TECHNICAL UNIVERSITY OF CLUJ-NAPOCA

**ACTA TECHNICA NAPOCENSIS**

**Series: Applied Mathematics, Mechanics, and Engineering**
**Vol. 68, Issue I, March, 2025**

# CASTIGLIANO'S THEOREM IN THE STUDY OF NONLINEAR ELASTIC STRUCTURAL ELEMENTS WITH POWER-LAW CONSTITUTIVE MODEL

Maria Luminita SCUTARU, Radu Georgian POPOESCU, Cristina MICLEA, Lucian DOMINTE

***Abstract:*** *The strong development of the industry in the last period has led to the strong diversification of the materials used in engineering applications and also to the increase of the forces and speeds that appear in these systems. As a result, numerous researches were undertaken that took into account the non-linear behavior of materials during the operation of mechanical systems. Constituent laws can no longer be considered linear laws. Numerous forms of constitutive laws have been proposed to cover the existing non-linearities. Among them, the power law type laws were noted, in which the dependence is of an exponential type. In the work, results were obtained regarding strain and stress appearing in standard elements used in engineering. Also, the possibility of using Castigliano's theorem was studied in the case of application to materials with non-linear behavior, with a power law constitutive law. The obtained results can be useful to engineers because they offer quick results and with good precision.*

**Key words:** nonlinear constitutive equation; power law; Castigliano's theorem; tension; compression; bending; shear; torsion

## 1. INTRODUCTION

The theory of elasticity, in its classic form, uses as the main hypothesis in a good part of its developments, the hypothesis of linearity that exists between stress and strain. Steel and other materials frequently used in engineering have, in most applications, this behavior and the results obtained using this hypothesis are generally good. There are still materials such as cast iron, copper, brass, lead, rubber, certain plastics and composite materials frequently used in modern technology, but for which this linear law no longer works. One of the most used models for the study of nonlinear materials is the power law. Non-linear constitutive models have started to be used more often recently due to the numerous and varied engineering applications in which such materials appear.

Traditional results from continuum media mechanics are used to present the main problems involved in the use of nonlinear materials are presented in [1]. Some types of materials with nonlinear behavior such as boron/aluminum fiber composite with metal matrix are described in [2,3]. In the paper [4], probabilistic models are used for the analysis of nonlinear systems. Micromechanical models remain very powerful models for the analysis of porous materials with phases with nonlinear behavior. In [6,7], a technique is developed that automatically adjusts the Jacobian based on the nonlinear constitutive law. In [8], nonlinear isotropic materials are studied. The paper [9] presents a method for the elastodynamic analysis of a one-dimensional nonlinear system. In [10], a useful study for post-buckling analysis is provided. The finite element method also offers in this case an important support for solving non-linear problems [11]. The technique is taken over by other authors in different versions [12-16]. Fiber-reinforced polymer composites, used more and more in real-world applications, present nonlinearities studied in [17,18].

**Received**: 10.02.25; **Similarities**: 05.03.25: **Reviewed**: 19.02./17.02.25: **Accepted**:12.03.25.

Models for hyperelastic materials are presented in [19,20].

More complex phenomena involving nonlinear materials are studied in [21]. Different types of constitutive laws are analyzed in [22,23]. Large, non-linear deformations are presented in [24]. Constitutive laws represent the starting point in the study of such problems [25-27]. The use of FEM has been used by numerous researchers [28–35].

Biomechanics problems where living materials are strongly nonlinear [36–40]. Numerous engineering applications involving nonlinear materials have been studied and the results communicated in interesting papers [41-50], while theoretical advances in the field of nonlinear materials are presented in papers [51-60].

In this paper, the authors aim to make a contribution to the development of methods for the study of non-linear materials, whose dependence on stress can be conveniently described with a constitutive law of the power law type.

## 2. POWER LAW STRAIN-STRESS MODEL

The description of the power low type constitutive law is made through the relations: [61]:

$$\varepsilon = \left( \frac{\sigma}{\sigma_0} \right)^n , \tag{1}$$

respectively,

$$\gamma = \left( \frac{\tau}{\tau_0} \right)^n . \tag{2}$$

In these relationships, $n$ is denoted by a real number that represents the exponent of the power-law, $\varepsilon$ is the normal strain, $\gamma$ is shear strain, $\sigma$ is denoted the normal stress, $\tau$ represents the shear stress. The constants of material $\sigma_0$ and $\tau_0$ have the same Young's modulus physical dimension [21]. These constants and the exponent $n$ can be determined experimentally for every material considered. Hooke's law is a particular case when $n$ = 1. When $n$ = 0 the material is totally rigid whereas when $n = \infty$ we are in the plastic domain.

In the classical theory of elasticity, it is considered, for most of the developments made, that the strain-stress relationship is a linear relationship. In reality, the linearity hypothesis is an approximation that describes very well most materials and their behavior within reasonable loading limits. However, in reality, materials are generally non-linear. Certain materials are strongly non-linear and for this reason it was found that special non-linear models must be developed for some of them. Among the multitude of models that were used in the study of these unusual materials, the constitutive law of the power-law type is counted. Although there are some studies on the behavior of these materials [21], an organized and systematic study has not been done until now. We mention the contribution [61] that completes the analysis of such systems. In the current work, we will deal with the possibility of using Castigliano's theorem to solve such problems.

