TECHNICAL UNIVERSITY OF CLUJ-NAPOCA

ACTA TECHNICA NAPOCENSIS

Series: Applied Mathematics, Mechanics, and Engineering

Vol. 68, Issue II, June, 2025

THE NONLINEAR DYNAMICS OF ESCAPEMENT MECHANISMS

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Abstract: *Clock mechanisms belong to the field of precision mechanics, which measure the passage of time by a high degree of uniformity of their movements. The key sub-assembly of the clock mechanism is an oscillator coupled with an escapement mechanism which introduces disturbances into the oscillatory process so that the oscillations are no longer free, but forced and nonlinear with a frequency that is subjected to change. This work is dedicated to the analysis of the clock mechanism dynamics using the theory of perturbation calculus. The approximate solution of the nonlinear differential equations that describe the motion of the clock's oscillator are determined by the method of two-time scale perturbation and represent universally applicable formulas for escapement errors. These formulas are verified by the computer simulations of the escapement mechanism operation.*

Key words: *clock, dynamics, escapement, mechanism, nonlinear, oscillation, perturbation, watch.*

1. INTRODUCTION

Time has long been an important subject of study in religion, philosophy, and science. Two contrasting viewpoints on time divide prominent philosophers:

- Sir Isaac Newton [1]: time is part of the fundamental structure of the universe – a dimension independent of events, in which events occur in sequence/
- Gottfried Wilhelm (von) Leibniz [2], Immanuel Kant [3]: time it is part of a fundamental intellectual structure – a priory intuition (together with space and number) within which humans sequence and compare events.

Albert Einstein introduced the relativity of time in his Special and General theories of relativity. Quantum mechanics unsuccessfully attempts to quantize time. In contemporary sciences, time is fundamental concept.

Despite the fact that nothing essential and definitive can be claimed about time, we have a strong and a priory intuition that flow of time is uniform. The measurement of time is based on this inner premonition of human soul. By the observing, tracking, recording and measuring any proper or favorable sort of uniform motion or, in general, any suitable kind of uniform process, we are actually measuring the flow of time. There are many uniform processes that can be used for measuring the flow of time [4]:

- Motions of Sun and Moon, (observed from the Earth), (day, week, month, solstice, year, saros cycle, eclipse cycle, etc.)
- Water clock (flow of water)
- Hourglass (flow of sand)
- Candle clock (burning a graduated candle).

It was Christiaan Huygens (1629.–1695), a Dutch physicist, mathematician, astronomer and inventor who discovered that mechanical oscillations are phenomena suitable for measuring the flow of time [5]. Christiaan Huygens designed the first clock with the pendulum and watch with the hairspring attached to the balance wheel, as oscillators [6]. In particular, the flow of time can be measured by the counting the oscillations number, since their frequency is relatively stable. All contemporary type of clocks (quartz - electronic, atomic and mechanical) share the same operational principle: they are counting the number of oscillations.

Received: 07.04.25; **Similarities**: 29.05.25: **Reviewed**: 07.04./ 07.04.25: **Accepted**:16.06.25.

It is necessary to briefly consider the construction of a mechanical watch, because this lecture is dedicated to mechanical watches and clocks. Each such watch, i.e. a mechanical instrument that measures the passage of time (Figure 1), contains of [7, and 8]:

1 Mainspring barrel
2 Gear train
3 Dial and watch hands
4 Oscillator (balance wheel and hairspring)
5 Escapement mechanism

