TECHNICAL UNIVERSITY OF CLUJ-NAPOCA

ACTA TECHNICA NAPOCENSIS

Series: Applied Mathematics, Mechanics, and Engineering
Vol. 67, Issue Special IV, August, 2024

USE OF CONSTRUCTAL LAW TO PROOF THE MAXIMUM WORK "PRINCIPLE" FOR BOTH BULK PLASTICITY AND CONTACT SURFACE FRICTION. APPLICATION TO ANISOTROPIC TRIBOLOGICAL BEHAVIOUR

Adinel GAVRUS

Abstract: *Along this scientific article, it is propose to proof the Maximum Work "Principle" used by materials plasticity theory starting from a constrained optimisation problem based on Constructal Law developed by prof. Adrian BEJAN of DUKE University (USA). This law postulate that all finite-size system searche to flow more and more easily over the time and to evolve in those configurations or shape distributing their imperfections to minimize losses or dissipations and entropy variation. On the mechanical point view, if add the thermodynamics principles (energy conservation and system's entropy evolution), starting from the first Newton law (describing equation of momentum balance and its equivalent form of virtual power principle), together with the natural tendency of all currents to flow from high to low values, it can be conclude that under external loadings, all material flow search to minimize the dissipated powers of both deformation and friction. So between all other virtual states corresponding to virtual mechanical variables values (stress, strain, strain rate) the real ones are those that minimize the total dissipated power. It is then obtained a minimization problem, which proofs mathematically that the Maximum Work "Principle", applied to both bulk plastic deformations and surfaces frictions for all continuum media: solid, fluid, mushy state,…, is a consequence of Constructal Law. It is an original mathematical proof corresponding to plasticity mechanics and tribology never presented until now by previous scientific studies of other researchers. Following the obtained sub-sequent theorems of potential convexity and associated normal rule laws, it is then possible to describe the materials elastic-plastic flow and the surface friction in a more general framework for both isotropic or anisotropic rheological and tribological properties. The validation of an elliptic anisotropic formulation of Coulomb law to predict the static and kinetic friction coefficients, as a function of different sliding directions, it is develop using experimental data given by a Pin-on-Plane Tribometer in the case of anisotropic metal/isotropic polymer contact interfaces (AA20124-351 rolling thick plate/UHMWPE cylindrical pion). Furthermore, it is estimate the differences observed between the friction shear orientations and the sliding velocity directions proven in the case of anisotropic contact surface the dependency of friction coefficient also with sliding direction.*

Key words: *Constructal Theory, Maximum Work "Principle" as a Consequence of an Optimisation Problem, Plasticity, Anisotropic Friction Formulation.*

1. INTRODUCTION

During the past decades, Prof. Adrian BEJAN of Duke University (USA) has developed fundamental scientific researches concerning application of thermodynamics theory to define and optimize complex finite systems time evolution and design based on the called Constructal Law [1-2]. So starting from natural needs of any finite system time flow to minimizes resistance and losses, searching simultaneously to maximize the speed to reach a stable dynamic equilibrium state, together with the tendency to maximize entropy state and to minimize the generated entropy, this law postulate that: "for a finite-size system to persists in time (to live or to survive), it must evolve in such a way that it provides easier access to the imposed (global) currents that flow through it" as to facilitate access as much as possible under the constraints flows or which cross them minimizing the corresponding losses [1]. It has also observed that all natural system search to optimize the imperfections distributions to facilitate the time system flow and to minimize the local resistance or to minimize the required process transformation or evolution powers. A lot of previous studies concerning thermal problems, fluid or porous media flow, nature, economic or society behaviour confirm this principle [2]. In a mathematical point of view The Constructal Theory can be seen as a definition and resolution of a general variational optimisation problem under predefined physical constraints and boundary conditions based on the fundamental principles of Newtons's equilibrium equations together with their equivalent formulations and on the first and second thermodynamics principles. Thus, all the mechanical variables defining the real bulk or surface mechanical state (velocities, stresses, strain and strain rate) are those that minimize the total dissipated power. Using previous recent author's scientific research works [3-5], with focus on bulk material forming regarding as applications and validations on plane compression, cylindrical crushing and forward extrusion, this article proposes along a synthetic computational framework to present the mathematical proof of the Maximum Work "Principle" used by materials plasticity as a direct consequence (Theorem) of the Constructal Principle, seen as being a more general postulate on thermodynamic point of view and valid for any type of bulk materials flow and also for surface continuum media interactions. It is well known that the Maximum Work Principle (MWP) is used by metallic plasticity theory [6] to obtain associated flow laws or to define constitutive rheological equations related to describe the material's strength and to develop analytical or numerical computations describing forming processes (forging, extrusion, rolling, stamping). Previous old works [6] try only to explain and to postulate the Maximum Working Principle by a purely phenomenological approach regarding polycrystalline metals structure starting from local slips of maximum density atomic planes, specific only for polycrystalline states as metals. By exploiting the Virtual Powers Principle (VPP) defining mechanic dynamic equilibrium, if is applied the Constructal Theory, the Maximum Work Principle can be obtained as a direct consequence of a general and original variational optimization problem which search to minimize the losses of material flow, see to minimize the sum of all the losses powers. This proof generalizes the PMW application to any type of continuum media and to all materials plastic flows: metals, polymers, fluid, mushy state proving also an equivalent form concerning contact friction stress work. Using the proposed mathematical framework can be proven the convexity properties and normal rule properties of both plastic and friction potential. It is also conclude that only rheological flow constitutive equations and tribological laws associated with a potential functional become to satisfy the second thermodynamics principle. The performed theory is valid by comparisons with computations based on the sub-sequent Upper and Lower Bound Theorems [6] obtained here as consequences of the proven PMW theorem, together with comparisons of the well-known slices method and Finite Element Modelling (FEM) results. Analytical computations concerning plane compression and cylindrical crushing [3-5] shown the feasibility of the proposed minimization problem formulated from the Constructal Law giving accurate approximate analytical solutions to define material flow [4-5].