## 3. STANDARD LOADING CASES

In previous papers and book the basic loading cases are studied and basic results are obtained considering the hypothesis of linear dependence strain-stress (Hook's law) [61]. In the following we will adapt the main results to the case when we work with a power law dependency. We present these results [61].

### 3.1. Tension. Compression

The simplest load and frequently encountered in practice is tension (pulling)/compression (squashing).Some real-world scenarios frequently encountered in engineering practice will be presented next, form an isotropic, homogeneous, with constant section beam.

a) In the study will be used the Bernoulli's hypothesis. This means that $\varepsilon = ct$ for a single, arbitrary section at each point of the middle average fiber. As a consequence, taking into account the Eq.(1), the stress is constant for all of the section's points:

$$\sigma = \sigma_0 (ct)^{\frac{1}{n}} . \tag{3}$$

If is written the equilibrium equation of the stresses acting on the section, it obtains:

$$N = \int_A \sigma dA = \sigma A \,. \tag{4}$$

$N$ is normal load in the section. So, the stress can be determined with the relation:

$$\sigma = \frac{N}{A}, \tag{5}$$

and the beam's lengthening/shortening is:

$$\Delta l = \varepsilon \cdot l = \left(\frac{\sigma}{\sigma_0}\right)^n l = \left(\frac{N}{\sigma_0 A}\right)^n l \quad ; \tag{6}$$

where $l$ is the length of the beam and $A$ the area of its section.

### 3.2. Torsion

The precedent hypothesis for the material are the same, that is a beam with a circular section and the length $l$. We denote with $R$ the section's radius and $\rho$ the radius of the elementary element $dA$, on which the stress $\tau$ is applied due to the torsional moment $M_r$. The equilibrium equation offer us:

$$M_r = \int_A \tau\rho dA \,. \tag{7}$$

The sliding angle is:

$$\gamma = \rho\theta \ , \tag{8}$$

It results:

$$\tau = \tau_0 \gamma^{\frac{1}{n}} \,. \tag{9}$$

Using Eqs. (8) and (9) in Eq. (7), the moment becomes:

$$M_r = \tau_0 \theta^{\frac{1}{n}} \int_A \rho^{\frac{n+1}{n}} dA = \tau_0 \theta^{\frac{1}{n}} I_r \ , \tag{10}$$

It is denoted:

$$I_r = \int_A \rho^{\frac{n+1}{n}} dA \ , \tag{11}$$

the cross-section's generalized polar moment. So, the stress is:

$$\tau = \frac{M_r \rho^{\frac{1}{n}}}{I_r}, \tag{12}$$

and the rotation angle:

$$\varphi = l\theta = \left(\frac{M_r}{\tau_0 I_r}\right)^n l, \tag{13}$$

### 3.3. Bending

The basics of the bending in Strength of Materials in the hypothesis of the linear elasticity are considered known. Using the Bernoulli's assumption, the elongation of a fiber situated at a distance $y$ from the neutral axis can be expressed by:

$$\varepsilon = \frac{y}{\rho}, \tag{14}$$

It is denoted with $\rho$ the average fiber's radius of curvature. The expression of stress is (see Eq.(1)):

$$\sigma = \sigma_0 \left(\frac{y}{\rho}\right)^{\frac{1}{n}} . \tag{15}$$

Considering equilibrium equation it results:

$$M_i = \int_A \sigma y dA = \frac{\sigma_0}{\rho^{\frac{1}{n}}} \int_A y^{\frac{n+1}{n}} dA = \frac{\sigma_0 I_z}{\rho^{\frac{1}{n}}} \ , \tag{16}$$

It is denoted:

$$I_z = \int_A y^{\frac{n+1}{n}} dA \ . \tag{17}$$

the generalized axial moment of the cross section. Using Eq.(16) it results:

$$\frac{1}{\rho} = \left(\frac{M_i}{\sigma_0 I_z}\right)^n . \tag{18}$$

From Eq.(15) and Eq. (18) it is obtained the stress:

$$\sigma = \frac{M_i y^{\frac{1}{n}}}{I_z} . \tag{19}$$

or:

$$\sigma = \frac{M_i}{W_z} \ , \tag{20}$$

In Eq.(20) it is noted:

$$W_z = \frac{I_z}{y^{\frac{1}{n}}} . \tag{21}$$

(the generalized axial strength modulus).

### 3.4. Shear

We consider that Juravski's [62,63] assumption are used (Figure 4). The section is considered divide in two parts. The part between sections I and II and the section across a plane parallel to the neutral plane is separated by the distance *y*, and the equilibrium equation is:

$$\int_{A_y}(\sigma + d\sigma)dA - \int_{A_{y1}}\sigma dA - \tau_{xy}dx = 0 . \tag{22}$$