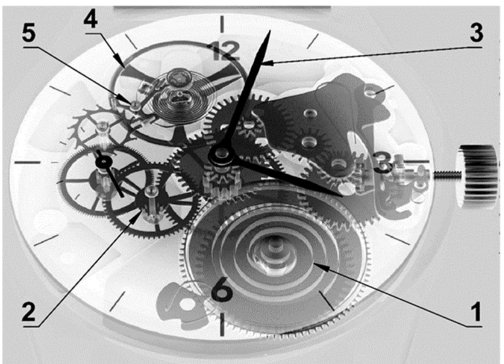

Fig. 1. Parts of every mechanical watch

The mainspring barrel supplies the watch with mechanical potential energy (elastic deformation energy) and, converting it into kinetic energy, enables the continuous operation of all moving parts of the mechanism. The gear train (transmission mechanism) is a system of meshing gears that transmits the drive energy from the mainspring barrel to the escapement. In addition, the gear train transmits its rotational movements to the hands, which record the elapsed time intervals on the dial. An oscillator is a physical pendulum or balance wheel with a spiral spring. The oscillator performs oscillations under the effect of restitution elastic force and is characterized by a relatively stable natural frequency, i.e., an approximately constant period of oscillation. The clock mechanism displays and measures the passage of time as an integer multiple of the period or half period of the oscillator's oscillations. The escapement mechanism, the heart of every watch, is the mechanism that regulates the angular velocity of clock or watch arbors. It is also called a locking-impulse mechanism because it achieves its function through these two, more or less separate activities. Under the influence of the driving torque, the gear train has a tendency to move faster, and with its locking function, the escapement prevents this tendency in the rhythm of balance wheel oscillations. Thus, the locking function of this regulator ensures the uniformity of the clock operation. In addition to the restitution force, the frictional torque (viscous air resistance and dry friction inside the support) acts on the oscillator, and thus tend to stop its motion. In order for the oscillator to oscillate continuously, it is necessary that the clock mechanism periodically compensates this energy loss, which is exactly what the impulse function of the escapement does. So, in the rhythm of its own oscillations, the escapement mechanism transfers certain energy to the oscillator, compensates the losses and thus ensures the continuity of its oscillation.

2. ESCAPEMENT MECHANISM

The escapement mechanism is an important component of watch and clock mechanisms with two different functions: the locking and the impulse function. The angular velocity of the clock and watch going train is regulated by the locking function while the dissipation of the oscillator energy is recompensed by the action of the impulse function [7, and 8]. The escapement counts the number of oscillations of the clock pendulum or watch balance wheel and thus measures the flow of time. Second, escapement delivers impulses to the clock and watch oscillators and recompenses the oscillator’s energy dissipated during its damped oscillations. Consequently, clock and watch oscillators perform driven damped oscillations which frequency and period is just approximately constant [7].

Parts of the Graham deadbeat escapement (Fig. 2):

1. anchor
2. anchor axis
3. escapement wheel
4. escapement wheel arbor
5. pendulum
6. left pallet
7. right pallet
8. left pallet locking surface
9. left pallet impulse surface
10. right pallet locking surface

11. right pallet impulse surface

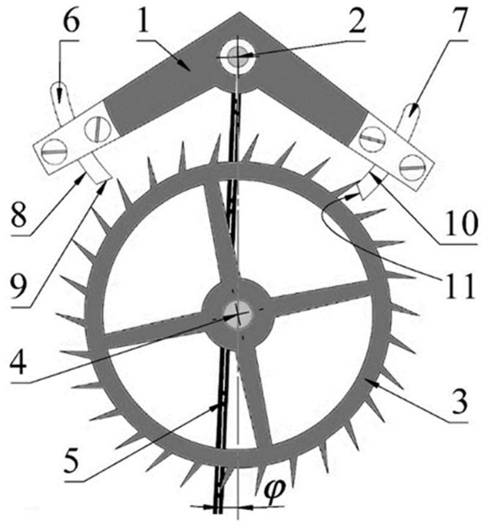

Fig. 2. The Graham deadbeat escapement

On each pallet, the locking function is accomplished first and then impulse one. When impulse function is finished on the left pallet, the locking function begins on the right pallet, etc. This activities are repeated rhythmically in accordance with the pendulum oscillations [7, and 10]. The external actions of an escapement on a clock or watch oscillator induct the frequency alteration of its oscillations. Thus the process of measurement of time flow generates the error to that measurement. This variation of oscillation frequency generated by the influence of the escapement to the oscillator is called an escapement error [7].

Classification of the escapement mechanism can be obtained by the following criteria [9]:

1. Whether the constructive and dynamic separation of impulse and locking function exist?
2. Whether the impulse function depends on a clock driving torque?
3. Whether the escapement interacts with the oscillator during the execution of the escapement locking function?

In accordance to the exposed criteria, all escapement mechanism can be categorized into following 4 classes [9]:

1. Recoil escapements

The impulse and locking functions are not separated; the impulse function depends on a driving torque; the escapement interacts with the oscillator during the execution of the locking function.

2. Deadbeat or frictional rest escapements

The impulse and locking functions are separated; other features are identical to the characteristics of the recoil escapements.

