TECHNICAL UNIVERSITY OF CLUJ-NAPOCA

ACTA TECHNICA NAPOCENSIS

Series: Applied Mathematics, Mechanics, and Engineering
Vol. 68, Issue I, March, 2025

THE LIMITS OF TAYLOR'S EXPANSION SERIES FOR THE STUDY OF ELASTIC CURVES IN ISOTROPIC BEAMS, STATIC DETERMINATE, WITH A CONSTANT MOMENT OF INERTIA

Adrian-Ioan BOTEAN

Abstract: *This paper proposes a detailed study on the evaluation of deflection and rotation for two distinct types of supports, aiming to analyze structural behavior under various loading conditions. To conduct this analysis, the Taylor series expansion method is employed, a mathematical tool that allows for the approximation of nonlinear functions by incorporating higher-order terms. Thus, in the context of deflection and rotation analysis, various loading scenarios are investigated, each with its specific characteristics, and the Taylor series expansion is applied to approximate structural behavior at points of interest, considering initial conditions and the characteristics of the support type. This methodology facilitates obtaining approximate solutions for deflection and rotation, particularly in regions where exact solutions are difficult to determine due to the complexity of geometry or load distribution. A central aspect of the study is the identification of the limitations of the Taylor series expansion method concerning the analyzed cases.*

Keywords: *structural deformations, Taylor series expansion, boundary conditions, nonlinear function approximation*

1. INTRODUCTION

The calculation of deflection and rotation is essential for verifying the rigidity of statically determinate structures. This process supports damage prevention by limiting deformations, enhances structural performance, and ensures durability, achieving an optimal balance between safety and material efficiency [1-6].

The evaluation of deflection y(x) and rotation φ(x) in statically determinate structures can be performed using a variety of analytical and numerical methods, each having specific applications depending on the complexity and type of structure being analyzed. The differential equation of the deformed neutral axis, based on the relation $d^2y(x)/dx^2 = \pm Mz(x)/E\cdot Iz$, represents a classical approach, providing a direct solution for evaluating the deformations of a beam under load. The double integral method allows for the integration of the bending moment over the entire length of the structure to determine y(x) and φ(x), while Mohr's theorem uses moment diagrams to calculate deflections at various points of the structure. Influence functions enable the evaluation of deflection at a specific point, depending on the load distribution. The principle of virtual work and Castigliano's theorem are based on energy principles [7], calculating deformations by integrating virtual forces or deriving the elastic energy stored in the material from which the structure is made. Another approach within energy methods is that the expression of the elastic curve is considered as an infinite sum of sinusoidal functions, which are successively evaluated using sinusoidal trigonometric series if it is considered a periodic function, or using Fourier series if it is considered a non-periodic function [8-10].

In the context of complex problems, the Finite Element Method (FEM) provides a numerical solution by breaking down the structure into finite elements [11, 12], while the graphical analysis of the funicular is used for simple structures, offering a visual solution for

Received: 29.01.25; **Similaritie**s: 05.03.25: **Reviewed**: 20.02./14.02.25: **Accepted**:17.03.25.

deformations. Standardized tables are often used to quickly determine deflections and rotations in the case of common beams, while extensometry [13], through direct measurements of specific linear deformations, validates theoretical calculations [14-17].

These methods, each with specific applicability, are essential in structural engineering to ensure the proper behavior of structures under load.

This paper initiates a detailed study on the evaluation of deflection and rotation for two distinct types of support, with the aim of analyzing the structural behavior under various loading conditions. To carry out this analysis, the Taylor series expansion is used, a mathematical method that allows for the approximation of nonlinear functions through higher-order terms.

Thus, in the context of deflection and rotation evaluation, different loading scenarios will be analyzed, each with its specific characteristics, and the Taylor series expansion method will be applied to approximate the structural behaviors at points of interest, taking into account the initial conditions and the support characteristics. This approach allows for obtaining approximate solutions for deflection and rotation, particularly in regions where exact solutions are difficult to obtain due to the complexity of the geometry or load distribution.