Considering the expression of stresses:

$$\sigma = \frac{M_i y^{\frac{1}{n}}}{I_z} . \tag{23}$$

and:

$$\sigma + d\sigma = \frac{(M_i + dM_i)y^{\frac{1}{n}}}{I_z} . \tag{24}$$

it obtains:

$$\int_{A_y}\frac{dM_i y^{\frac{1}{n}}}{I_z}dA = \tau_{xy}bdx \ . \tag{25}$$

With the notation:

$$S_z = \int_{A_y} y^{\frac{1}{n}}dA \ , \tag{26}$$

it obtains:

$$\tau_{xy} = \frac{TS_z}{bI_z} \ . \tag{27}$$

## 4. CASTIGLIANO'S THEOREM

### 4.1. DEFORMATION ENERGY

a) **Internal energy.** According to the nature of the request, the internal energy is given by the expressions:

$$L_i = \int_V dV \int_0^{\sigma}\sigma d\varepsilon \, , \tag{28}$$

or:

$$L_i = \int_V dV \int_0^{\tau}\tau d\gamma \, . \tag{29}$$

Considering the constitutive power law it obtains:

$$d\varepsilon = n\frac{\sigma^{n-1}}{\sigma_0}d\sigma \, , \tag{30}$$

and:

$$d\gamma = n\frac{\tau^{n-1}}{\tau_0}d\tau \, , \tag{31}$$

Using Eq.(30) and (31), the expressions of the internal energy are obtained:

$$L_i = \frac{n}{(n+1)\sigma_0^n}\int_V \sigma^{n+1}dV \ , \tag{32}$$

and:

$$L_i = \frac{n}{(n+1)\tau_0^n}\int_V \tau^{n+1}dV \ , \tag{33}$$

For tension/compression, if Eq. (5) is taken into account, we obtain:

$$L_i = \frac{n}{(n+1)\sigma_0^n A^n}\int_0^l N^{n+1}dl \, , \tag{34}$$

where, if we consider constant the external tension/compression force *N*, we get:

$$L_i = \frac{nN^{n+1}l}{(n+1)\sigma_0^n A^n} \, , \tag{35}$$

For pure bending it results:

$$L_i = \frac{n}{(n+1)\sigma_0^n I_z^n}\int_0^l M_i^{n+1}dl \, , \tag{36}$$

and, if the bending moment is considered constant:

$$L_i = \frac{nM_i^{n+1}l}{(n+1)\sigma_0^n I_z^n} \, , \tag{37}$$

Finally, for the torsion of beams with a circular section, we obtain:

$$L_i = \frac{n}{(n+1)\tau_0^n I_r^n}\int_0^l M_r^{n+1}dl \, , \tag{38}$$

and, if the twisting moment is constant:

$$L_i = \frac{nM_r^{n+1} l}{(n+1)\tau_0^n I_r^n}, \tag{39}$$

b) **Complementary energy.** Analogously, if the nature of the loads is taken into account, the relationships are obtained, respectively:

$$L_c = \int_V dV \int_0^\sigma \varepsilon d\sigma, \tag{40}$$

$$L_c = \int_V dV \int_0^\tau \gamma d\tau, \tag{41}$$

If Eq.(1) and Eq.(2) are taken into account, the relationships are obtained: $P_j \ (j \neq k)$ .

$$L_c = \frac{1}{(n+1)\sigma_0^n} \int_V \sigma dV, \tag{42}$$

and:

$$L_i = \frac{1}{(n+1)\tau_0^n} \int_V \tau dV. \tag{43}$$

Thus, if we are dealing with tension/compression, it can be written:

$$L_c = \frac{1}{(n+1)\sigma_0^n A^n} \int_0^l N^{n+1} dl, \tag{44}$$

where, if we consider constant the external tension/compression force $N$, we get:

$$L_c = \frac{N^{n+1} l}{(n+1)\sigma_0^n A^n}, \tag{45}$$

If we are dealing with pure bending, then:

$$L_c = \frac{1}{(n+1)\sigma_0^n I_z^n} \int_0^l M_i^{n+1} dl, \tag{46}$$

and, if the bending moment is considered constant:

$$L_c = \frac{M_i^{n+1} l}{(n+1)\sigma_0^n I_z^n}, \tag{47}$$

When twisting the beams with a circular section, you get:

$$L_c = \frac{1}{(n+1)\tau_0^n I_r^n} \int_0^l M_r^{n+1} dl, \tag{48}$$

and, if the twisting moment is constant:

$$L_c = \frac{M_r^{n+1} l}{(n+1)\tau_0^n I_r^n}, \tag{49}$$

In the case of compound loads, the sum of the energies is made.

## 4.2. GENERALIZED CASTIGLIANO'S THEOREM

It is considered a non-linear elastic system actuated by independent forces $P_1, P_2, \ldots, P_m$. By giving to a force $P_k$ an elementary increase $dP_k$, keeping the other forces constant, the complementary energy increases by:

$$dL_c = \frac{\partial L_c}{\partial P_k} dP_k \quad . \tag{50}$$

The complementary energy increases only due to the displacement of the $P_i$ force, being equal to:

$$dL_c = \delta_k dP_k \quad . \tag{51}$$

The other forces $P_j \ (j \neq k)$ by their displacement do not produce complementary mechanical work. Castigliano's generalized theorem results:

$$\delta_k = \frac{\partial L_c}{\partial P_k} \quad . \tag{52}$$

Therefore, if the complementary energy of a certain elastic system (therefore also nonlinear) is expressed as a function of external (independent) forces, then the derivative of this energy in relation to one of the forces is equal to the displacement projected on the force at the point of its application .

The theorem is demonstrated analogously in the case of the rotations that are produced by an externally applied couple. The theorem applies to:

- calculation of displacements of nonlinear and linear systems;

- raising the undeterminacy of systems in the form of:

$$\frac{\partial L_c}{\partial X} = 0 \ . \qquad (53)$$

where X is the statically undeterminate reaction.