3. Isodynamic or constant escapements

The impulse function does not depend on a driving torque.

4. Detached escapements

The impulse and locking functions are separated from each other; the escapement does not interact with the oscillator during the execution of the locking function

2.1 Diagrams of torque interactions for recoil anchor escapement

The recoil anchor escapement is shown on Fug.3 and diagram of torque interactions for this type of escapement and oscillator is shown on Fig 4.

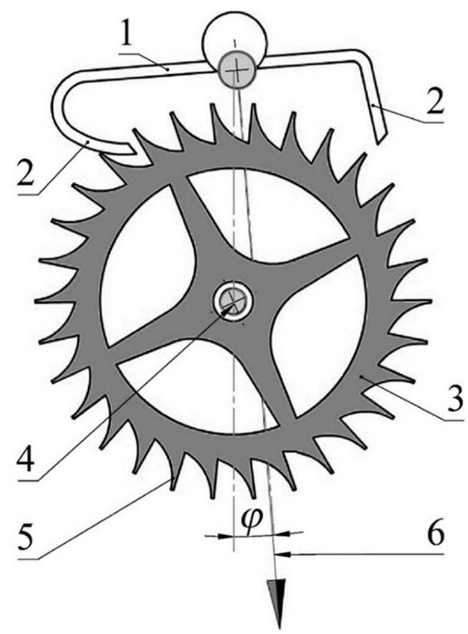

Fig. 3. Recoil anchor escapement

In average, S=Glsinφ and M have the same direction, which means that restitution torque is increased. Consequently, this type of escapement diminishes the period of the pendulum oscillations and accelerates the clock rate. The effect is called tachy-chronous: ταχύς – fast, shortens [9].

The recoil anchor escapement was invented by an English clockmaker William Clement and British scientist Robert Hook, 1635 –1703, in 1670. That same year the clockmaker Joseph Knibb, 1640 – 1711, built the first clock with the Clement-Hook anchor escapement in Oxford. This type of escapement is considered as the first modern clock rate regulator and it is still installed in many contemporary wall clocks [9].

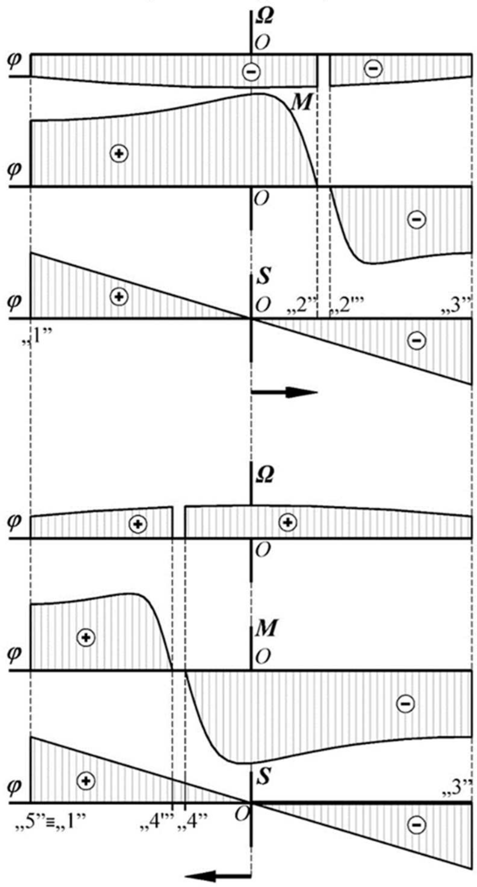

Fig. 4. Diagrams of torque interactions for recoil anchor escapement

3. THE THEORY OF PERTURBATION

Theory of regular perturbation comprises mathematical methods (Fig.5) for finding an approximate solution to a problem, by starting from the exact solution A_0 of a related, simpler but less realistic problem [11]. Thus, the approximate solution is represented as perturbation series that quantifies the deviation from the exact solvable but simpler problem

$$A = A_0 + \varepsilon^1 A_1 + \varepsilon^2 A_2 + \varepsilon^3 A_3 + \ldots . \qquad (1)$$

A_0 - the solution of the exact solvable initial (simpler but less realistic) problem

A_1, A_2, A_3 … - the first, second, third and the higher order terms which may be found iteratively by some systematic procedure.

