modulus is not a constant value and is correlate with the local stress level.

Analyzing the displacement of mechanical bodies from a macroscopic point of view, two types of displacements can be defined:

Volume displacement, δ_v, caused by the action of distributed stress throughout the volume. If the stress is compressional and uniformly applied to a part of constant cross-section, the volume displacement in the axial direction is defined as the difference between the initial dimension, l_0, and the final dimension, l. This displacement is uniform in the direction of load, the value of the volume displacement is $\delta_v = \Delta l$, and the relative displacement ε is constant provided A and E are constants.

However, this volume displacement is insufficiently defined by absolute or relative shortening because, as it can be seen from figure 1, the preservation of volume causes swelling that compensates for the reduction in length.

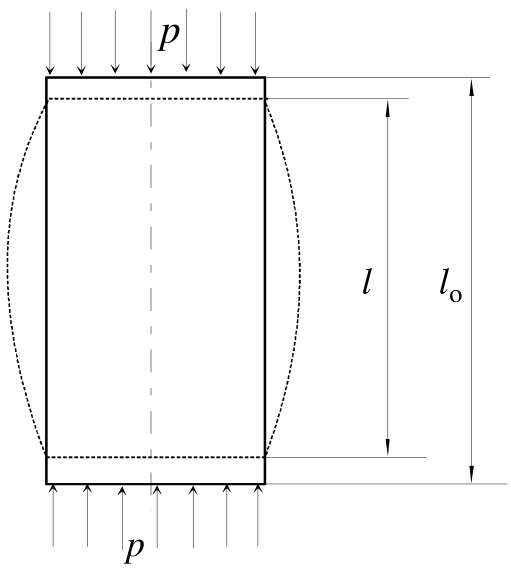

Fig.1. Volume displacement.

Contact displacement, δ_c, caused by direct interaction between bodies by contact between curved surfaces, which would initially make contact on a geometrical form without area (i.e., a line or a point). The presence of the normal force F in contact results in the development of a contact area around the initial point or line of contact. For spherical bodies, the contact is characterized by a contact radius, a, a contact pressure, p, and a rigid-body approach, δ, due to occurring local contact displacements.

Two types of displacements can occur simultaneously on a part and can also superimpose. Depending on the shape, dimensions, mechanical properties of the material and stress, one or the other component may become predominant. Knowledge of them allows a more judicious design of elements. This paper makes an assessment of these displacements for the case of rubber spheres. The subject of displacement of spheres of reduced rigidity is also studied in [1-3], which provides formulas for the calculation of displacements.

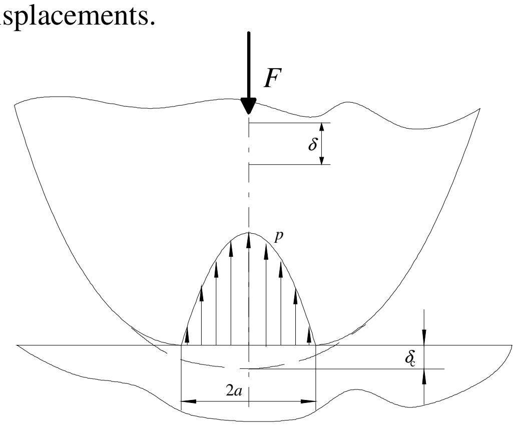

Fig. 2. Contact displacement, pressure distribution and contact area in the contact between an elastic sphere and a rigid half-space.

The well-known Hertz formulas allow the calculation [4] of contact displacements, which, for a contact between two spherical bodies bounded by surfaces of radii R_1 and R_2, made from linearly elastic materials, yield the radius of the circular contact area, a, the maximum Hertz pressure, p_0:

$$a = \sqrt[3]{\frac{3}{2} \cdot \frac{\eta F}{\Sigma k}}, \quad p_0 = \frac{1}{\pi}\sqrt[3]{\frac{3}{2}\left(\frac{\Sigma k}{\eta}\right)F}, \tag{3}$$

and the relative approach between the bodies in contact:

$$\delta = \frac{1}{2}\sqrt[3]{\frac{9}{4}\eta^2 \Sigma k F^2}, \tag{4}$$

where $\eta = \frac{1-\nu_1^2}{E_1} + \frac{1-\nu_2^2}{E_2}$, $\Sigma k = \frac{1}{R_1} \pm \frac{1}{R_2}$, and 1 and 2 are indexes referring to the contacting bodies.