This scientific paper proposes, as a fundamental novelty, especially concerning MWP proof for all types of a continuum media, also the validation of the proposed theoretical framework through formulation of anisotropic friction concerning metal/isotropic polymer contact interfaces of an anisotropic AA20124-351 rolling thick plate/UHMWPE cylindrical pin contact sliding. Furthermore, it is also estimate the differences observed between the friction shear orientations and the sliding friction velocity directions. It also important to underline that in mechanics point of view, during several materials forging, sheet forming, mechanics systems/mechanisms, a hard coupling between elastic or plastic deformation and surface friction occurs during material flow and materials body interactions.

2. THEORETICAL FRAMEWORK

According to the theory of continuum media mechanics for all material plastic flow defined by a body volume Ω, the dynamic mechanical equilibrium can be written from the Virtual Powers Principle (VPP) expressed in an equivalent form of the Newton's law by:

$$\int_{\Omega}[\sigma]:[\dot{\varepsilon}^*]dV + \int_{\partial\Omega} -\vec{\tau}\cdot\Delta\overrightarrow{v}^*\, dS = \int_{\partial\Omega'}\vec{T}\cdot\vec{v}^d dS' + \int_{\partial\Omega''}\vec{T}^d\cdot\vec{v}^* dS'' + \int_{\Omega}\rho\vec{f}\cdot\vec{v}^* dV - \int_{\Omega}\rho\frac{d\vec{v}*}{dt}\cdot v*dV \quad (1)$$

The VPP principle is valid for any field of admissible virtual velocities $\vec{v}^*$ undergoing corresponding boundary conditions and taking into account a Cauchy stress tensor $[\sigma]$, a friction stress vector $\vec{\tau}$ acting on the contact interfaces or on the surface of velocities discontinuities $\partial\Omega$, a virtual strain rate tensor $[\dot{\varepsilon}^*]=\left\{[grad(\vec{v}*)]+[grad(\vec{v}*)]^T\right\}/2$, a specific loading $\vec{T}$, an imposed velocities $\vec{v}^d$ on the body border part $\partial\Omega'$, a specific imposed loads $\vec{T}^d$ on the body border part $\partial\Omega''$, a specific mass forces $\vec{f}$ and a specific density ρ. The real velocity field $\vec{v}$ of material flow gives the material dynamic equilibrium described by the Power Work Principle (PWP):

$$\int_{\Omega}[\sigma]:[\dot{\varepsilon}]dV + \int_{\partial\Omega} -\vec{\tau}\cdot\Delta\overrightarrow{v}\, dS = \int_{\partial\Omega'}\vec{T}\cdot\vec{v}^d dS' + \int_{\partial\Omega''}\vec{T}^d\cdot\vec{v} dS'' + \int_{\Omega}\rho\vec{f}\cdot\vec{v} dV - \int_{\Omega}\rho\frac{d\vec{v}}{dt}\cdot\vec{v}dV \quad (2)$$

2.1 Theorem of « Maximal Work Principle »

Using the Constructal Law, in the case of a deformable material evolution during a forming process, it can postulate that the plastic flow together with contact surface friction evolution place such that it minimizes the sum of the dissipated powers corresponding to the bulk plastic deformation, to the friction at the contact surfaces or to the shear stresses working on interfaces' velocities discontinuities and to the imposed loads, regarded also as system losses. Consequently, the real values of all the kinematic and mechanical variables (velocities, stresses, strains, strain rates) are those, which minimize the total dissipated power. Thus, for any state of material flow defined by virtual mechanics variables, the real ones must minimize the obtained functional defining the total dissipation power.

In conclusion, for all plastic materials or continuum media (metallic or non-metallic, polymers, mushy state or fluids state) and for any virtual state, the real strain rate tensor $[\dot{\varepsilon}]$ and the real Cauchy stress tensor $[\sigma]$ corresponding to the real flow can be obtained by minimizing the sum of the dissipated bulk plastic power, of the dissipated friction power or of the dissipated velocities discontinuities power and of the imposed loads power, regarded that complementary losses power.