## 5. RESULTS

### 5.1. THE DEFORMATION TO BASIC LOAD

a) For tension/compresion it can be written:

$$\Delta l = \frac{\partial L_c}{\partial P} = \frac{\partial}{\partial P}\left[\frac{1}{(n+1)\sigma_0^n A^n}\int_0^l N^{n+1} dl\right] = \frac{1}{(n+1)\sigma_0^n A^n}\int_0^l N^n \frac{\partial N}{\partial P} dl \ . \qquad (54)$$

For $N = P =$ ct, it results:

$$\Delta l = \frac{N^n l}{\sigma_0^n A^n}. \qquad (55)$$

For bending, if the bending moment $M = M_i = ct$ is considered, we obtain, in the same way:

$$f = \frac{\partial L_c}{\partial P} = \frac{1}{\sigma_0^n I_z^n}\int_0^l M_i^{\ n} \frac{\partial M_i}{\partial P} dl = \left(\frac{M_i}{\sigma_0 I}\right)^n \frac{l^2}{2}. \qquad (56)$$

and:

$$\varphi = \frac{\partial L_c}{\partial M} = \frac{1}{\sigma_0^n I_z^n}\int_0^l M_i^{\ n} \frac{\partial M_i}{\partial M} dl = \left(\frac{M_i}{\sigma_0 I}\right)^n l. \qquad (57)$$

For the torsional stress, it is obtained, analogously:

$$\varphi = \frac{\partial L_c}{\partial M} = \frac{1}{\tau_0^n I_r^n}\int_0^l M_r^{\ n} \frac{\partial M_r}{\partial M} dl = \left(\frac{M_r}{\tau_0 I_r}\right)^n l. \qquad (58)$$

### 5.2. DEFORMATION OF A BEAM LOADED WITH A CONCENTRATED FORCE

Using the presented method, the displacement in the middle of the beam will be calculated:

$$f = \frac{\partial L_c}{\partial P} = \frac{2}{\sigma_0^n I_z^n}\int_0^{\frac{l}{2}} M_i^{\ n} \frac{\partial M_i}{\partial P} dl \ . \qquad (59)$$

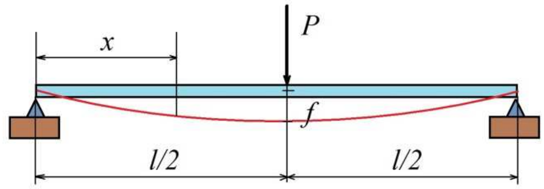


Figure 1. Beam loaded with a concentrated force

Because:

$$M_i = \frac{P}{2} x \ . \qquad (60)$$

it results:

$$\frac{\partial M_i}{\partial P} = \frac{x}{2} \ . \qquad (61)$$

and:

$$f = \frac{2}{\sigma_0^n I_z^n}\int_0^{\frac{l}{2}} \left(\frac{Px}{2}\right)^n \frac{x}{2} dx = \frac{P^n l^{n+2} x}{(n+2)\sigma_0^n I_z^n 2^{2n+2}} \ . \qquad (62)$$

If we consider $n = 1$, we find the well-known relationship from the Strength of Materials course:

$$f = \frac{Pl^3}{48 E I_z^n} \ . \qquad (63)$$

### 5.3. DEFORMATION OF AN ANGLED BEAM

The vertical ($v$) and horizontal (u) displacement of the free end is calculated.

$$v = \frac{\partial L_c}{\partial P} \qquad (64)$$

where:

$$L_c = \frac{1}{(n+1)\sigma_0 I_z^n}\left[\int_0^l (Px)^{n+1} dx + \int_0^h (Pl)^{n+1} dy\right] \qquad (65)$$

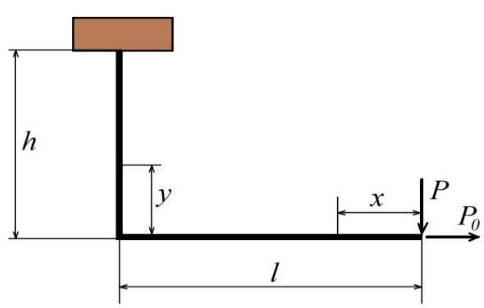


**Figure 2. Angled beam**

It obtains:

$$v=\frac{P^n}{\sigma_0^n I_z^n}\left(\frac{l^{n+2}}{n+2}+l^{n+2}h\right) \quad . \qquad (66)$$

$$u=\frac{\partial L_c}{\partial P_0} \qquad (67)$$

where:

$$L_c=\frac{1}{(n+1)\sigma_0^n I_z^n}\left[\int_0^l (Px)^{n+1}dx+\int_0^h (P_0 y-Pl)^{n+1}dy\right] \qquad (68)$$

After performing the calculations, you will get:

$$L_c=\frac{1}{(n+1)\sigma_0^n I_z^n}P^{n+1}l^{n+2}+\frac{1}{(n+1)\sigma_0^n I_z^n P_0}\left[(P_0h-Pl)^{n+2}-(-Pl)^{n+2}\right] \qquad (69)$$

Then it results for *u*:

$$u=\frac{\left[-(P_0h-Pl)^{n+2}+(-Pl)^{n+2}\right]}{(n+1)(n+2)\sigma_0^n I_z^n P_0^2}+\frac{\left[P_0(n+2)(P_0h-Pl)^{n+2}h\right]}{(n+1)\sigma_0^n I_z^n P_0^2} \quad . \qquad (70)$$