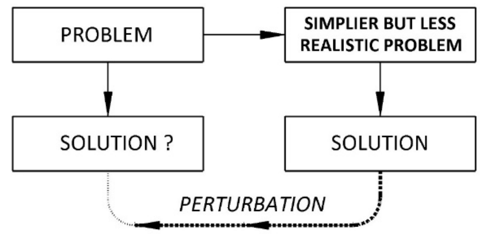

Fig. 5. The Perturbation Technique

It is necessary that perturbation series represent an asymptotic expansion of the solution, which means that

$$\varepsilon^1 A_1 > \varepsilon^2 A_2 > \varepsilon^3 A_3 > \ldots . \qquad (2)$$

If it is no so, terms are disordered since some of them grow without bound and perturbation series does not converge to the solution of the problem. These terms that grow without bound are called secular terms. They are caused by the terms of resonant forcing in differential equation.

The particularly great value of perturbation theory is that it can provide approximate solutions to very complex problems in analytical forms. Perturbation theory is used in a wide range of scientific fields, and reaches its most sophisticated and advanced forms in quantum field theory. Apart from quantum mechanics, perturbation theory has numerous applications in mathematics - solving algebraic and differential equations, in physics, especially in celestial mechanics for calculating the trajectories of celestial bodies, then in chaos theory, thermodynamics, chemistry, etc [11]. In this lecture, one perturbation methods will be presented: the two-time scale perturbation [12]. For this method, the solutions will be expressed in the form:

$$A = A_0 + \varepsilon^1 A_1 + O(\varepsilon^2) . \qquad (3)$$

Where $O(\varepsilon^2)$ indicate the big O notation for the order of the error in the approximate solution.

4. THE ESCAPEMENT ERRORS DETERMINED BY THE TWO-TIME SCALE PERTURBATION METHOD

The two-time scale perturbation is an alternative to the method of regular perturbation by which the appearance of secular terms can be avoided. It comprises techniques used to construct uniformly valid approximations to the solutions of perturbation problems, both for small as well as large values of the independent variables. This is done by introducing fast-scale and slow-scale variables for an independent variable, and subsequently treating these variables, fast and slow, as if they are independent. The equation which describes the driven damped oscillations of the watch balance wheel (Fig. 6) is

$$J\ddot{\varphi} + c\dot{\varphi} + k\varphi = M(\varphi) \quad (4)$$

in which J is the momentum of inertia of the balance wheel, c is damping coefficient, k is spiral spring constant. The initial conditions are $\varphi(0) = \Phi_0$, $\dot{\varphi}(0) = 0$.

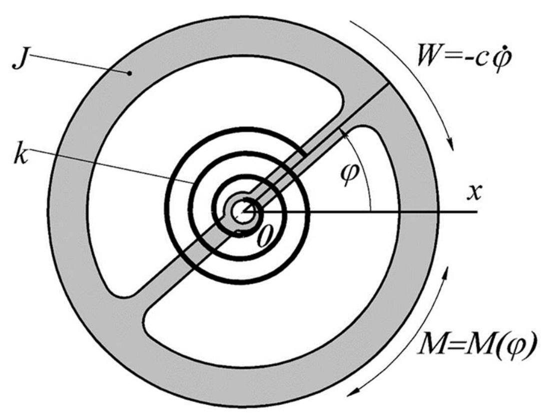

Fig. 6. The Balance wheel with the hairspring

Equation Eq. 4 can be transformed to:

$$\ddot{\varphi} + 2\xi\omega_0\dot{\varphi} + \omega_0^2\varphi = \mu(\varphi) \quad (5)$$

where $\omega_0 = \sqrt{k/J}$, $\xi = c/(2J\omega_0)$, $\xi \in (0,\ 1)$ and $\xi \ll 1$, $\mu(\varphi) = M(\varphi)/J$. This nonlinear equation is solved by the two-time scale perturbation method.