It is then possible to provide easier access of the continuum media flow minimizing the required power. In addition, the plasticity of the material is generally governed in terms of the stress tensor and of the shear friction vector through the definition of a plastic criterion defined by a multi-variables scalar function $\Phi_p([\sigma])-\sigma_0=0$ together with a similar form regarding the friction term i.e. $\Psi_f(\vec{\tau})-\tau_f=0$ where σ_0 is the equivalent stress defining intrinsic material rheology and τ_f is the equivalent stress friction defining intrinsic surface contacts tribology. In this case, from all other admissible virtual velocities fields $v*$ (different with respect to the real one) characterized by a virtual strain rate tensor $[\dot{\varepsilon}^*]\neq[\dot{\varepsilon}]$ and a virtual plastic constraints tensor $[\sigma^*]$ ($\Phi_p([\sigma^*])-\sigma_0=0$ and $\Psi_f(\vec{\tau})-\tau_f=0$), the real plastic flow state can be obtained by minimizing the functional defined through the virtual total dissipated and losses power:

$$P_d = Min(P^*{}_d)\ with$$
$$P^*{}_d = \int_{\Omega}[\sigma^*]:[\dot{\varepsilon}^*]dV + \int_{\partial\Omega} -\vec{\tau}^*\cdot\Delta\overrightarrow{v}^*\, dS + \int_{\partial\Omega''} -\vec{T}^d\cdot\overrightarrow{v}^*\, dS'' \quad (3)$$

Using the boundary conditions concerning material loadings and kinematics, it is obtain a variational minimization problem under specified constraints. As shown by the Constructal Theory the real flow state can be regarded as the optimal solution. So all virtual flow states require to follow the below power inequality:

$$\int_{\Omega}[\sigma]:[\dot{\varepsilon}]dV + \int_{\partial\Omega} -\vec{\tau}\cdot\Delta\overrightarrow{v}\,dS + \int_{\partial\Omega''} -\vec{T}^{d}\cdot\overrightarrow{v}\,dS'' \leq$$
$$\int_{\Omega}[\sigma^*]:[\dot{\varepsilon}^*]dV + \int_{\partial\Omega} -\vec{\tau}^*\cdot\Delta\overrightarrow{v}^*\,dS + \int_{\partial\Omega''} -\vec{T}^{d}\cdot\overrightarrow{v}^*\,dS'' \quad (4)$$

In the case of quasi-static conditions and consistent materials, it can be neglect the mass and the inertial forces.

So the VPP principle can be then write in the following simplified form:

$$\int_{\Omega}[\sigma]:[\dot{\varepsilon}]dV + \int_{\partial\Omega} -\vec{\tau}\cdot\Delta\overrightarrow{v}\,dS + \int_{\partial\Omega''} -\vec{T}^{d}\cdot\overrightarrow{v}\,dS'' = \int_{\Omega}[\sigma]:[\dot{\varepsilon}^*]dV + \int_{\partial\Omega} -\vec{\tau}\cdot\Delta\overrightarrow{v}^*\,dS + \int_{\partial\Omega''} -\vec{T}^{d}\cdot\overrightarrow{v}^*\,dS'' \quad (5)$$

Using (3) and (4), for any body Ω and for any virtual stress state $[\sigma^*]$ ($\Phi_p([\sigma^*])$-$\sigma_0 = 0$) or any virtual friction stress $\vec{\tau}^*$ ($\Psi_f(\vec{\tau}^*)$ -$\tau_f = 0$) of surface body $\partial\Omega'$ is given the equivalent form:

$$\int_{\Omega}([\sigma^*]-[\sigma]):[\dot{\varepsilon}^*]dV + \int_{\partial\Omega'}(\vec{\tau}-\vec{\tau}^*)\cdot\Delta\overrightarrow{v}^*\,dS' \geq 0 \quad (6)$$

This condition requires positive values for each integral term i.e.:

$$([\sigma^*]-[\sigma]):[\dot{\varepsilon}^*] \geq 0, \forall[\sigma^*], \Phi_p([\sigma^*]) - \sigma_0 = 0; \; -(\vec{\tau}^* - \vec{\tau})\cdot\Delta\vec{v}^* \geq 0, \forall\vec{\tau}^*, \Psi_f(\vec{\tau}^*) - \tau_f = 0 \quad (7)$$

If one considers that there exists a virtual state of the stresses or friction for which the terms are negative, one could build another identical virtual state with real state of stresses, outside the field, for which the terms of the integrals are negative (therefore with zero integrals values outside this domain) and one would obtain a negative value for the inequality (6) in contradiction with the required positive value. So, in the opposite sense, taking into account the real plastic flow characterized by the velocity field *v*, the strain rate tensor$[\dot{\varepsilon}]$and the Cauchy stress tensor $[\sigma]$, any other state of the admissible stresses must verify the following inequalities:

$$([\sigma]-[\sigma^*]):[\dot{\varepsilon}] \geq 0, \forall[\sigma^*], \Phi_p([\sigma^*]) - \sigma_0 = 0; \; -(\vec{\tau} - \vec{\tau}^*)\cdot\Delta\vec{v} \geq 0, \forall\vec{\tau}^*, \Psi_f(\vec{\tau}^*) - \tau_f = 0 \quad (8)$$

Starting from the above two relationships (7) and (8), it is possible to prove mathematically the convex shape of the potential functions defining both the plastic and friction criteria. Subsequently it is also proven the known property of normal rule.