An essential aspect of this study is the identification of the applicability limits of the Taylor series expansion method for the cases under investigation. Therefore, the goal is to establish the validity domains of this method, considering that the Taylor series can provide precise approximations only when structural deformations are small and the material behavior is assumed to be linear. The proposed study will contribute to a deeper understanding of the method's applicability in different loading scenarios and support types, establishing the conditions under which the Taylor series expansion can be an effective and useful alternative for analyzing structural behavior in civil and mechanical engineering [18].

The Taylor series expansion allows for the representation of functions through higher-order polynomials, ensuring an accurate local evaluation of the solutions to the differential equations that describe the structural deformation behavior under various loading and support conditions [19-23].

By expanding functions into Taylor series, it is possible to model the structural behavior of beams through a polynomial representation in the vicinity of a point of interest. This technique simplifies the analysis of complex problems by reducing the computational complexity and providing a clear interpretation of mechanical phenomena. Specifically, Taylor series are used to linearize the behavior of nonlinear materials near an operating point, facilitating the application of both analytical and numerical methods for determining the approximate structural response [24-30].

The application of this method to a discrete segment of a beam allows for obtaining detailed information about the distribution of deformations and stresses. The Taylor series expansion provides a formal framework for describing the local behavior of the structure, enabling the precise evaluation of the effects of applied forces and moments on its deformation and stability. Additionally, the identification of critical zones becomes more accessible, contributing to the geometric and functional optimization of structures to prevent exceeding limit states [31-40].

In terms of stability analysis, Taylor series are used to approximate the solutions of differential equations that describe the equilibrium of beams under complex loads. These solutions provide essential information about the stability conditions and load limits of structures, thus contributing to ensuring their performance under extreme operating conditions [41-45].

Thus, from a scientific perspective, the application of Taylor series in the analysis of deformations in straight beams represents a robust and versatile method, integrating both theoretical and practical aspects for evaluating structural behavior. This approach enables engineers to develop solutions optimized for safety and efficiency, significantly contributing to the advancement of modern structural design and analysis [46-50].

Given a function f defined on an interval I, and differentiable at the point a∈I. The Taylor formula for the function f at the point a is:

$$f(x) = f(a) + \frac{x-a}{1!} \cdot f'(a) + \cdots +$$
$$+ \frac{(x-a)^n}{n!} \cdot f^{(n)}(a) + R_n(x), x \in I \quad (1)$$

In the sequence ($R_n(x)$) for x∈X⊂I is convergent to zero, that is $\lim_{x \to 0} R_n(x) = 0$, x∈X⊂I, then the series:

$$f(a) + \frac{x-a}{1!} \cdot f'(a) + \cdots +$$
$$+ \frac{(x-a)^n}{n!} \cdot f^{(n)}(a) + \cdots \quad (2)$$

called the Taylor series of the function f at the point a, is convergent for x∈X⊂I to f(x), therefor:

$$f(x) = f(a) + \frac{x-a}{1!} \cdot f'(a) + \cdots +$$
$$+ \frac{(x-a)^n}{n!} \cdot f^{(n)}(a) + \cdots \quad (3)$$

Relation (3) is called the formula for expanding the function f(x) into a Taylor series around the point a.

It is observed that series (2) is convergent for x=a. It is of interest for there to exist points x≠a for which series (2) is also convergent. A sufficient condition for the existence of a set of convergence that includes points other than a is provided by the following theorem:

The Taylor series of the function f around the point a is convergent in a neighborhood V of a if the derivatives of any order $f^{(n)}$ are uniformly bounded in V, that is, $\left|f^{(n)}(x)\right| \leq M, M > 0$, for any x∈V and any natural number n.

This can be demonstrated if the remainder R_n , in its Lagrange form, is:

$$R_n(x) = \frac{(x-a)^{n+1}}{(n+1)!} \cdot f^{(n+1)}(\xi), \xi \in (a, x) \subset V$$

thus

$$|R_n(x)| < \left|\frac{(x-a)^{n+1}}{(n+1)!}\right| \cdot M$$

however, $|R_n(x)| \to 0$ as $n \to \infty$, because the series with the general term:

$$u_n = \left|\frac{(x-a)^{n+1}}{(n+1)!}\right| \cdot M$$

is convergent for any x∈R.

$$\lim_{n \to \infty} \frac{u_{n+1}}{u_n} = \lim_{n \to \infty} \left|\frac{x-a}{n+1}\right| = 0$$

If in relation (3) a is replaced with 0 and f is infinitely differentiable at the point 0∈I, it follows:

$$f(x) = f(0) + \frac{x}{1!} \cdot f'(0) + \cdots +$$
$$+ \frac{(x)^n}{n!} \cdot f^{(n)}(0) + \cdots \quad (4)$$

called the Mac-Laurin series.