If $P_0=0$, Eq. (70) is undetermined. Using l'Hospitales' rule, we get:

$$v=-\frac{P^n l^n h^2}{2\sigma_0^n I_z^n} \quad . \qquad (71)$$

## 5.5 THE UNDETERMINATION OF A BEAM

Let X be an undeterminate static reaction (Fig.3). The bending moments in the two regions of the bar are:

$$M_1=Xx \quad ; \quad M_2=X\left(\frac{l}{2}+x\right)-\frac{Px}{4} \quad . \qquad (72)$$

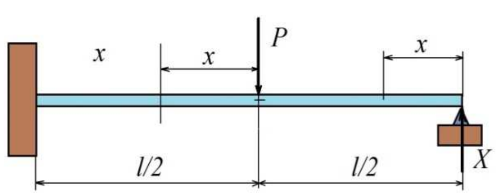


**Fig. 3. Undetermined beam**

Write the zero displacement condition in support 2:

$$\frac{1}{\sigma_0^n I_z^n}\int_0^{\frac{l}{2}}(Xx)^n\, xdx+\frac{1}{\sigma_0^n I_z^n}\int_0^{\frac{l}{2}}\left[X\left(\frac{l}{2}+x\right)-\frac{Px}{4}\right]^n\left(\frac{l}{2}+x\right)dx=0 \qquad (73)$$

If the integration is done, it results after elementary transformations:

$$\left(\frac{X}{P}\right)^n\left(\frac{X}{P}-1\right)^2+\left(2\frac{X}{P}-1\right)^{n+1}\left(2\frac{X}{P}-\frac{2n+3}{n+1}\right)-\left(\frac{X}{P}\right)^{n+1}\left(\frac{X}{P}-\frac{n+2}{n+1}\right)=0 \qquad (74)$$

Introducing the new variable:

$$t=\frac{X}{P} \quad . \qquad (75)$$

the following equation is obtained:

$$t^n(t-1)^2+(2t-1)^{n+1}\left(2t-\frac{2n+3}{n+1}\right)-t^{n+1}\left(t-\frac{n+2}{n+1}\right)=0 \qquad (76)$$

which will give the unknown force X, after solving the algebraic equation using a numerical method.

## 6. DISCUSSION AND CONCLUSIONS

In the classic theory of elasticity, a basic hypothesis is the linearity of the dependence between strain and stress, the hypothesis on which a large part of this theory is built. Of course, in the real world this hypothesis is only a very convenient approximation for the development of the theory. In reality, practically all materials are non-linear, but for materials mainly used in engineering applications, the hypothesis of small deformations and Hooke's law work excellently. However, there are also materials that are strongly non-linear (cooper) or that are subjected to loads high enough for the non-linearity to develop significant negative effects. The need to use these materials, for various reasons, as well as the development of the industry through the appearance of structures of high complexity or with large dimensions and loaded with great forces, led to the need to study such non-linear materials. Different nonlinearity models were considered and used. Among them, the models that use a constitutive power law type law stand out. Some applications using this law are shown in the references used. In the current work, we were concerned with the possibility of using Castigliano's theorem for materials whose constitutive law is of the power law type. Some examples illustrate the possibility of using this theorem in the specific conditions given by leaving the linear law and replacing it with an exponential law. Some differences are highlighted that appear by comparison with the classical calculation when the strain-stress dependence is linear. The cases presented in this paper can provide a starting point for future research on the behavior of such materials. The results can be extended to other types of constitutive laws, in the case of some engineering applications that claim this. These applications become extremely necessary in the contemporary context of the development of new and composite materials. The obtained results can be easily used in future applications, being able to offer quick and simple solutions, desired by any designer.

## 7.REFERENCES

[1]Drapaca, C.S.; Sivaloganathan, S.; Tenti, G. *Nonlinear constitutive laws in viscoelasticity.* Mathematics and Mechanics of Solids, Vol. 12, No. 5, pp. 475-501, 2007.

[2]Nigam, H.; Dvorak, G.J.; Baheieldin, Y.A. *An Experimental Investigation of Elastic-Plastic Behavior od a Fibrous Boron Aluminum Composite.Matrix-Dominated Mode.* International Journal of Plasticity , 1994, *10* (1) , pp.23-48.

[3]Gil, C.M.; Lissenden, C.J.;Lerch, B.A. *Unusual nonlinear response of some metallic materials.* Mechanics of Materials, 1999, *31* (9) , pp.565-577

[4]Teferra, K.; Deodatis, G. *Variability response functions for beams with nonlinear constitutive laws. Probabilistic Engineering Mechanics*, Vol. 29, pp.139-148, 2012.

[5]Michel, J.C.; Suquet, P. *The Constitutive Law of Nonlinear Viscous and Porous* Materials. *Journal of the Mechanics and Physics of Solids*, Vol. 40, No. 4, pp.783-812, 1992.

[6]Cantón-Sánchez, R.; Romero, I. *Dimensionally reduced nonlinear solids with general loads and constitutive laws: Theory and finite element formulation for rod-like bodies.* International Journal of Solids and Structures, 2021, *210*, pp.273-288.

[7]Wackerfuss, J.; Gruttmann, F. *A nonlinear Hu-Washizu variational formulation and related finite-element implementation for spatial beams with arbitrary moderate thick cross-sections.* Computer Methods in Applied Mechanics and Engineering, 2011, *200* (17-20) , pp.1671-1690

[8]Lainé, E; Vallée, C and Fortuné, D. *Nonlinear isotropic constitutive laws: choice of the three invariants, convex potentials and constitutive inequalities.* International Journal of Engineering Science, Vol. 37, No. 15, pp.1927-1941, 1999.