As the damping ξ is small, the specific torque μ(φ) represents a small disturbance of the free damped oscillations of the balance wheel. In order to highlight this property and make the calculation flow clearer, the function μ_1 (φ) is formally introduced so that:

$$\mu(\varphi) = \xi\mu_1(\varphi) \quad (6)$$

Independent variable t is replaced by two variables t_1 и t_2 so that $t_1 = t$ and $t_2 = \xi t$ for $\xi \in (0, 1)$. The approximate solution $\varphi(t_1, t_2, \xi)$ of the differential equation Eq. 4 is given in the form:

$$\ddot{\varphi}(t_1,t_2,\xi) \approx \dot{\varphi}_0(t_1,t_2) + \xi\dot{\varphi}_1(t_1,t_2) \quad (7)$$

The initial conditions $\varphi(0) = \Phi_0$ and $\dot{\varphi}(0) = 0$ for both functions $\varphi_0(t_1, t_2)$ and $\varphi_1(t_1, t_2)$ are given by the expressions:

$$\begin{cases} \varphi_0(0,0) = \phi_0 \\ \dot{\varphi}_0(0,0) = 0 \\ \varphi_1(0,0) = 0 \\ \dot{\varphi}_1(0,0) = 0 \end{cases} . \quad (8)$$

When all the necessary differentiations of Eq. 8 are accomplished, and neglect the terms containing ξ^2, the differential equation Eq. 5 becomes:

$$\begin{cases} \frac{\partial^2\varphi_0}{\partial t^2} + \omega_0^2\varphi_0 + \\ + \xi\left(\frac{\partial^2\varphi_1}{\partial t_1^2} + \omega_0^2\varphi_1 + 2\frac{\partial^2\varphi_0}{\partial t_1\partial t_2} + 2\omega_0\frac{\partial^2\varphi_0}{\partial t_1}\right) = \\ = \xi\mu_1(\varphi) \end{cases} . \quad (9)$$

In order for the left side of equation Eq. 9 to be equal to zero, the both equations must be satisfied:

$$\frac{\partial^2\varphi_0}{\partial t^2} + \omega_0^2\varphi_0 = 0 \quad (10)$$

$$\begin{cases} \dfrac{\partial^2 \varphi_1}{\partial t_1^2} + \omega_0^2 \varphi_1 = \\ = \mu_1(\varphi) - 2\dfrac{\partial^2 \varphi_0}{\partial t_1 \partial t_2} - 2\omega_0 \dfrac{\partial^2 \varphi_0}{\partial t_1} \end{cases} . \quad (11)$$

Initial conditions are: $\varphi_0 = \Phi_0$, $\partial\varphi_0/\partial t_1 = 0$, $\varphi_1 = 0$ and $\partial\varphi_1 / \partial t_1 = -\partial\varphi_0 / \partial t_2$.

The solution for the equation Eq. 10 is $\varphi_0(t_1) = \Phi\sin(\omega_0 t_1 + \gamma)$ and for the equation Eq. 11 the solution must be required in the form:

$$\begin{cases} \varphi(t_1, t_2) = \Phi(t_2)\sin[\omega_0(t_1) + \gamma(t_2)] = \\ = \Phi\sin\psi \end{cases} . \quad (12)$$

where amplitude $\Phi(t_2)$ and phase difference $\gamma(t_2)$ of oscillation are both function of time t_2. Regarding Eq. 6 and Eq. 12, the specific torque $\mu(\varphi)$ can be expressed as:

$$\begin{cases} \mu(\varphi) = \mu[\Phi(t), \psi(t)] = \mu(\Phi, \psi) = \\ = \xi\mu_1[\Phi(t), \psi(t)] = \xi\mu_1(\Phi, \psi) \end{cases} . \quad (13)$$

When all the necessary differentiations of Eq. 12 are accomplished and regarding Eq. 13, differential equation Eq. 11 becomes:

$$\begin{cases} \dfrac{\partial^2 \varphi_1}{\partial t_1^2} + \omega_0^2 \varphi_1 = \\ = \mu_1(\Phi, \psi) - 2\Phi\dfrac{\partial \gamma}{\partial t_2}\omega_0 \sin\psi - \cdot \\ - 2(\omega_0 \dfrac{\partial \Phi}{\partial t_2} + \Phi\omega_0)\omega_0 \cos\psi \end{cases} \quad (14)$$