2. METHOD

From the perspective of the topics addressed in this study, the function f(x) will be denoted by y(x) and will represent the continuous and smooth function that expresses the deformation of the deformed average fiber, as shown in Figure 1.

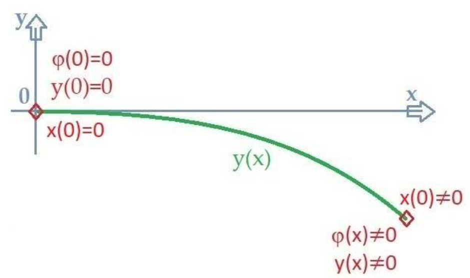

Fig. 1. The deformed average fiber for a cantilever beam.

The function y(x) can be expressed using the Taylor series as follows:

$$y(x) = \sum_{n=0}^{\infty} \frac{1}{n!} \cdot \frac{d^n y(0)}{dx^n}\Big|_{x=0} \cdot x^n \quad (5)$$

or,

$$y(x) = y(0) + \varphi(0) \cdot x + \frac{1}{2!} \cdot \frac{d^2 y(0)}{dx^2} \cdot x^2 +$$
$$+ \frac{1}{3!} \cdot \frac{d^3 y(0)}{dx^3} \cdot x^3 + \frac{1}{4!} \cdot \frac{d^4 y(0)}{dx^4} \cdot x^4 +$$
$$+ \frac{1}{5!} \cdot \frac{d^5 y(0)}{dx^5} \cdot x^5 + \cdots \quad (6)$$

where: y(0) and φ(0) represent the displacement and rotation at the reference point.

Assuming that a load q is applied, distributed per unit length L, and with the maximum amplitude at x=0, it follows that:

$$E \cdot I_z \cdot \frac{d^5y(0)}{dx^5} = q \Rightarrow \frac{d^5y(0)}{dx^5} = \frac{q'(x)}{E \cdot I_z} \quad (7)$$

where: E represents Young's modulus, and I_z is the axial moment of inertia.

The linear variation law of the distributed load q can be expressed through the fourth-order derivative as:

$$E \cdot I_z \cdot \frac{d^4y(0)}{dx^4} = q(x) \Rightarrow \frac{d^4y(0)}{dx^4} = \frac{q(x)}{E \cdot I_z} \quad (8)$$

The third-order derivative expresses the variation of the shear force $T_y(x)$, while the second-order derivative characterizes the evolution of the bending moment $M_z(x)$.

$$E \cdot I_z \cdot \frac{d^3y(0)}{dx^3} = T_y(x) \Rightarrow \frac{d^3y(0)}{dx^3} = \frac{T_y(x)}{E \cdot I_z} \quad (9)$$

$$E \cdot I_z \cdot \frac{d^2y(0)}{dx^2} = M_z(x) \Rightarrow \frac{d^2y(0)}{dx^2} = \frac{M_z(x)}{E \cdot I_z} \quad (10)$$

The first-order derivative represents the tangent to the deformed neutral axis (or the rotation of the cross-sectional section).

$$E \cdot I_z \cdot \frac{dy(0)}{dx} = \varphi(x) \Rightarrow \frac{dy(0)}{dx} = \frac{\varphi(x)}{E \cdot I_z} \quad (11)$$

Equations (9) and (10) are derived in accordance with the sign conventions for positive values of the bending moment $M_z(x)$ and shear force $T_y(x)$, as shown in Figure 2. The sign of the distributed load q depends on the direction of the ordinate y. Furthermore, relations (8), (9), and (10) represent the differential relationships between forces, as expressed in the specialized literature in the following form:

$$-q(x) = \frac{dT_y}{dx} = \frac{d^2M_z}{dx^2} \quad (12)$$

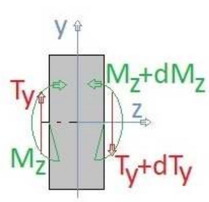

Fig. 2. The sign convention for T_y (shear force) and M_z (bending moment).