Consequently, the strain rate should be proportional to the gradient of the plastic criterion with respect to each components of the stress and in the same sense the sliding velocities orthogonal on the associated friction potential surface i.e:

$$[\dot{\varepsilon}] = \lambda_p \partial\Phi_p/\partial[\sigma], \lambda_p \geq 0, \quad \Delta\vec{v} = -\lambda_f \partial\Psi_f/\partial\vec{\tau}, \lambda_f \geq 0 \quad (9)$$

Based on the proven property of the convexity concenring plastic criterion $\Phi_p([\sigma^*])$ and friction criterion $\Psi_f(\vec{\tau}^*)$the two inequalities expressed by (9) can be extended to the virtual stress and friction shear respecting $\Phi_p([\sigma^*]) - \sigma_0 \leq 0$ and $\Psi_f(\vec{\tau}^*) - \tau_f \leq 0$. It is then possible to conclude that for any state of virtual stresses is required:

$$([\sigma]-[\sigma^*]):[\dot{\varepsilon}] \geq 0, \forall[\sigma^*], \Phi_p([\sigma^*]) - \sigma_0 \leq 0; \; -(\vec{\tau} - \vec{\tau}^*)\cdot\Delta\vec{v} \geq 0, \forall\vec{\tau}^*, \Psi_f(\vec{\tau}^*) - \tau_f \leq 0 \quad (10)$$

The first inequality is known in the plasticity theory as the Maximum Work Principle [3-6]. It is proved here that this can be obtained as a consequence of Constructal Law [1-2] and of the Mechanical Principle of Virtual Powers. The second inequality reflects the proof of a same principle also to the friction stress state.

2.2 Synthesis of Practical Consequences

Using the theoretical proofs presented above, it can be concluded that the Maximum Work Principle can be applied for any type of continuous media: fluid, solid or mushy-state material, metallic or non-metallic materials, polymers, etc, as well as to define in a similar form the "Principle" of Maximum Cork concerning the friction stresses. Specific expressions concerning the convex property of plasticity or of friction criteria $\Phi_p([\sigma])$ - $\sigma_0 = 0$, $\Psi_f(\vec{\tau})$ - $\tau_f = 0$ can be proposed and used to define the isotropic or anisotropic plastic behavior, respectively tribological laws defining sliding anisotropic friction at the contact interfaces. It is also possible to obtain and proven as consequence of previous properties the theorems of the Lower and Upper Bound (Figure 1).

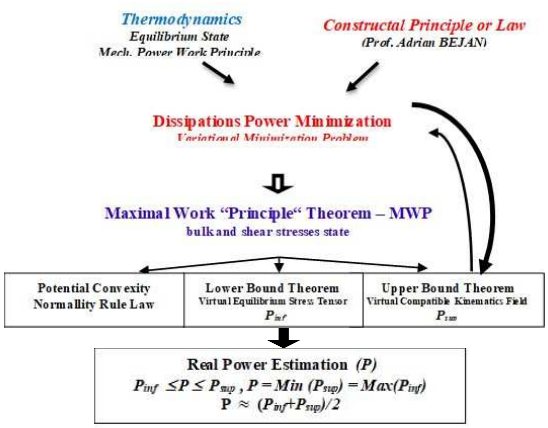


Fig.1. Flowchart drawing the formulation of the variational minimization problem concerning the dissipated power *P* based on the Constructal Theory and the proof of the of Maximum Work "Principle" (MWP) together with the corresponding theorems [3-5]

These two theorems can also be considered as variational optimization formulations and are frequently used to obtain analytical estimations of the real power or real loads applied during a forming process using particular assumptions.

They are also used to compare and validate the results obtained by Numerical Simulation as the Finite Element Modeling (FEM). Regarding the Upper Bound Theorem, it is observed that it is indeed obtained an equivalent form of the variational minimization problem (3).

It can be possible then to conclude on the coherence and equivalence between the Maximum Work Principle and the Minimization of the total dissipations powers or losses defined by Constructal Theory.

3. FORMULATION OF ANISOTROPIC FRICTION

Starting from the above proof of Maximum Work "Principle" as a consequence of Constructual Law, at the contact interface between two bodies [3-5; 7-8] it is proposed to express the anisotropy of a Coulomb friction law corresponding to an elastic-plastic contact regime using anisotropic material's surface morphology caused by for example by a rolling manufacturing process.

In this case, two principal morphologic surface direction are obtain on the material surface: rolled direction (DL) and transversal one (DT) having micro-roughness oriented along DL direction. Starting form first developments of anisotropic Coulomb friction presented in [7], recently applied for a pin-on-disc tribometer, using idea proposed by author of present scientific paper along published research's works [8-9], a depth quantitative analysis, to can understanding and proof application of MWP also for a friction law formulation, is presented below.

Thus, on a general mechanics point of view, in a local plane zone of the contact interface, materialized by a plane tangent to it, is possible to define the friction law linking the shear stress $\vec{\tau} = \tau_x \vec{x} + \tau_y \vec{y}$ having the composantes τ_x, τ_y along x axis (longitudinal direction - DL), respectivelly y axis (transversal direction - DT) with the relative sliding velocity defined from a friction potential expressed by:

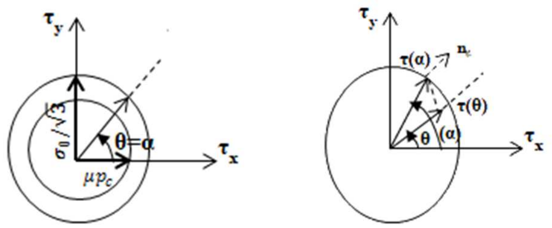
$$\begin{aligned} &\psi_f(\vec{\tau}) = \psi_f(\tau_x, \tau_y) \leq 0, \vec{v}_g = -\lambda_g (\partial\psi_f / \partial\vec{\tau}) \\ &\psi_f(\vec{\tau}) = 0 \Rightarrow \lambda_g > 0; \psi_f(\vec{\tau}) < 0 \Rightarrow \lambda_g = 0 \end{aligned} \quad (11)$$