In the case of highly deformable bodies, such as rubber spheres, contact displacements and volume displacements can have comparable values, as opposed to steel spheres, where the contact displacement is much larger than the volume displacement, due to significant stresses in the vicinity of a relatively small contact area, and also due to the high value of the Young's modulus of elasticity. The modulus of elasticity of rubbers is at least a few orders of magnitude smaller than that of steel. Thus, the literature provides data on the modulus of elasticity of rubber, [5-6], indicating values of the order of megapascals.

When measuring the displacement of a real body, it is practically impossible to measure only the contact displacement or only the volume displacement; instead, the total displacement, δ_{tot} is actually measured. In order to separate this total displacement into contact displacement and volume displacement, a rigorous definition of these types of displacement is required.

The contact displacement of a sphere, δ_c, is defined as the value of the displacement corresponding to the body approach. For a contact between a rubber sphere and a rigid plane, in the absence of volume displacement, the contact displacement would be as shown in figure 3.

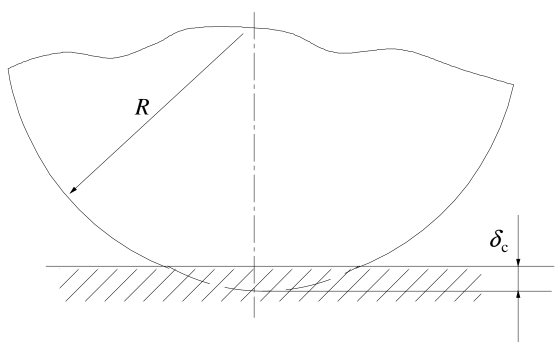

Fig. 3. Contact displacement definition for a contact between a rubber sphere and a rigid plane.

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Volume displacement of a sphere is the displacement of the body in the direction of stress, under the effect of compression. According to figure 4, in a hemisphere pressed against a rigid, fixed plane, under the effect of pressure, p, will occur both contact displacements, δ_c, and volume compression displacements, δ_v, resulting in:

$$R = H + \delta_v + \delta_c = H + \delta_{tot}, \qquad (5)$$

where δ_{tot} is the total, cumulative displacement and H is the final dimension of the hemisphere in the direction of stress.

However, the assessment of volume displacements of the hemisphere must take into account the fact that the cross section, normal in the direction of load, is variable, and therefore the stiffness is also variable. The Hooke's law, [7] yields: ¶

$$\frac{F}{A} = E\frac{\Delta l}{l_0}, \qquad (6)$$

where: F - normal load and A - cross-sectional area.

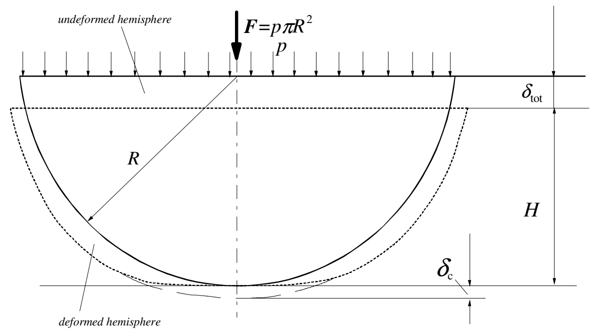

Fig. 4. Representation of contact and volume displacements for a hemisphere.

Applying this formula to an element dx from the sphere, figure 5, yields the relation

$$\frac{F}{\pi\, r_{(x)}^2} = E\,\frac{\Delta(d_x)}{dx} \qquad (7)$$

where $r(x) = \sqrt{R^2 - x^2}$.

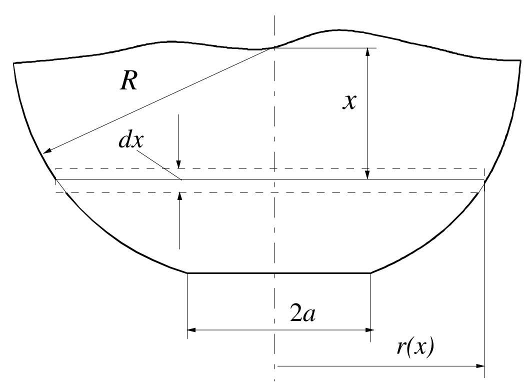

Fig. 5. Definition of the calculus elements of the volume displacement of the hemisphere.