According to Figure 1, by substituting relations (7) through (11) into relation (6), it results in:

$$y(x) = \frac{1}{2!} \cdot \frac{M_z(x)}{E \cdot I_z} \cdot x^2 + \frac{1}{3!} \cdot \frac{T_y(x)}{E \cdot I_z} \cdot x^3 +$$

$$+\frac{1}{4!} \cdot \frac{q(x)}{E \cdot I_z} \cdot x^4 + \frac{1}{5!} \cdot \frac{q'(x)}{E \cdot I_z} \cdot x^5 \quad (13)$$

It should be emphasized that the origin of the orthogonal coordinate system can be arbitrarily chosen at any considered point. Generally, it is recommended that this point be defined to the left of the considered interval so that the calculation volume is as small as possible. It should also be mentioned that the number of terms included in the Taylor series is finite and depends on the order of the function governing the variation of the distributed load.

This consideration is crucial because, in practice, a truncated Taylor series is used to approximate the behavior of the system. The more terms included, the more accurate the approximation becomes, but the computational effort also increases. The choice of the number of terms is typically determined by the required accuracy and the nature of the load distribution.

In a similar manner, using Taylor series, the rotation φ(x), the bending moment $M_z(x)$, and the shear force $T_y(x)$ can be expressed as follows:

$$\varphi(x) = \varphi(0) + \frac{M_z(0)}{E \cdot I_z} \cdot x + \frac{1}{2!} \cdot \frac{T_y(x)}{E \cdot I_z} \cdot x^2 +$$

$$+\frac{1}{3!} \cdot \frac{q(x)}{E \cdot I_z} \cdot x^3 + \frac{1}{4!} \cdot \frac{q'(x)}{E \cdot I_z} \cdot x^4 \quad (14)$$

$$M_z(x) = M_z(0) + T_y(0) \cdot x + \frac{1}{2!} \cdot q(x) \cdot x^2 +$$

$$+\frac{1}{3!} \cdot q'(x) \cdot x^3 \quad (15)$$

$$T_y(x) = T_y(0) + q(x) \cdot x + \frac{1}{2!} \cdot q'(x)x^2 \quad (16)$$

3. RESULT AND DISCUSSION

The validation of the Taylor series application method, as well as the establishment of its applicability limits, is carried out by going through several progressively complex examples for two distinct support cases: a cantilever beam (as shown in Figure 1) and a simply supported beam (as shown in Figure 3).

A first example is the one considered in the introductory part of this study: a cantilever beam, fixed at the left end, on which a uniformly distributed load q is applied.

At the left end, the boundary conditions are: x=0, φ(0)=0, y(0)=0, $M_z(0)=-q \cdot L^2/2$,

$T_y(0)=q\cdot L$. The uniformly distributed load is a negative quantity, directed in the negative y coordinate direction.

From relation (13), we obtain:

$$y(x) = \frac{1}{2!}\cdot\frac{M_z(0)}{E\cdot I_z}\cdot x^2 + \frac{1}{3!}\cdot\frac{T_y(0)}{E\cdot I_z}\cdot x^3 + \\ +\frac{1}{4!}\cdot\frac{q}{E\cdot I_z}\cdot x^4 = \\ = \frac{q\cdot x^2}{24\cdot E\cdot I_z}\cdot(-6\cdot L^2 + 4\cdot x\cdot L - x^2) \quad (17)$$

For x=L, from relation (17), we obtain:

$$y(x)_{max} = -\frac{q\cdot L^4}{8\cdot E\cdot I_z} \quad (18)$$

From relation (14), it follows:

$$\varphi(x) = \frac{M_z(0)}{E\cdot I_z}\cdot x + \frac{1}{2!}\cdot\frac{T_y(x)}{E\cdot I_z}\cdot x^2 + \\ +\frac{1}{3!}\cdot\frac{q}{E\cdot I_z}\cdot x^3 = \\ = \frac{q\cdot x}{6\cdot E\cdot I_z}\cdot(-3\cdot L^2 + 3\cdot x\cdot L - x^2) \quad (19)$$

For x=L, from relation (19), it follows:

$$\varphi(x)_{max} = -\frac{q\cdot L^3}{6\cdot E\cdot I_z} \quad (20)$$

Relations (18) and (20) are identical to the solutions provided in the specialized literature (certa eadem derivate directe - c.e.d.d.).