[9]Ehrhardt, M. *Absorbing Boundary Conditions for Hyperbolic Systems. Numerical Mathematics-Theory Methods and Applications,* 2010, *3* (3) , pp.295-337

[10]Yang, Y.B.; Leu, L.J. *Constitutive Laws and Force Recovery Procedures in Nonlinear-Analysis of Trusses. Computer Methods in Applied Mechanics and Engneering,* Vol. 92, No. 1 , pp. 121-131, 1991.

[11]Mrówczynski, D.; Gajewski, T. Garbowski, T. *Application of the generalized nonlinear constitutive law in 2D shear flexible beam structures. Archives of Civil Engineering*, Vol. 67, No. 3, pp. 157-176, 2021.

[12]Pedersen, P.; Taylor, J.E. *Optimal Design Based on Power-Law Nonlinear Elasticity.* IUTAM Symposium on Optimal Design with Advanced Materials, 1993, p.51-66.

[13]Steigemann, M. *Power-Law Solutions of Anisotropic Multi-material Elasticity Problems.* Journal of Elasticity, 2015, *118* (1) , pp.63-87

[14]Marin, M; Carrera, E and Vlase, S. *An extension of the Hamilton variational principle for piezoelectric bodies with dipolar structure.* Mechanics of Advanced Materials and Structures, Vol. 30, No. 12, pp. 2453-2457, 2023.

[15]Nami, M.R. ; Janghorban, M. Free *Vibration of Functionally Graded Size Dependent Nanoplates Based on Second Order Shear Deformation Theory using Nonlocal Elasticity Theory.* Iranian Journal of Sciences and Technology-Transaction of Mechanical Engineering, 2015**,** *39* (M1) , pp.15-28.

[16]Codarcea-Munteanu, L.; Marin, M.; Vlase, S. *The study of vibrations in the context of porous micropolar media thermoelasticity and the absence of energy dissipation.* Journal of Computational Applied Mechanics, Vol. 54, No. 3, pp. 437-454, 2023.

[17]Giménez-Palomares, F; Jiménez-Mocholí, AJ; Espinós-Capilla, A. *Obtaining Stress Distribution in Bending through Virtual Laboratories.* 8th International Technology, Education and Development Conference (INTED) 2014, pp.6209-6218.

[18]Katouzian, M.;, Vlase, S.; Marin,M. *Elastic moduli for a rectangular fibers array arrangement in a two phases composite.* Journal of Computational Applied Mechanics, Vol. 55, No. 3, pp. 538-551, 2024.

[19]Korepanov, VV; Matveenko, VP; Shardakov, I.N. *Numerical analysis of singular solutions of two-dimensional problems of asymmetric elasticity.* Mechanics of Solids, 2013**,** *48* (4) , pp.397-404.

[20]Elishakoff, I; Gentilini, C.; Viola, E. *Three-dimensional analysis of an all-round clamped plate made of functionally graded materials.* Acta Mechanica, 2005**,** *180*, (1-4) , pp.21-36.

[21]Braun, M. Bishoff, M.; Ramm, E. *Nonlinear Shell Formulations for Complete 3-Dimensional Constitutive Laws including Composites and Laminates.* Computational Mechanics, Vol. 15, No. 1, pp.1-18, 1994.

[22]Kim, J.H.; Lee, M.G.; Kang, T.J. *Development of nonlinear constitutive laws for anisotropic and asymmetric fiber reinforced composites.* Polymer Composites, Vol. 29, No. 2, pp. 216-228, 2008.

[23]Bardella, L. *A phenomenological constitutive law for the nonlinear viscoelastic behaviour of epoxy resins in the glassy state.* European Journal of Mechanics A-Solids, Vol. 20, No. 6, pp. 907-924, 2001.

[24]Cai, W.; Wang, P.; Wang, Y.J. *Linking fractional model to power-law constitutive model with applications to nonlinear polymeric stress-strain responses.* Mechanics of Time-Dependent Materials, Vol. 27, No. 3, pp. 875-888, 2023.

[25]Katouzian, K.; Vlase, S.; Scutaru, M.L. *Finite Element Method-Based Simulation Creep Behavior of Viscoelastic Carbon-Fiber Composite.* Polymers, Vol. 13, No. 7, Art. No. 1017, 2021.

[26]Katouzian, M.; Vlase, S.; Calin, M.R. *Experimental procedures to determine the viscoelastic parameters of laminated composites.* Journal of Optoelectronics and Advanced Materials, 2011, 13 (9-10) , pp.1185-1188.

[27]Launay, B. A Simplified Nonlinear Model for *Describing the Viscoelastic Porperties of Wheat-Flour Doughs at High Shear Strain.* Cereal Chemistry, 1990, *67* (1) , pp.25-31.

[28]Suárez-Antola, R. Power *Law and Stretched Exponential Responses in Composite Solids*. International Conference on Materials Science, Machinery and Energy Engineering (MSMEE 2013), Materials Science, Machinery and Energy Engineering, 2014, 853 , pp.9-16.

[29]Ponte Castaneda, P.; Suquet, P. *Nonlinear composites.* Advanced in Applied Mechanisc, Vol 34, 1998, *34* , pp.171-302.

[30]Bowyer, E.P.; Krylov, V.V. *Experimental investigation of damping flexural vibrations in glass fibre composite plates containing one- and two-*

*dimensional acoustic black holes*. Composite Structures, 2014, *107*, pp.406-415.