The total volume displacement of a hemisphere can be written as:

$$\Delta l = \int_0^{\sqrt{R^2-a^2}} \frac{F}{\pi E} \frac{dx}{R^2-x^2} . \qquad (8)$$

.

Results

$$\Delta l = -\frac{F}{\pi E} \int_0^{\sqrt{R^2-a^2}} \frac{dx}{R^2-x^2} =$$

$$= -\frac{F}{\pi E} ln \left| \frac{R-x}{R+x} \right|_0^{\sqrt{R^2-a^2}} , \qquad (9)$$

and

$$\delta_{v\ hemisphere} = -\frac{F}{2\pi RE} ln \left| \frac{R-\sqrt{R^2-a^2}}{R+\sqrt{R^2-a^2}} \right| . \qquad (10)$$

This formula allows the calculation of the volume displacement of a hemisphere in the simplified case of a material whose modulus of elasticity does not vary with the load level, which is unfortunately not true in the case of rubber-like materials. For the latter materials, a more accurate evaluation of the volume displacement can be made by assessing the law of variation of the modulus of elasticity with stress or by experimental measurements. Also, it can be used a secant value of the elastic modulus of elasticity.

3.EXPERIMENTS AND RESULTS

The purpose of this experiment was to highlight and differentiate contact and volume displacements, for rubber spheres. The experiment consists in compression of one or two rubber spheres on a previously reported experimental rig, [8], which allows drawing the correlation curve between axial stress force (in the direction of the line of the centers of the two spheres) and the relative approach between the spheres. Two types of experiments were conducted, each using a positioning device that allows a relative positioning of spheres with respect to the line of centers in the direction of force. These devices consist of two supporting elements (lower and upper) in which the spheres are positioned. These elements are guided relatively by mini-columns. The spheres rest in devices either on flat surfaces or in hemi spherical cavities that copy their surfaces.

Simultaneously with measuring the contact force and the relative proximity of the spheres, it was inserted into the contact, between the two spheres or between a sphere and the conjugate plane surface (of the device) of an ultra-low pressure paper film (Fujifilm Prescale).

As a result of the action of non-zero contact pressure, the contact area is highlighted on paper by a colored area.

An image analysis assessment of the contact size area was performed. The colored areas (figure 6) were scanned simultaneously with a graduated scale, then imported into a CAD program where an equilateral triangle including the colored area was drawn. A circle inscribed in this triangle defines the contact area. Measuring the diameter of this circle and using the representation scale, given by the ratio between the real size of the graduated scale and its size in the scanned image, determines the real value of the contact area diameter.

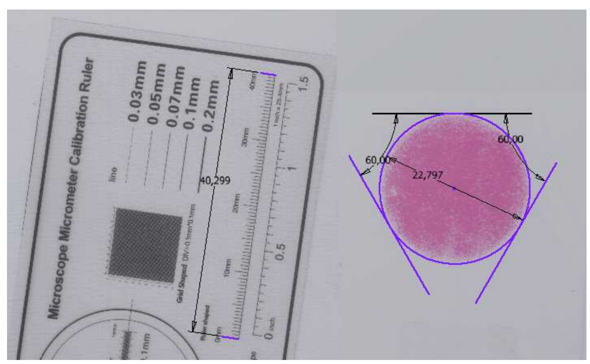

Fig. 6. Examples of contact areas highlighted with ultra-low-pressure paper and scaling method.

The results are presented in Table 1. According to the above, the value of the radius of the contact area, *r*, is correlated with the relative approach, respectively:

$$r = \sqrt{R^2 - (R-\delta_1)^2}, \qquad (11)$$

where: R=54,4 mm is the radius of the sphere and δ_1 the displacement of a single contact.

This can be verified by comparing the values for the radius of the contact, obtained with Hertz's formula and that obtained using the above formula. This comparison is shown for a contact between a sphere and a plane in figure 7.a, for steel and in figure 7.b for progressively reduced values for the modulus of elasticity. The very good match between the two values, calculated for steel, is explained by the high rigidity of the steel and therefore a negligible volume displacement compared to the contact ones. As the rigidity decreases, the deviation is increasing.