A second example is a cantilever beam, fixed at the left end, on which a linearly distributed load q is applied, having a maximum value at the left end and a zero value at the right end.

Here, to reduce the volume of calculations, it can be assumed that the origin of the coordinate system is at the free end, where the boundary conditions are: x=0, $T_y(0)=0$, $M_z(0)=0$, $\varphi(0)\neq0$, $y(0)\neq0$. From this perspective, the governing variation law for the load distribution is $q(x)=q\cdot x/L$. At the left end, the boundary conditions are x=L, $\varphi(L)=0$, $y(L)=0$, $M_z(L)=-q\cdot L^2/6$, $T_y(L)=q\cdot L/2$.

According to relation (14), it follows:

$$\varphi(L) = \varphi(0) - \frac{1}{4!}\cdot\frac{q}{L\cdot E\cdot I_z}\cdot L^4 \Rightarrow$$

$$\varphi(0) = \frac{q\cdot L^3}{24\cdot E\cdot I_z} \quad (21)$$

and from relation (13), it follows:

$$y(L) = y(0) + \varphi(0)\cdot L - \frac{1}{5!}\cdot\frac{q}{L\cdot E\cdot I_z}\cdot L^5 \Rightarrow$$

$$y(0) = -\frac{q\cdot L^4}{24\cdot E\cdot I_z} + \frac{q\cdot L^4}{120\cdot E\cdot I_z} \Rightarrow$$

$$y(0) = -\frac{q\cdot L^4}{30\cdot E\cdot I_z} \quad (22)$$

Relations (21) and (22) are identical to the solutions provided in the specialized literature (c.e.d.d.).

Continuing, a series of examples will be presented in which the beam is simply supported at both ends, as shown in Figure 3. The boundary conditions at the two ends are as follows: at the left end, x(0)=0, y(0)=0, $\varphi(0)\neq0$, and at the right end, $x(0)\neq0$, y(0)=0, $\varphi(0)\neq0$.

Thus, if it is assumed that a concentrated force acts at the midpoint of the span between the supports (at L/2), directed in the negative y coordinate direction, the shear force is given by $T_y(0)=F/2$ for $0\leq x\leq L/2$.

From relation (13), it follows:

$$y(x) = \varphi(0)\cdot x + \frac{T_y(0)}{E\cdot I_z}\cdot\frac{x^3}{3!} \Rightarrow$$

$$y(x) = \varphi(0)\cdot x + \frac{F\cdot x^3}{12\cdot E\cdot I_z} \quad (23)$$

From relation (14), it follows:

$$\varphi(x) = \varphi(0) + \frac{T_y(0)}{E\cdot I_z}\cdot\frac{x^2}{2!} \Rightarrow$$

$$\varphi(x) = \varphi(0) + \frac{F\cdot x^2}{4\cdot E\cdot I_z} \quad (24)$$

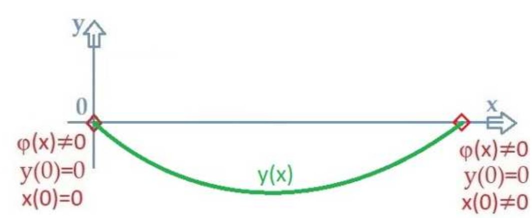

Fig. 3. Deformed neutral axis for a simply supported beam.