Here $\lambda_g = \|\vec{v}_g\| / \|\partial\psi_f / \partial\vec{\tau}\|$ to can obtain the same unitar sliding vector $\vec{n}_g = -\vec{v}_g / \|\vec{v}_g\|$ defyning at the normal vector of friction potential as it is imposed by the Work Maximal "Principle" i.e.:

$$\vec{n}_g = -\vec{v}_g / \|\vec{v}_g\| = (\partial\psi_f / \partial\vec{\tau})(\|\partial\psi_f / \partial\vec{\tau}\|) \quad (12)$$

It can easily shown that for the case of isotropic friction of the Coulomb type, if its is noted the contact pressure p_c and the Coulomb coefficient μ we have $\tau = \|\vec{\tau}\| = \mu p_c$ during the sliding and one could easily verify that an isotropic friction potential of type $\psi_f(\vec{\tau}) = \|\vec{\tau}\| - \mu p_c$ gives $\vec{\tau} = -\mu p_c \vec{v}_g / \|\vec{v}_g\|$.

In this case if θ is the angle of the direction of sliding in the direction of the normal to the loading surface and α the angle of the orientation of the tangential stress with respect to the x axis is proven although using (12) that $\vec{\tau} \cdot \vec{n}_g = \tau = \mu p_c$ i.e. $\alpha = \theta$ (Fig. 2a)



a) b)

Fig. 2. Friction diagrams using the shear stresses $\vec{\tau}$ directions defined by α and sliding direction $\vec{n}_g$ defined by θ a) isotropic friction α=θ, b) anisotropic friction $\alpha \neq \theta$ [8]

This allows us to consider that during anisotropic friction $\alpha \neq \theta$ (Fig. 2b) and the Coulomb coefficients will be a function of the angle θ i.e.:

$$\vec{\tau} \cdot \vec{n}_g = \tau(\theta) = \mu(\theta) p_c \tag{13}$$

Analogously with quadratic formulation of a bulk plastic anisotropy (circular Von-Mises or elliptic Hill criteria for material bulk anisotropy), it can be choosen to express the friction anisotropy by an elliptical quadratic function written in the form [7]:

$$\psi_f(\vec{\tau}) = \psi_f(\tau_x, \tau_y) = G_f \tau_x^2 + H_f \tau_y^2 - \lambda^2 \tag{14}$$

Here G_f, H_f and λ are the anisotropic friction coefficients. Starting from the normal rule (12), if a sliding along x axis: v_{gy}=0, $\mu = \mu_x$ it is obtain:

$$\tau_x = \mu_x p_c,\ \tau_y = 0; v_{gx} = -\frac{\|\vec{v}_g\|}{\|\partial\psi / \partial\vec{\tau}\|} 2 G_f \tau_x \tag{15}$$

If sliding along y axi : v_{gx}=0, $\mu = \mu_y$ and :

$$\tau_y = \mu_y p_c,\ \tau_x = 0; v_{gy} = -\frac{\|\vec{v}_g\|}{\|\partial\psi / \partial\vec{\tau}\|} 2 H_f \tau_y \tag{16}$$

The elliptic coefficient's expressions are given by:

$$\begin{aligned} G_f \mu_x^2 p_c^2 - \lambda^2 = 0 \ \Rightarrow\ G_f = \lambda^2 / \mu_x^2 p_c^2 \\ H_f \mu_y^2 p_c^2 - \lambda^2 = 0 \ \Rightarrow\ H_f = \lambda^2 / \mu_y^2 p_c^2 \end{aligned} \tag{17}$$

It is then possible to writte the anisotropic quadratic criterion in the form:

$$\psi_f(\tau_x, \tau_y) = \left(\tau_x^2 / \mu_x^2\right) + \left(\tau_y^2 / \mu_y^2\right) - p_c^2 = 0 \tag{18}$$

Taking into account the orientation angle α of shear stress, defining by $tg(\alpha) = \tau_y / \tau_x$, $\tau_x = \tau\cos\alpha$ and $\tau_y = \tau\sin\alpha$, where $\tau = \sqrt{\tau_x^2 + \tau_y^2}$, it is obtained:

$$\tau = p_c / \sqrt{\left(\cos^2\alpha / \mu_x^2\right) + \left(\sin^2\alpha / \mu_y^2\right)} \tag{19}$$

with:

$$\begin{aligned} \tau_x = \cos\alpha / \sqrt{\frac{\cos^2\alpha}{\mu_x^2} + \frac{\sin^2\alpha}{\mu_y^2}} \\ \tau_y = \sin\alpha / \sqrt{\frac{\cos^2\alpha}{\mu_x^2} + \frac{\sin^2\alpha}{\mu_y^2}} \end{aligned} \tag{20}$$

The normal vector on the criterion surface, parallel with the sliding direction becomes:

$$\vec{n}_g = \frac{\partial \psi}{\partial \tau} \Big/ \left\| \frac{\partial \psi}{\partial \tau} \right\| = \begin{pmatrix} \dfrac{\tau_x}{\mu_x^2} \\ \dfrac{\tau_y}{\mu_y^2} \end{pmatrix} / \sqrt{\frac{\tau_x^2}{\mu_x^4} + \frac{\tau_y^2}{\mu_y^4}} \qquad (21)$$

$$\vec{n}_g = \begin{pmatrix} \dfrac{\cos \alpha}{\mu_x^2} \\ \dfrac{\sin \alpha}{\mu_y^2} \end{pmatrix} / \sqrt{\frac{\cos^2 \alpha}{\mu_x^4} + \frac{\sin^2 \alpha}{\mu_y^4}} \qquad (22)$$