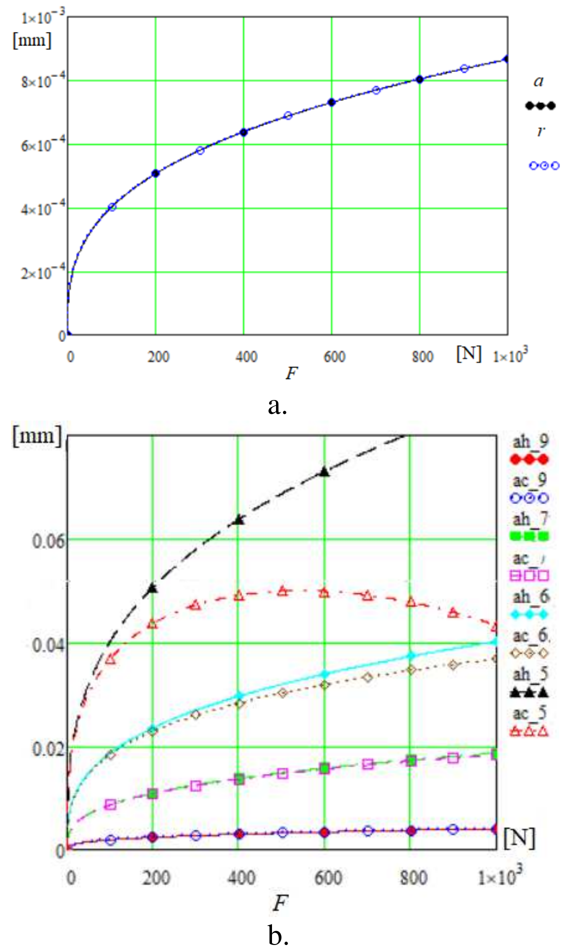

Fig. 7. Comparison of the radius of the contact area calculated with Hertz's formula and from the contact displacement with formula 12. (a. Values for $E = 2.1 \cdot 10^{11}$ [Pa], b. Curves ah_9, ah_7, ah_6, ah_5 are Hertzian radius for $E = 2.1 \cdot 10^9$ [Pa], $E = 2.1 \cdot 10^7$ [Pa], $E = 2.1 \cdot 10^6$ [Pa], $E = 2.1 \cdot 10^5$ and ac_9, ac_7, ac_6, ac_5 are the radius r for $E = 2.1 \cdot 10^9$ [Pa], $E = 2.1 \cdot 10^7$ [Pa]· $E = 2.1 \cdot 10^6$ [Pa] respectively $E = 2.1 \cdot 10^5$ [Pa].)

The diameter of the contact area, if the displacement is only the contact displacement (and the volume displacement was negligible) would be $d_c = 2r$.

The contact displacement component is given by:

$$\delta_c = R - \sqrt{R^2 - r^2}, \qquad (12)$$

where r is the radius of the measured contact area.

The volume displacement, δ_v, is given by the difference between the total approach displacement, δ_1, and the contact displacement:

$$\delta_v = \delta_1 - \delta_c. \qquad (13)$$

Table 1.

Experimental displacement results.

Contact type	Load, [N]	$2a$, [mm]	δ_{tot}, [mm]	δ_1, [mm]
Plane-Sphere-Sphere-Plane	16.72	10.55	3	0.75
	41.58	13.84	6	1.50
	74.80	16.83	9	2.25
	120.56	19.51	12	3
	180.40	22.63	15	3.75
Cavity-Sphere-Sphere--Cavity	38.72	15.38	3	1.50
	101.20	20.76	6	3
	197.12	25.74	9	4.50
	334.40	30.85	12	6
	519.20	35.53	15	7.50

By applying these calculations to the measurement data presented above, the values from Table 2 and represented in Figure 8 are obtained:

Table 2.

Global, contact and volume displacements for hemispheres.