For this type of support, φ(x) is zero at the location L/2, i.e., at the point where the nodal force F is applied. Thus, from relation (24), it follows:

$$x = \frac{L}{2} \Rightarrow \varphi(x) = 0 \Rightarrow \varphi(0) = -\frac{F\cdot L^2}{16\cdot E\cdot I_z} \quad (25)$$

By substituting expression (25) into (24), the following result is obtained:

$$\varphi(x) = -\frac{F \cdot L^2}{16 \cdot E \cdot I_z} + \frac{F \cdot x^2}{4 \cdot E \cdot I_z} \Rightarrow$$

$$\varphi(x) = \frac{F}{16 \cdot E \cdot I_z} \cdot (-L^2 + 4 \cdot x^2) \quad (26)$$

And by substituting expression (25) into (23), the following result is obtained:

$$y(x) = -\frac{F \cdot L^2}{16 \cdot E \cdot I_z} \cdot x + \frac{F \cdot x^3}{12 \cdot E \cdot I_z} \Rightarrow$$

$$y(x) = \frac{F \cdot x}{48 \cdot E \cdot I_z} \cdot (-3 \cdot L^2 + 4 \cdot x^2) \quad (27)$$

For x=L/2, from relation (27), it follows:

$$y\left(\frac{L}{2}\right) = -\frac{F \cdot L^3}{48 \cdot E \cdot I_z} \quad (28)$$

Relations (25) and (28) are identical to the solutions provided in the specialized literature (c.e.d.d.).

If instead of the nodal force F, a uniformly distributed load q is applied over the entire length L of the beam, directed in the negative y coordinate direction, the shear force is given by $T_y(0)=q \cdot L/2$.

In this case, from relation (13), it follows:

$$y(x) = \varphi(0) \cdot x + \frac{1}{3!} \cdot \frac{T_y(x)}{E \cdot I_z} \cdot x^3 + $$

$$+ \frac{1}{4!} \cdot \frac{q(x)}{E \cdot I_z} \cdot x^4 \Rightarrow$$

$$y(x) = \varphi(0) \cdot x + \frac{q \cdot L \cdot x^3}{12 \cdot E \cdot I_z} - \frac{q \cdot x^4}{24 \cdot E \cdot I_z} \quad (29)$$

and from relation (14), it follows:

$$\varphi(x) = \varphi(0) + \frac{T_y(0)}{E \cdot I_z} \cdot \frac{x^2}{2!} + \frac{1}{3!} \cdot \frac{q(x)}{E \cdot I_z} \cdot x^3 \Rightarrow$$

$$\varphi(x) = \varphi(0) + \frac{q \cdot L \cdot x^2}{4 \cdot E \cdot I_z} - \frac{q \cdot x^3}{6 \cdot E \cdot I_z} \quad (30)$$

For this type of support, φ(x) is zero at the location L/2. Thus, from relation (30), it follows:

$$x = \frac{L}{2} \Rightarrow \varphi(x) = 0 \Rightarrow \varphi(0) = -\frac{q \cdot L^3}{24 \cdot E \cdot I_z} \quad (31)$$

By substituting relation (31) into (29), the following result is obtained:

$$y(x) = -\frac{q \cdot L^3}{24 \cdot E \cdot I_z} + \frac{q \cdot L \cdot x^3}{12 \cdot E \cdot I_z} -$$

$$- \frac{q \cdot x^4}{24 \cdot E \cdot I_z} =$$

$$= \frac{q}{24 \cdot E \cdot I_z} \cdot (-L^3 + 2 \cdot L \cdot x^3 - x^4) \quad (32)$$

For x=L/2, from relation (32), it follows:

$$y\left(\frac{L}{2}\right) = -\frac{5 \cdot q \cdot L^4}{384 \cdot E \cdot I_z} \quad (33)$$

Relations (31) and (33) are identical to the solutions provided in the specialized literature (c.e.d.d.).

Continuing, it is assumed that the simply supported beam is loaded with a nodal force F, applied at a distance L/3 from the left end. In this case, the shear force is $T_y(0)=2 \cdot F/3$ for $0 \le x \le L/3$.