Or the vector defyning the sliding direction can be written in the $\vec{n}_g = \cos(\theta)\vec{x} + sin(\theta)\vec{y}$ and consequently using previous relationship (11) it is possible to estimate the orientation angle α of sliding shear stress by:

$$tg(\alpha) = \left(\mu_y^2 / \mu_x^2\right) tg(\theta) \qquad (23)$$

Taking into account a Coulomb friction law $\vec{\tau} \cdot \vec{n}_g = \mu(\theta) p_c = \mu(\alpha) p_c$, the equations (21), (22) and (23) lead to the expression of Coulomb coefficients variation μ with the angle orientations θ or α in the follwing form:

$$\mu(\alpha) = \frac{\sqrt{\cos^2 \alpha / \mu_x^2 + \sin^2 \alpha / \mu_y^2}}{\sqrt{\cos^2 \alpha / \mu_x^4 + \sin^2 \alpha / \mu_y^4}} = \mu(\theta) \qquad (24)$$

$$\mu(\theta) = \sqrt{\mu_x^2 \cos^2 \theta + \mu_y^2 \sin^2 \theta} = \mu(\alpha)$$

In the case where it is defined the friction law by a law of type $\tau(\theta) = \mu'(\theta) p_c$, using the resultant of shear stress composante $\vec{\tau}$, it is possible to show that:

$$\vec{\tau} \cdot \vec{n}_g = \tau \cos(\alpha - \theta); \mu'(\theta) = \mu(\theta) / \cos(\alpha - \theta) \qquad (25)$$

These permits to distinguish the Coulomb coefficients estimated by measurement of the shear stress under sliding direction and those estimated from the measurement of the shear stress resultant. On experimental point of view, this problem is still open, because results will be fonction on the type of loads cells used by tribometers. Contrarly, if $\cos(\alpha - \theta) \approx 1$, for the case where $\alpha - \theta$ is smallest that 10°-15°, it can be neglected the impact of measured shear force type and only differences of Coulomb coefficients magnitudes with sliding orientations are keep important.

4. TRIBOLOGICAL PIN/UHMWPE-ON-PLATE/AA20124-351 TESTS AND ANISOTROPIC FRICTION

Pin-on-Plane friction tests were carried out on a UMT Tribometer - Multi Specimen Test System of CETR Research Center - UPB Bucharest (Romania) (Fig. 3). Coulomb coefficients were measured during a linear movement for five constant sliding speeds v_g: 0.005, 0.05, 0.5, 1 and 5 mm/s of a hard polymer cylindrical pin (UHMWPE) with a 2 mm diameter on an anisotropic rolled plate AA2024-T351 alloy with 10 mm thickness and surface dimensions defined by 30mm x 25mm, along five different directions: θ = 0°, 30°, 45°, 60° and 90°, using three normal constant forces F_n: 3N, 5N and 7N, corresponding respectively to three different average contact pressures p_c: 0.95 MPa, 1.59 MPa and 2.22 MPa estimated by division of normal forces F_n with the pin circular surface.

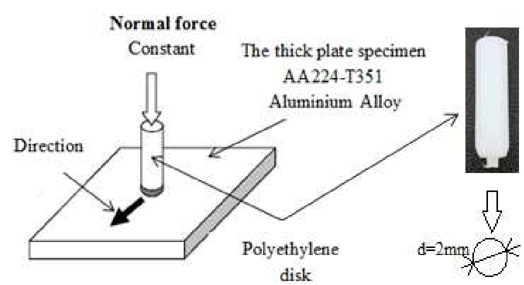


Fig.3. Pin-on-Plan Tribometer - UMT Multi Specimen Test and Principle of Linear Friction Slidings [8-9]

To estimate the Coulomb coefficients, the classic relationships $\mu = F_f/F_n$ were used. Assuming that the circular contact area is sufficiently small, this ratio is practically equal to the ratio of the tangential shear stress along the sliding direction with an average normal contact pressure defined by $p_c = 4F_n/\pi d^2$. As can be regarded in [8-9] any experimental curves corresponding to the variation of tangential forces and corresponding Coulomb coefficient with the time, not present stick-slip mechanism. All the results show also a weak influence of the relative sliding speed v_g and of the average contact pressure p_c on the values of the static Coulomb coefficients μ_s and kinetic μ_k ones (maximum variation 10%-15% close to the tribometer forces measurements uncertainty). A weak peak is reached especially for the sliding speed of 0.5 mm/s. Contrarily, it is observed more influence of the sliding direction on friction coefficients values, around of 50% - 80%. This phenomenon describes an anisotropic friction characterized by the variation of Coulomb coefficients with sliding direction. So for a sliding along the median direction DD ($\theta = 45°$), the value of friction coefficient is close to the quadratic media of values corresponding to LD ($\theta = 0°$) and TD ($\theta = 90°$) (error of 0.8%-3%) i.e. $\mu(45°) \approx \left[\left(\mu^2(0°)+\mu^2(90°)\right)/2\right]^{1/2}$ with shear stress τ orientation $\alpha = 60°$.