[31]Xu, YJ; Wu, QL; Yao, F. *Creep behavior of bagasse fiber reinforced polymer composites.* Bioresource Technology, 2010, *101* (9), pp.3280-328.

[32]Teodorescu Draghicescu Horatiu, Scutaru M.L., Rosu D, Calin M.R., Griogore P, *New Advanced Sandwich Composite with twill weave carbon and EPS*, Journal of Optoelectronics and Advanced Materials-Rapid Ccommunications, 2013, Vol 15, (3-4), pg 199-203

[33]Herve, E.; Dendievel, R..; Bonnet, G. Steady-State Power-Law Creep in Inclusion *Matrix Composite-Materials.* Acta Metallurgica Materialia, 1995**,** *43* (11), pp.4027-4034.

[34]Ionita, D.; Arghir, M. *Consideration on the Lagrange Inhterpolation Method Applicable in Mechanical Engineering.* Acta Technica Napocensis. Series Applied Mathematics, Mechanics and Engineering, 2021, 64 (2), pp.329-332.

[35]Vlase, S.; Teodorescu-Draghicescu, H.; Calin, M.R.; Serbina, L. *Simulation of the elastic properties of some fibre-reinforced composite laminates under off-axis loading system.*Optoelectronics and Advanced Materials-Rapid Communications, 2011, *5* (3-4), pp.424-429

[36]Wang, C.; Zhang, W.; Kassab, G.S. *The validation of a generalized Hooke's law for coronary arteries.* American Journal of Physiology-Heart and Circulatory Physiology, Vol. 294, No.1, pp. H66-H73, 2008, DI 10.1152/ajpheart.00703.2007

[37]May-Newman, K.; Yin, F.C.P. *A constitutive law for mitral valve tissue.* Journal of Biomechanical Engineering-Transactions of the ASME, Vol. 120, No. 1, pp. 38-47, 1998, DI 10.1115/1.2834305.

[38]Caggiano, L.R.; Holmes, J.W. *A Comparison of Fiber Based Material Laws for Myocardial Scar*. Journal of Elasticity. Vol 145, No. 1-2, pp. 321-337, 2021, DI 10.1007/s10659-021-09845-5.

[39]Marcucci, L.; Bondi, M.; Randazzo, G.; Reggiani, C.; Natali, A.; Pavan, P.G. *Fibre and extracellular matrix contributions to passive forces in human skeletal muscles: An experimental based constitutive law for numerical modelling of the passive element in the classical Hill-type three element model. PLOS ONE*, Vol. 14, No. 11, Art. No. e0224232,2019, DI 10.1371/journal.pone. 0224232.

[40]Agarwal, A. *An improved parameter estimation and comparison for soft tissue constitutive models containing an exponential function.* Biomechanics and Modeling in Mechanobiology, Vol. 16, No. 4, pp. 1309-1327, 2017, DI 10.1007/s10237-017-0889-3.

[41]Nemani, A.V.; Ghaffari, M.; Nasiri, A. *Advancements in Additive Manufacturing for Copper-Based Alloys and Composites: A Comprehensive Review.* Journal of Manufacturing and Materials Processing, 2024, *8* (2).

[42] Ishfaq, K.; Asad, M.; Harris, M.; Alfaify, A.; Anwar, S. ; Lamberti, L.; Scutaru, M.L. *EDM of Ti-6Al-4V under Nano-Graphene Mixed Dielectric: A Detailed Investigation on Axial and Radial Dimensional Overcuts*
NANOMATERIALS, 2022, 12 (3)

[43] Seshagirirao, D.V.; Raju, S.; Mantrala, K.M. *Tensile, hardness and metallurgical analysis of laser cured Co-Cr-Mo alloy samples.* Advances in Materials and Processing Technologies, 2023, *9* (4), pp.1448-1458.

[44] Vlase,S.; Teodorescu-Draghicescu, H.; Calin, M.R. *Behavior of multiphase fiber-reinforced polymers under short time cyclic loading.* Optoelectronics and Advanced Materials-Rapid Ccommunications, 2011, 5 (3-4), pp.419-423

[45] Vlase, S.; Teodorescu-Draghicescu, H.; Scutaru, M.L. *Advanced Polylite composite laminate material behavior to tensile stress on weft direction.* Journal of Optoelectronics and

Advanced Materials, 2012, *14* (7-8) , pp.658-663.

[46]Noll, W. On the Continuity of the Solid and Fluid States. Indiana University Mathematics Journal, 1955, 4(1):3-81. doi: 10.1512/IUMJ.1955.4.54001

[47]Luo, J.; Gao, J.J.; Qi, S.Y. *Study on microstructure and mechanical properties of Ni60+WC/ Ni35/ AISI1040 functional surface gradient structure of remanufacturing chute plate for the mining scraper by a low cost high power $CO_2$laser cladding technique.* Materials Research Express, 2020, *7*(8)

[48]Ziaee, S. *Dynamic stability of porous functionally graded nanotubes via nonlocal strain and velocity gradient theory.* Journal of the Brazilian Society of Mechanical Sciences and Engineering, 2023, *45* (12)

[49]Costa, A.; Miranda, R.; Yapp, D. *Analysis of beam material interaction in welding of titanium with fiber lasers.* Materials and Manufacturing Processes, 2007**,** *22* (7-8) , pp.798-803

[50]Teodorescu-Draghicescu, H.; Vlase, S.; Mihalcica, M. *Advanced Pultruded Glass Fibers-Reinforced Isophtalic Polyester Resin.* MATERIALE PLASTICE, 2015, *52* (1) , pp.62-64.