From relation (13), it follows:

$$y(x) = \varphi(0) \cdot x + \frac{T_y(0)}{E \cdot I_z} \cdot \frac{x^3}{3!} \Rightarrow$$

$$y(x) = \varphi(0) \cdot x + \frac{F \cdot x^3}{9 \cdot E \cdot I_z} \quad (34)$$

and from relation (14), it follows:

$$\varphi(x) = \varphi(0) + \frac{T_y(0)}{E \cdot I_z} \cdot \frac{x^2}{2!} \Rightarrow$$

$$\varphi(x) = \varphi(0) + \frac{F \cdot x^2}{3 \cdot E \cdot I_z} \quad (35)$$

In this case, relation (35) is a simple statically indeterminate equation because, based on the boundary conditions, the rotation φ(0) cannot be determined. Additionally, the section where φ(x) becomes zero is unknown. The indeterminacy is resolved by expressing the bending moment as follows:

$$0 \le x \le \frac{L}{3} \Rightarrow M_z(x) = \frac{2 \cdot F}{3} \cdot x \quad (36)$$

$$\frac{L}{3} < x \le L \Rightarrow M_z(x) = \frac{2 \cdot F}{3} \cdot x - F \cdot \left(x - \frac{L}{3}\right) \quad (37)$$

The rotation φ(x) is obtained by integrating the bending moment $M_z(x)$:

$$\varphi(x) = \int_0^x \frac{M_z(x)}{E \cdot I_z} \cdot dx \quad (38)$$

For

$$x \le \frac{L}{3} \overset{(38)}{\Longrightarrow} \varphi(x) = \frac{1}{E \cdot I_z} \cdot \int \frac{2 \cdot F \cdot x}{3} dx =$$

$$= \frac{1}{E \cdot I_z} \cdot \left(-\frac{F \cdot x^2}{3 \cdot E \cdot I_z} + C_1\right) \quad (39)$$

and for

$$x > \frac{L}{3} \overset{(38)}{\Longrightarrow}$$

$$\varphi(x) = \int \frac{1}{E \cdot I_z} \cdot \left[\frac{2 \cdot F}{3} \cdot x - F \cdot \left(x - \frac{L}{3}\right)\right] \cdot dx$$

$$= \frac{1}{E \cdot I_z} \cdot \left[\frac{F \cdot x^2}{3} - \frac{F \cdot \left(x - \frac{L}{3}\right)^2}{2} + C_2\right] \quad (40)$$

From the continuity and smoothness condition of the deformed neutral axis, it follows that $C_1=C_2=C$, and from the boundary conditions defined at the support in the right end, the following is obtained:

$$C = -\frac{5 \cdot F \cdot L^2}{81} \quad (41)$$

For x=0, from relation (39), it follows:

$$\varphi(0) = \frac{C}{E \cdot I_z} = -\frac{5 \cdot F \cdot L^2}{81 \cdot E \cdot I_z} \quad (42)$$

By substituting relation (42) into (35), for $0 \leq x \leq L/3$, the following is obtained:

$$\varphi(x) = \varphi(0) + \frac{F \cdot x^2}{3 \cdot E \cdot I_z} \Rightarrow$$

$$\varphi(x) = -\frac{5 \cdot F \cdot L^2}{81 \cdot E \cdot I_z} + \frac{F \cdot x^2}{3 \cdot E \cdot I_z} \quad (43)$$

Thus, from relation (43), for x=L/3, we have:

$$\varphi\left(\frac{L}{3}\right) = -\frac{2 \cdot F \cdot L^2}{81 \cdot E \cdot I_z} \quad (44)$$

The Taylor series for the rotation φ(x) around the point x=L/3 is:

$$\varphi(x) = \varphi\left(\frac{L}{3}\right) + \left(x - \frac{L}{3}\right) \cdot \varphi'\left(\frac{L}{3}\right) +$$

$$+\frac{1}{2!} \cdot \left(x - \frac{L}{3}\right)^3 \cdot \varphi''\left(\frac{L}{3}\right) + \cdots \quad (45)$$

The deflection can be expressed using relation (34) and has the form:

$$y(x) = \varphi(0) \cdot x + \frac{F \cdot x^3}{9 \cdot E \cdot I_z} \Rightarrow$$

$$y(x) = \frac{F}{81 \cdot E \cdot I_z} \cdot (-5 \cdot L^2 \cdot x + 9 \cdot x^3) \quad (46)$$

and for x=L/3, from relation (46), it follows:

$$y\left(\frac{L}{3}\right) = -\frac{4 \cdot F \cdot L^3}{243 \cdot E \cdot I_z} \quad (47)$$

In the section where the rotation φ(x) is zero (considering the coordinate x_o), the deflection $y(x)=y_{max}$. Thus, from relation (40), a second-order equation is obtained in the following form:

$$-27 \cdot x_o^2 + 54 \cdot L \cdot x_o - 19 \cdot L^2 = 0 \quad (48)$$

for which the real solution is $x_o=0.455 \cdot L$.