The function $\mu_1(\Phi, \psi)$ is periodic with angular frequency ω_0 and can be expanded into a Fourier trigonometric series. This procedure must be carried out in order to observe the appearance of resonant forcing terms which are generated by the fundamental harmonics of the periodic escapement torque. Thus, the Fourier expansion of the function $\mu_1(\Phi, \psi)$ is:

$$\begin{cases} \mu_1(\Phi, \psi) = \\ = \dfrac{a_0(\Phi)}{2} + a_1(\Phi)\cos\psi + b_1\Phi\sin\psi + \\ + \sum_{n=1}^{\infty} a_n(\Phi)\cos(n\psi) + \sum_{n=1}^{\infty} b_n\Phi\sin(n\psi) \end{cases} . \quad (15)$$

In which the coefficients a_0, a_n and b_n are determined by the following expressions:

$$\begin{cases} a_0(\Phi) = \dfrac{1}{\pi}\int_0^{2\pi} \mu_1(\Phi, \psi) d\psi, \\ a_0(\Phi) = \dfrac{1}{\pi}\int_0^{2\pi} \mu_1(\Phi, \psi)\cos(n\psi) d\psi, \\ a_0(\Phi) = \dfrac{1}{\pi}\int_0^{2\pi} \mu_1(\Phi, \psi)\sin(n\psi) d\psi, \end{cases} . \quad (16)$$

The fundamental harmonics $a_1(\Phi)\cos\psi$ and $b_1(\Phi)\sin\psi$ are especially emphasized since they represent the resonant forcing terms by which the secular terms are generated. It was already emphasized that these secular terms must be avoided from the solution of differential equation Eq. 14 since they grow without bound and perturbation series does not converge to the solution of the problem [11].

Regarding Eq. 15 and Eq. 16, differential equation Eq. 14 becomes:

$$\begin{cases} \dfrac{\partial^2 \varphi_1}{\partial t_1^2} + \omega_0^2 \varphi_1 = \\ = \dfrac{a_0(\Phi)}{2} + \\ + \left[a_1(\Phi) - 2(\dfrac{\partial \Phi}{\partial t_2} + \Phi\omega_0)\omega_0 \right]\cos\psi + \\ + \left[b_1(\Phi) - 2\Phi\dfrac{\partial \gamma}{\partial t_2}\omega_0 \right]\cos\psi + \\ + \sum_{n=1}^{\infty} a_n(\Phi)\cos(n\psi) + \sum_{n=1}^{\infty} b_n\Phi\sin(n\psi) \end{cases} \quad (17)$$

The resonant forcing terms in equation Eq. 17 are eliminated by the following expressions:

$$a_1(\Phi) - 2(\frac{\partial \Phi}{\partial t_2} + \Phi\omega_0)\omega_0 = 0 \quad (18)$$

$$b_1(\Phi) - 2\Phi\frac{\partial \gamma}{\partial t_2}\omega_0 = 0 \quad (19)$$

In accordance with Eq. 16, and $t_2 = \xi t$, Eq. 18 can be transformed to:

$$\begin{cases} \dot{\Phi} = \frac{d\Phi}{dt} = \\ = -\xi\omega_0\Phi + \\ + \frac{1}{2\pi\omega_0}\int_0^{2\pi}\mu(\Phi,\psi)\cos\psi \quad d\psi \end{cases}. \quad (20)$$

In accordance with Eq. 16, and and $t_2 = \xi t$, Eq. 19 can be transformed to:

$$\begin{cases} \dot{\gamma} = \frac{d\gamma}{dt} = \\ = -\frac{1}{2\pi\omega_0\Phi}\int_0^{2\pi}\mu(\Phi,\psi)\sin\psi \quad d\psi \end{cases}. (21)$$

The sub integral functions in formulas Eq. 20 and Eq. 21 depend on the phase coordinate ψ and the oscillation amplitude Φ, and the limits of integration are 0 and 2π. The mentioned expressions can also be defined as functions of the angular coordinate φ and the oscillation amplitude Φ. According to the formulas used to transform the coordinate ψ into the angular coordinate φ:

$$\begin{cases} \phi = \Phi\sin\psi, d\phi = \\ = \Phi\cos\psi, d\psi, \sin\psi = \frac{\phi}{\psi}\cos\psi = \\ = \frac{1}{\Phi}\sqrt{\Phi^2 - \phi^2}, d\psi = \pm\frac{d\phi}{\sqrt{\Phi^2-\phi^2}} \end{cases}. (22)$$

the expressions Eq. 20 and Eq. 21 becomes:

$$\begin{cases} \dot{\Phi} = \frac{d\Phi}{dt} = \\ = -\xi\omega_0\Phi + \frac{1}{2\pi\omega_0\Phi}\oint_\Phi \mu(\phi)d\phi \end{cases}. \quad (23)$$

$$\begin{cases} \dot{\gamma} = \frac{d\gamma}{dt} = \\ = -\frac{1}{2\pi\omega_0\Phi^2}\int_0^{2\pi}\frac{\mu(\phi)\phi \quad d\phi}{\sqrt{\Phi^2-\phi^2}} \end{cases}. \quad (24)$$