Contact type	Load, [N]	r, [mm]	δ_1, [mm]	δ_c, [mm]	δ_v, [mm]
Plane-Sphere-Sphere-Plane	16.72	5.27	0.75	0.517	0.233
	41.58	6.92	1.50	0.895	0.605
	74.80	8.41	2.25	1.334	0.916
	120.56	9.76	3	1.810	1.190
	180.40	11.32	3.75	2.465	1.285
Cavity-Sphere-Sphere--Cavity	38.72	7.69	1.50	1.110	0.39
	101.20	10.38	3	2.060	0.94
	197.12	12.87	4.50	3.238	1.262
	334.40	15.43	6	4.796	1.204
	519.20	17.77	7.50	6.604	0.896

From the analysis of the graphs presented in Figure 8 it can be concluded that:

- Total and contact displacement increase with loading.
- Contact displacement increases faster than volume displacement, which is to be expected due to lower local rigidity.

- The volume displacement, in the case of the experiment with a series of 4 contacts, increases with loading, but the growth rate gradually attenuates, and, in the case of two spheres arranged in spherical cavities, the volume displacement decreases at higher load and the displacement passing into contact displacement. The latter unnatural result can be attributed to the spherical cavities preventing the volume displacement of the spheres.

To verify this, a new experiment was carried out by pressing a single sphere between two rigid plane plates. The results obtained are shown in Table 3 and Figure 9.

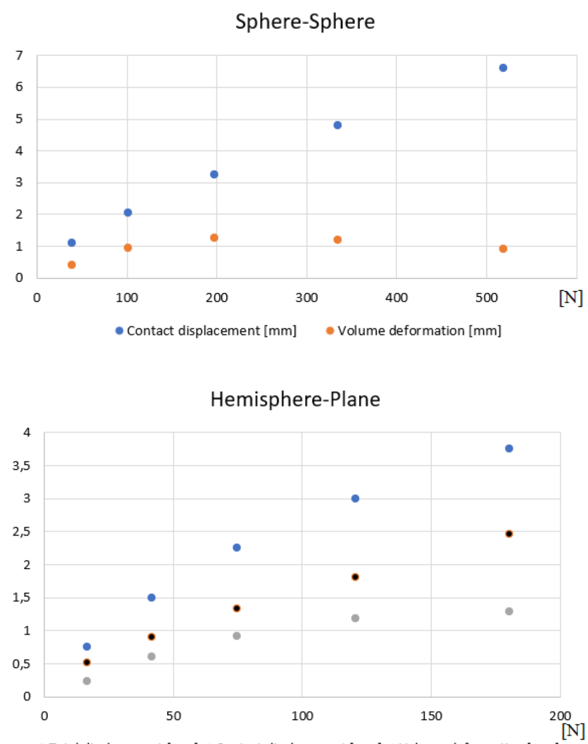

Fig. 8. Results concerning contact and volume displacement of half-spheres.

This experiment allowed the control of the contact force, the total displacement and the diameter of the contact area, as shown above for the other experiments performed, but also the measurement of the median diameter of the deformed ball.

Compared to previous experiments, the latter has the particularity that it presents the simpler case of two similar ball-plane contacts.

It is noticed that, in this case, the tendency to decrease of the volume displacement no longer occurs, which was expected. plane

Table 3.

The results of compression of a sphere between two rigid plane plates.

Contact type	Load, [N]	r, [mm]	δ_1, [mm]	δ_c, [mm]	δ_v, [mm]
Plane-Sphere-Plane	34.76	8.9	1.5	1.18	0.32
	85.36	12.41	5	2.05	0.94
	156.2	14.98	4.5	3.18	1.31
	242.9	17.04	6	4.3	1.69
	352	18.75	7.5	5.68	1.82

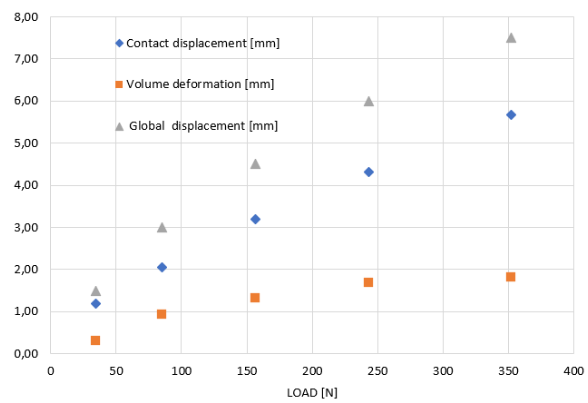

Fig. 9. Total, contact and volume displacements for a rubber sphere compressed between two rigid flat plates.