Relations (44) and (47) are identical to the solutions provided in the specialized literature (c.e.d.d.).

4. CONCLUSIONS

The use of Taylor series expansion allows for the formulation of general equations for determining the deformation and rotation of the beam's cross-section, in a manner similar to methods based on integrating the differential equation of the deformed neutral axis.

These methods include direct integration and the Clebsch method for determining the integration constants. Unlike energy, graphical, or graph-analytical methods, this approach offers the advantage of reduced computational complexity, as it enables the derivation of shear force and bending moment equations without requiring the graphical representation of their variation.

The deflection and rotation equations can be easily written for each segment of the beam by expressing the Taylor series expansion at the beginning of that segment.

In this study, the following cases were considered: the first example corresponds to the case introduced in the introductory section, namely a cantilever beam fixed at the left end, subjected to a uniformly distributed load q; the second example consists of a cantilever beam fixed at the left end, subjected to a linearly distributed load with a maximum value at the left extremity and a zero value at the right extremity; the third example is a simply supported beam with supports at both ends, loaded at the midpoint of the span between the supports with a concentrated force F acting in the negative direction of the ordinate y; in the fourth example, instead of the nodal force F, a uniformly distributed load q is applied along the entire length L of the beam, oriented in the negative direction of the ordinate y; finally, in the fifth example, the simply supported beam is subjected to a nodal force F, applied at a distance of L/3 from the left extremity.

For the first four studied cases, the obtained results are identical to those provided in the specialized literature.

In the case of the fifth example, compared to the first four, there is a difficulty in determining the rotation. This indicates that a primary limitation of using the Taylor series arises in the case of asymmetric loadings for simply supported beams.

To solve this issue, the differential equation of the deformed neutral axis was integrated, and by applying the boundary conditions as well as the condition of continuity and smoothness of the deformed neutral axis, the calculation relation for the integration constant CCC was derived. Furthermore, from the expression of the rotation in the first segment of the beam, the rotation at the leftmost section could be evaluated.

In general, the use of the Taylor series for complex applications can be recommended, provided that the limitation highlighted in the previous paragraph is taken into account.

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LIMITELE DEZVOLTĂRII ÎN SERIE TAYLOR PENTRU STUDIUL FIBREI MEDII DEFORMATE ÎN GRINZI IZOTROPE, STATIC DETERMINATE, CU MOMENT DE INERȚIE CONSTANT

Rezumat: În această lucrare se propune un studiu detaliat asupra evaluării săgeții și rotirii în cazul a două tipuri distincte de rezemare, având drept obiectiv analiza comportamentului structural sub diverse condiții de încărcare. În vederea realizării acestei analize, se utilizează metoda dezvoltării în serii Taylor, un instrument matematic ce permite aproximarea funcțiilor neliniare prin includerea termenilor de ordin superior. Astfel, în contextul analizei săgeții și rotirii, sunt investigate scenarii variate de încărcare, fiecare având particularități specifice, iar dezvoltarea în serii Taylor este aplicată pentru aproximarea comportamentului structural în punctele de interes, luând în considerare condițiile inițiale și caracteristicile tipului de reazem. Această metodologie facilitează obținerea unor soluții aproximative pentru săgeată și rotire, în special în regiunile unde soluțiile exacte sunt dificil de determinat din cauza complexității geometriei sau a distribuției sarcinilor. Un element central al studiului îl constituie identificarea limitărilor metodei dezvoltării în serii Taylor în raport cu cazurile analizate.

Cuvinte cheie: *deformatii structural, seriile Taylor extinse, conditii la limita, functii neliniare de aproximare*

Adrian-Ioan BOTEAN, Lecturer, PhD.Eng., Mechanical Engineering Department, Faculty of Automotive, Mechatronics and Mechanics, Technical University of Cluj-Napoca, 28 Memorandumului, 400114 Cluj-Napoca, Romania; adrian.ioan.botean@rezi.utcluj.ro