Equation Eq. 23 describes the change of an oscillations amplitude and Eq. 24 the change of oscillation phase angle shift as a functions of angular coordinateφ. Last two differential equations describes the dynamic of the watch balance wheel. They are universally applicable for all types of escapement mechanisms, both for stationary and non-stationary oscillations.

Finally, it is important to emphasize that equations Eq. 21 and Eq. 23 defines the variation of oscillation frequency generated by the influence of the escapement to the oscillator and represent the mathematical description of an important horological phenomenon known as escapement error.

5. SIMULATIONS

Motion analysis disclosed in this chapter is accomplished to prove the correctness of the theoretical formulas for the escapements errors derived from the perturbation theory. Besides significance in theory of oscillations and clock mechanisms, the exposed method can be broadly applied for all mechanisms which operation are triggered by movement or depend on sensor states [10].

Event-based motion analysis is approach complementary to the motion analysis based on time and it serves to solve more complex kinematic and dynamic problems by the using of the SolidWorks application. Since external actions of an escapement on a watch oscillator are triggered by the part movement or state of the escapement-oscillator assembly, it is wise to choose this method of simulation rather than the method based on flow of operational time.

In particular, the event based motion analysis is applied to simulate and describe the complex dynamic behavior of the escapement-oscillator assembly and thus to determine the numerical values of the escapement error.

The correctness of the following two formulas are considered:

Formula which describes the error of detached escapement which includes English lever and Swiss lever escapements, as the most famous in contemporary watch mechanism:

$$\begin{cases} R = \\ = -\dfrac{\omega_0}{4Q} \dfrac{\sqrt{\Phi_0^2 - (\varphi_0 + a)^2} - \sqrt{\Phi_0^2 - (\varphi_0 - a)^2}}{a} \end{cases} \quad (25)$$

Formula which describes the error of recoil escapement which includes verge escapement, different anchor and grasshopper escapements:

$$R = \frac{\omega_0}{2Q} \frac{\sqrt{\Phi_0^2 - \varphi_M^2}}{\varphi_M} \quad (26)$$

In both formulas Φ_0 is the stationary amplitude, φ_0 is the angular centrum of impulse, α is a half of angular interval $[\varphi_0 - \alpha,\ \varphi_0 + \alpha]$ in which escapement delivers the energy to the balance wheel and φ_M is the meshing angle between escapement pallets and teeth of the escapement wheel.

Since escapement error is generated by the influence of the escapement to the watch oscillator (balance wheel), the entire 3D solid model of the escapement and balance wheel assembly (Fig. 7) must be built and considered during computer simulation of its motion. It must be emphasized that these 3D models are abstract, adapted to describe parameters of the exposed mathematical formulas clearly.

The proper validation of the previous formulas by the SolidWorks event based simulation needs correct and precise selection of the following motion analysis parameters:

- The balance wheel mass momentum of inertia $J = 1.8 \cdot 10^{-4} kgm^2$,
- Spiral spring constant $k = 4\pi^2 {v_0}^2 \cdot J = 0.11369784\ Nm/rad$,
- Damping coefficient $c = 2\pi \cdot J \cdot v_0/Q = 2.261947 \cdot 10^{-5}\ Nm/\left(\frac{rad}{s}\right)$,
- Detent escapement torque $M = J \cdot \pi^3 \cdot {v_0}^2 \cdot {\Phi_{ST}}^2/Q \cdot \alpha = 0.189363273 Nm$
- Recoil escapement torque $M = J \cdot \pi^3 \cdot {v_0}^2 \cdot {\Phi_{ST}}^2/Q \cdot \varphi_M = 0.001402691 Nm$