Assuming the material as incompressible, the initial volume of the sphere and the volume of the deformed sphere should be identical. In order to calculate the volume of the deformed sphere, however, the profile of its deformed surface must be known. For this profile, the following elements are known, figure 10:

1. The beginning and end points A, B of the free profile of the deformed sphere are also on the outer diameter of the contact area of the ball with the rigid flat compression surfaces.
2. Point C given by the deformed median diameter of the sphere is also half the height of the deformed sphere.
3. The profile of the deformed sphere rest tangent to the two rigid, parallel planes.

One curve that meets these conditions is the ellipse. For this reason, the deformed profile of the sphere will be approximated by an ellipse.

Relating to a coordinate system, as shown in Figure 10, and approximating the deformed profile of the sphere by a cylinder of radius, r, and height $2(R-\delta_1)$, bounded externally by a

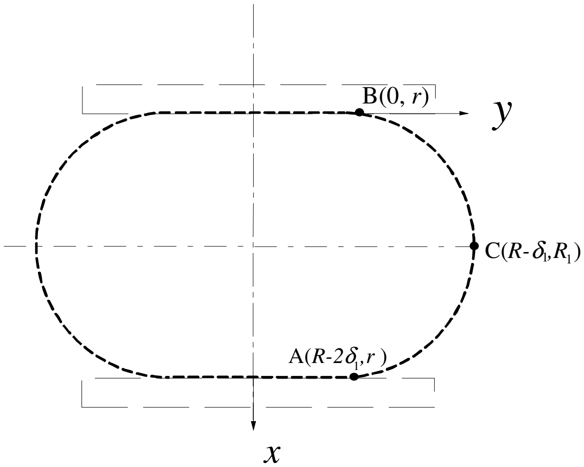

Fig. 10. Schematization of the undeformed and deformed sphere between two rigid plane plates.

toroidal surface with the generator described by an ellipse, defined by the function

$$y(x)=r+(R_1-r)\sqrt{1-\frac{[x-(R-\delta_1)]^2}{(R-\delta_1)^2}}, \quad (14)$$

where R_1 is the effective radius of the deformed sphere, specified in figure 10, the volume of the deformed sphere can be calculated analytically:

$$V_{def}=\pi\int_0^{2R-2\delta_1} y^2(x)dx. \quad (15)$$

The obtained results are given in Table 4.

Table 4.

Displacement and volume results for a single compressed sphere between rigid plane plates.¶

r, [mm]	$2R_1$, [mm]	δ_1, [mm]	V, [mm^3 ·10^4]	V_{def} [mm^3 ·10^4]	Relative error, [%]
7.94	55	1.5	8.429	9.075	7.66
10.38	55.80	3	8.429	9.093	7.87
12.78	56.60	4.5	8.429	9.062	7.51
14.69	57.20	6	8.429	8.868	5.21
16.64	58.70	7.5	8.429	8.871	5.24

Because the value of Young's modulus of rubber is not a constant and is correlated with the stress value only an average value, E_m, for the modulus of elasticity, in formula 10, can be calculated and, substituting in this formula $\delta_{v\ hemisphere}$ with the experimentally measured values, from Table 3, the values of the mean modulus of elasticity of the sphere material can be calculated using the formula:

$$E_m=-\frac{\frac{F}{2\pi R}ln\left|\frac{R-\sqrt{R^2-a^2}}{R+\sqrt{R^2-a^2}}\right|}{\delta_{v\ hemisphere}}. \quad (16)$$

The results of this mean modulus for the experimental loads, r and $\delta_{v\ hemisphere}$, can be found in Table 5.

Table 5.

Mean values for Young's modulus for a single compressed sphere between rigid plane plates.

Contact type	Load, [N]	r, [mm]	$\delta_{v\ semisphere}$, [mm]	E_m [MPa]
Plane-Sphere-Plane	34.76	8.9	0.32	2.23
	85.36	12.41	0.94	1.49
	156.2	14.98	1.31	1.67
	242.9	17.04	1.69	1.3
	352	18.75	1.82	1.21

The obtained values for E_m are of the same order of magnitude as the experimental values mentioned in literature, for rubbers, [9], and have similar tendencies with rubber Young's modulus, to decrease with load.