It is then proven an anisotropic friction phenomenon. In agreement with an elliptic quadratic anisotropic friction model (8), this one is write from the values of the Coulomb coefficients following the sliding directions of 0° and 90° and from the local average contact pressure p_c in the form:

$$\left[\tau_x^2 / \mu^2(0°)\right]+\left[\tau_y^2 / \mu^2(90°)\right]=p_c^2 \qquad (26)$$

Starting from the relationships (13), (21), (22) and (23) it can be obtain for each sliding direction θ the estimation of Coulomb coefficients by following expression:

$$\mu(\theta)=\left[\mu^2(0°)\cos^2(\theta)+\mu^2(90°)\sin^2(\theta)\right]^{1/2} \qquad (27)$$

Concerning the α direction of shear stress, this one is estimate by:

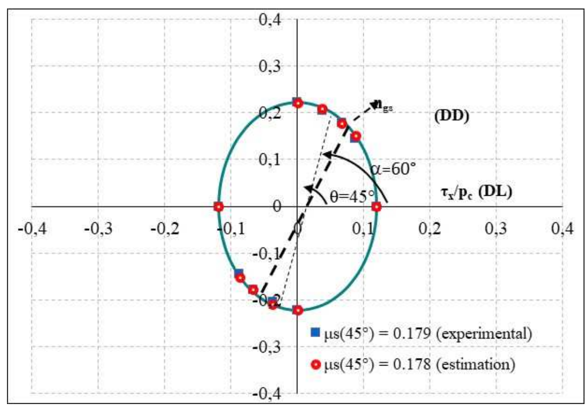
$$tg(\alpha)=\left[\mu(90°)/\mu(0°)\right]tg(\theta) \qquad (28)$$

The anisotropic friction model defining by (27) and (28) use for Coulomb coefficient computation a norm of shear stress vector projection along the sliding direction defined by:

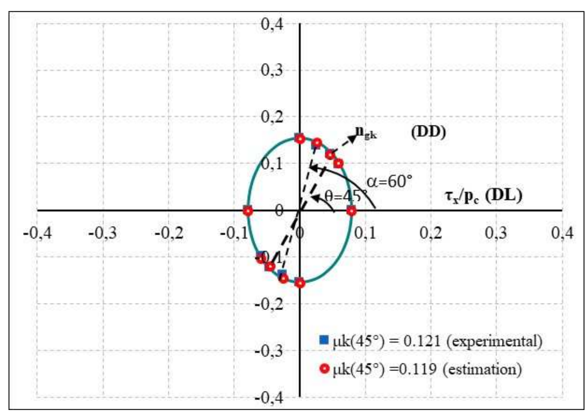
$$\vec{\tau}\cdot\vec{n}_g=\mu(\theta)p_c \qquad (29)$$

It can be conclude that elliptic anisotropic model estimates Coulomb coefficient $\mu(\theta)$ with a smallest prediction error around of 5% - 10%. It is also show that $(\alpha-\theta) \leq 15°$ and consequently if use of relationship (25) $\mu'(\theta)/\mu(\theta) \leq 1.035$ (prediction error of 3.5%). The uncertainty of the use of real tangential force or of the component oriented towards the sliding direction θ on estimation of Coulomb friction coefficients is very weak.

Figure 4 gives graphs of elliptic anisotropic friction criterion obtained for the five studied sliding directions and the two categories of Coulomb coefficients: static and kinetic, confirming the very good agreement existing between the computation model (elliptic curve) and experiment (red points).



a)



b)

Fig. 4. Elliptic Anisotropic Model corresponding to pion-plan sliding tribological tests of UHMWPE pion /AA2024-T351 thick plate along different directions for a local pressure p_c = 1.59 MPa : a) static friction μ_s, b) kinetic friction μ_k

For the estimation of the tangential shear stress resultant, the following formula are used:

$$\tau(\theta) = \mu'(\theta) p_c = \mu(\theta) p_c / \cos(\alpha - \theta) \qquad (30)$$

Recent studies concerning the differences bewteen the tangential force, i.e. friction shear stress and sliding velocities are proven in [10].

5. CONCLUSION

Along this research work, it is proof along an original theoretical framework the application of PWM, generally used to describe plastic flow, also for contact friction mechanics, regarded as the macroscopic scale.

This PWM "principle" can be consider now as a direct consequence of the general principle of Constructal Theory. The last one is write quantitatively by the minimization of the total losses or dissipated energy, in particular in the case of deformable material's body. From this minimization problem is prove mathematically the "Principle" of Maximum Work regarding both bulk stresses and friction shear. Then the normal rule and convexity properties of associated plastic or friction potentials becomes a sub-consequence of the proposed variational optimization problem.

Application concerning the anisotropic friction formulation together with validation of an anisotropic friction based on experimental Pin-on-Plane tribological tests show in an original way, the feasibility and robustness of Maximum Work Theorem that can be apply also to contact interfaces. Furthermore, it is prove that for an anisotropic contact surface the friction coefficient is also dependent of the sliding direction.

6. REFERENCES

[1] A. Bejan. **Shape and Structure: From Engineering to Nature**, UK: Cambridge Univ., 2000.

[2] A. Bejan, S. Lorente. **Design with Constructal Theory**, New Jersey: John Willey & Sons, 2008.