[51]Ali, S.Z.; Ho, W.O; Udrea, F. *A High Temperature SOI CMOS $NO_2$ Sensor.* Proceedings of the 14th International Symposium on Olfaction and Electronic Nose, 2011, 1362 , pp.53-+

[52]Flannery, A.F.; Mourlas, N.J.; Kovacs, G.T.A. *PECVD silicon carbide as a chemically resistant material for micromachined transdu-cers.* 9th International Conference on Solid-State Sensors and Actuators, Sensors and Actuators A-Physical, 1998, 70 (1-2) , pp.48-55.

[53]Chen, Y.H.; Uchino, K .; Viehland, D. *Substituent effects in $0.65Pb(Mg_{1/3}Nb_{2/3})O_3$-$0.35PbTiO_3$ piezoelectric ceramic.* Journal of Electroceramics, 2001**,** *6* (1) , pp.13-19.

[54]Tao, H.C.; Fan, L.Z.; Qu, X.H. *In situ synthesis of $TiO_2$-graphene nanosheets composites as anode materials for high-power lithium ion batteries.* Electrochimica Acta, 2012**,** *69* , pp.328-333

[55]Loewke, K.; Meyer, D.; Nemat-Nasser, S. *Signal identification in smart composite materials using the two-dimensional fast Fourier transform.* Smart Materials and Structures, 2005¸ *14* (5) , pp.895-903

[56]Sawada, K.; Tabuchi, M.; Kimura, K. *Analysis of long-term creep curves by constitutive equations.* 11th International Conference of Creep and Fracture of Engineering Materials and Structures, May 04-09, 2008, Bad Berneck, Germany, 2009, Vol. 510-11, pp. 190-194, DI 10.1016/j.msea.2008.06.057

[57]Aharonov, E.; Scholz, C.H. *The Brittle-Ductile Transition Predicted by a Physics-Based Friction Law.* Journal of Geophysical Research Solid Earth, Vol.124, No. 3, pp. 2721-2737, 2019, DI 10.1029/2018JB016878

[57a] Raj, S.V.; Freed, A.D. A phenomenological description of primary creep in class M materials. Materials Science and Engineering A-Structural Materials Properties Microstructure and Processing, Vol. 283, No. 1-2, pp.196-202, 2000, DI 10.1016/S0921-5093(99)00790-X.

[58]Fabrizio, M.; Lazzari, B.; Rivera, J.M. *Asymptotic behaviour for a two-dimensional thermoelastic model.* Mathematical Methods in the Applied Sciences, Vol. 30, No. 5, pp. 549-566, 2007, DI 10.1002/mma.798

[59]Zhang, W.; Wang, C.; Kassab, G.S. *The mathematical formulation of a generalized Hooke's law for blood vessels.* Biomaterials, Vol. 28, No. 24, pp. 3569-3578, 2007, DI 10.1016/j.biomaterials.2007.04.030

[60]Vlase, S.; Marin, M. *Standard Deformations of Nonlinear Elastic Structural Elements with Power-Law Constitutive Model.* Mathematics 2024, *12*, 3992. https://doi.org/10.3390/math12243992.

[61]Timoshenko, S. *Elements of Strength of Materials.* Van Nostrand; 5th edition,1968.

[62]Timoshenko, S.P.; Woinowsky-Krieger, S. *Theory of Plates and Shells*, McGraw-Hill, New York, 1959

## Teorema lui Castigliano în studiul elementelor structurale elastice neliniare cu modelul constitutiv al legii puterii

**Rezumat.** *Dezvoltarea puternica a industriei in ultima perioada a dus la diversificarea puternica a materialelor utilizate in aplicatii ingineresti si, de asemenea la cresterea fortelor si vitezelor care apar in aceste sisteme. Ca urmare s-au intreprins numeroase cercetari care au tinut seama de comportarea nelineara a materialelor in timpul functionarii sistemelor mecanice. Legile constitutive nu mai pot fi considerate legi lineare. S-au propus numeroase forme de legi coinstitutive care sa acopere neliniariutaile existente. Printre acestea s-au remarcat legile de tipul power law, in care dependenta este de tip exponential. In cadrul lucrarii s-au obtinut rezultate privind strain si stress aparute in elemente standard utilizate in inginerie. De asemenea a fost studiata posibilitatea utilizarii teoremei lui Castigliano in cazul aplicarii la materiale cu comportare nelineara, cu lege constitutiva de tip power law. Rezultatele obtinute pot fi utile inginerilor caci ofera rezultate rapide si cu o buna precizie.*

**Cuvinte cheie:** ecuație constitutivă neliniară; legea puterii; Teorema lui Castigliano; tensiune; comprimare; aplecare; Forfecare; Torsiune

**Maria Luminita SCUTARU,** Professor, TRANSILVANIA University of Brasov, Department of Mechanical Engineering, lscutaru@unitbv.ro

**Radu Georgian POPESCU**, PhD Student, TRANSILVANIA University of Brasov,Department of Mechanical Engineering

**Cristina MICLEA**, PhD Student, TRANSILVANIA University of Brasov,Department of Mechanical Engineering

**Lucian DOMINTE**, PhD Student, TRANSILVANIA University of Brasov,Department of Mechanical Engineering