4. CONCLUSIONS

1. In the case of spheres made of materials of low rigidity, volume displacements may be comparable or higher in order of magnitude to local, contact displacements.
2. The presence of large volume displacements is obviously associated with significant volume stresses.
3. Both local contact displacements and volume displacements increase with load.
4. Local contact displacements grow faster than volume displacements whose growth rate is progressively attenuating. This is due to the fact that increasing contact deformity increases local rigidity.
5. The correct assessment of the nature and amplitude of the components of the displacement (volume or contact) allows a more judicious design of the parts shape.

6. Evaluation of the local and volume components of the total displacement can help to design optimal shape of parts.
7. The obtained value for the mean Young's modulus from table 5, validate that the obtained results for the rubber spheres displacements with the assumed hypothesis are satisfactory.

5.REFERENCES

[1] Tatara, Y., 1991, *On Compression Of Rubber Elastic Sphere Over A Large Range Of Displacements, Part 1: Theoretical Study*, Journal of Engineering Materials Technology, 285-291, Vol 3, doi.10.1115/1.2903407. ISSN 0094-4289.

[2] E.,Diaconescu, M.,Glovnea, F.,Ciutac, 2009, *Circular Contact Of Incompressible Nonlinear-Elastic Bodies*, Proccedings of the ASME/STLE International Joint Tribology Conference, October 19-21.

[3] Tamaşag, I., Beşliu-Băncescu, I., Muscă I., 2023, *A Simple Non-Contact Method To Evaluate High Deformable Bodies Deformation.* Tehnomus Journal, New Technologies and Products in Machine Manufacturing Technologies, No.30, p. 95-101

[4] Johnson, K.L., 1985, *Contact Mechanics*, Cambridge University Press, 452p.

[5] Harris, C. 2018, *Shock and Vibration Handbook*. Cyril M. Harris, editor, Allan G. Piersol, editor.-5th ed. cap. 33, Ronald J. Schaefer *Mechanical Properties of Rubber*, 18p.

[6] https://www.chevalier-cleret.com/event/young-modulus-for-rubber.html

[7] Tripa, P., Hluşcu, M., 2006, *Strength of materials. Fundamentals and applications* Timişoara, ed. Mirton, 2006, 395p (in Romanian).

[8] Sîrghie, E., Muscă, I., Românu, I. C. 2021, *Experimental Study On Loading-Unloading Rubber Spheres Deformation.* Tehnomus Journal, New Technologies and Products in Machine Manufacturing Technologies, No.28, p. 57-63.

[9] Gough, J., 2000, *Stress-Strain Behaviour of Rubber.* Ph.D. Thesis, Queen Mary and Westfield College University of London, 263p.

Considerații privind deformațiile volumice și de contact

Lucrarea prezintă unele considerații teoretice și numerice privind deformarea sferelor din cauciuc și încearcă să facă diferența între deformarea locală și deformarea volumică. În apropierea contactului concentrat al corpurilor, apare o deformare locală importantă, datorită contextului sarcinii aplicate pe o arie redusă contact a corpurilor și, de asemenea, rigidității locale scăzute. Această sarcină acționează și asupra întregului material al corpului și, dacă valoarea rigidității sale nu este foarte mare, poate apărea o deformare importantă a volumului.

Lucrarea începe cu o prezentare teoretică și o evaluare analitică a deformărilor. Apoi sunt prezentate câteva experimente de comprimare (dispozitive, experiemnte și rezultate) pentru: două sfere, comprimate între două suprafețe plane și paralele (trei contacte dintre care unul sferă-sferă și două sferă-plan), două sfere, comprimate între două corpuri cu cavități semisferice (un contact sferă-sferă) şi o sferă comprimată între două suprafeţe rigide plane și paralele (două contacte sferă-plan). În timpul acestor experimente, au fost înregistrați trei parametri: sarcina globală normală, deformarea globală normală a sferelor și aria relativă de contact între sfere sau între sferă și plan. Având în vedere că valorile ariei de contact sunt în corelare cu valorile deformările relative locale, au fost calculate deformările și, prin diferență, au fost evaluate și deformările de volum. Din aceste rezultate se poate concluziona că: sferele de cauciuc prezintă valori mari și comparative ale deformărilor locale și de volum.

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