[3] A. Gavrus. *The Maximum Work Principle Regarded as a Consequence of an Optimization Problem Based on Mechanical Virtual Power Principle and Application of Constructal Theory*, AIP Proc. 1896, (2017).

[4] A. Gavrus. *Application of Constructal Theory to Write Mechanical Maximum Work Principle and Equilibrium State of Continuum Media Flow as a Solution of a Variational Optimization Problem*, Int. J. of Model. & Optim. 8(4), 227-231 (2018).

[5] A. Gavrus. *Computational Framework Concerning the Formulation of Maximum Work Principle used in Plasticity, Materials Forming and Tribology as a Consequence of a Variational Optimization Problem Defined From the Constructal Law*, IOP Conf. Ser.: Mater. Sci. Eng. 968, 8 pages (2020).

[6] B. Avitzur. **Metal Forming: Processes and Analysis**, New-York: McGraw-Hill, 1968

[7] P. Montmitonnet, *Lois de frottement et déformations plastiques*, Matériaux & Techniques 1-2-3 (1993) pp. 8-21.

[8] E. Pirva, A. Gavrus, A. Tudor, N. Stoica, S. Cananau, *Anisotropie et Caractère Fractal du Frottement pour des Contacts Tribologiques AA2024-T351/ UHMWPE*, Chapter Book **Tribologie des procédés et fonctionalisation des surfaces**, pp. 53-63, Presses des Mines, P. Montmitonnet, I. Lahouij (Ed), Paris, 2019.

[9] E. Pirva, A. Tudor, A. Gavrus, N. Stoica, S. Cananau, *Some aspects regarding the influence of the anisotropy of an AA2021-T351 rolled thick plate on its tribological behaviour*, Mechanics & Industry, 20 (6), 10 Pages, 2019.

[10] A. Wijata, B. Stanczyk, J. Awrejcewicz, *Empirical mathematical model for describing anisotropic dry friction forces*, Tribology International, 192, 2024.

UTILISATION DE LA LOI CONSTRUCTUALE POUR PROUVER LE « PRINCIPE » DE TRAVAIL MAXIMAL POUR LA PLASTICITÉ ET LE FROTTEMENT DES SURFACES DE CONTACT. APPLICATION AU COMPORTEMENT TRIBOLOGIQUE ANISOTROPE

Dans ce travail scientifique il est proposé de démontrer le « Principe » du Travail Maximal utilisé par la théorie de la plasticité des matériaux à travers un problème d'optimisation sous contraintes formulé à partir de la Loi Constructuale développée par le Prof. Adrian BEJAN de DUKE University (USA). Cette loi postule que tout système de taille finie cherche à s'écouler de plus en plus facilement au fil du temps et à évoluer dans telles configurations ou formes en répartissant au mieux leurs imperfections et minimisant les pertes ou dissipations ensemble avec la variation d'entropie. Du point de vue mécanique, si l'on ajoute les principes de la thermodynamique (définissant la conservation de l'énergie et l'évolution de l'entropie des systèmes), à partir de la première loi de Newton (décrivant l'équation de l'équilibre de la quantité de mouvement et sa forme équivalente du Principe des Puissances Virtuelles), ainsi que la tendance naturelle de tout type d'écoulement passant de valeurs élevées à valeurs faibles, on peut conclure que sous des charges externes, tous les flux d'écoulement des matériaux cherchent à minimiser les puissances dissipés de déformation et de frottement. Ensuite, entre tous les autres états virtuels correspondant aux valeurs des variables mécaniques virtuelles (contraintes, déformations, vitesses de déformation) ceux réels sont ceux qui minimisent la puissance totale dissipée. On obtient alors un problème de minimisation, qui prouve mathématiquement que le «Principe » de Travail Maximal, appliqué à la fois aux déformations plastiques et aux frottements de surfaces, pour tout type de milieux continus: solide, fluide, matériaux pâteux,…, est une conséquence de la Loi Constructale. Il s'agit d'une preuve mathématique originale correspondant à la mécanique de la plasticité et aussi à la tribologie jamais présentée jusqu'à présent par des études scientifiques antérieures d'autres chercheurs en dehors de l'auteur. En suivant les théorèmes obtenus de convexité des potentiels correspondants et des lois de normalités associées, il est alors possible de décrire l'écoulement plastique et le frottement des matériaux dans un cadre plus général au niveau des propriétés rhéologiques et tribologiques tant isotropes que anisotropes. La validation d'une formulation elliptique anisotrope de la loi de Coulomb et d'un modèle analytique pour prédire les coefficients de frottement statiques et cinétiques en fonction de différentes directions de glissement, est présentée à l'aide de données expérimentales fournies par un tribomètre Pion-Plane dans le cas des interfaces de contact métal/polymère anisotrope (plaque épaisse laminé AA20124-351/pion cylindrique UHMWPE). De plus, sont également estimer les différences observées entre les orientations des forces de cisaillement par frottement et le vecteur vitesse de glissement. Il est prouvé ainsi que pour des surfaces de contact anisotropes le coefficient de frottement dépendent aussi de la direction de glissement.

Adinel GAVRUS, INSA Rennes/Rennes Univ., Mechanical and Civil Engineering Laboratory LGCGM (EA 3913), 20 Av. des Buttes de Coesmes, 35043 Rennes Cedex, France, adinel.gavrus@insa-rennes.fr