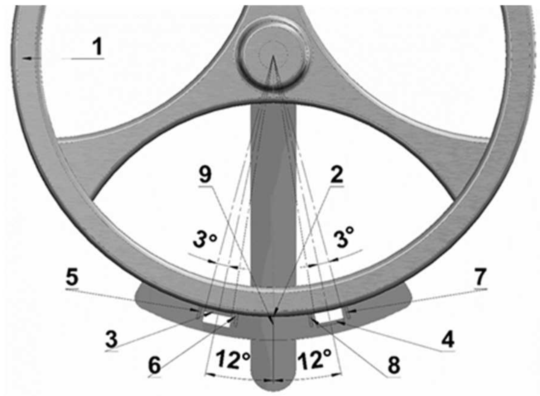

Fig. 7. 3D solid model of detent escapement and balance wheel assembly

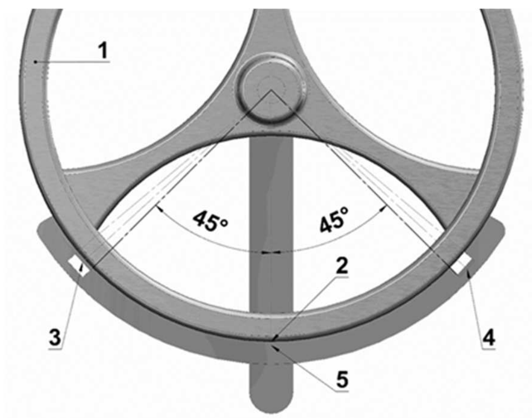

Fig. 8. 3D solid model of recoil escapement and balance wheel assembly

Parts of the 3D solid model of detent escapement and balance wheel assembly (Fig.7.):
1 – balance wheel,
2 – five different proximity sensors,
3 – left pallet,
4 – right pallet,
5–8 – sensor triggers which activate especially defined proximity sensors by which the position of the pallets 3 and 4 can be changed and controlled,
9 – sensor trigger which detects the transition of the balance wheel through the equilibrium position φ=0.

Parts of the 3D solid model of recoil escapement and balance wheel assembly (Fig.8.):
1 – balance wheel
2 – sensor which changes the direction of the escapement torque
3 – left pallet with the sensor trigger
4 – right pallet with the sensor trigger

5 – sensor trigger on (5) detect the transition of the balance wheel through the equilibrium position φ=0.

The results of the simulation:

- For the detached escapement, the angular frequency and period of non-driven damped oscillations obtained by simulation are: $\omega_{01} = 25{,}13266316\ rad/s, T_{01} = 2\pi/\omega_{01} = 0{,}25000078\ s$
- For the recoil escapement, the angular frequency and period of non-driven damped oscillations obtained by simulation are: $\omega_{02} = 25{,}13262744\ rad/s, T_{02} = 2\pi/\omega_{02} = 0{,}25000113\ s.$
- For the detached escapement, the angular frequency and period of driven damped oscillations obtained by simulation are: $\omega_1 = 25{,}12986985\ rad/s, T_1 = 2\pi/\omega_1 = 0{,}250028565\ s.$
- The detached escapement error obtained by simulation is: $R_1 = \omega_1 - \omega_{01} = -0{,}00279331\ rad/s.$
- The theoretical value of the detached escapement error is: $R_{1\mathrm{T}} = -0{,}002795454\ s$
- The relative difference between values of detached escapement errors obtained by theory of perturbation and computer simulation is $\delta_1 = \left|\frac{R_{1\mathrm{T}}-R_1}{R_{1\mathrm{T}}}\right| = 7{,}66959 \cdot 10^{-4} < 0.08\%$.
- For the recoil escapement, the angular frequency and period of driven damped oscillations obtained by simulation are $\omega_2 = 25{,}2413366\ rad/s, T_2 = 2\pi/\omega_2 = 0{,}24892443 s$
- The recoil escapement error obtained by simulation is $R_2 = \omega_2 - \omega_{02} = 0{,}10870916\ rad/s$
- The theoretical value of the recoil escapement error is $R_{2\mathrm{T}} = +0{,}108827469\ rad/s$
- The relative difference between values of detached escapement errors obtained by theory of perturbation and computer simulation is $\delta_2 = \left|\frac{R_{2\mathrm{T}}-R_2}{R_{2\mathrm{T}}}\right| = 0{,}001087 < 0.11\%.$

6. CONCLUSION

The extremely small previously calculated values for δ_1 and δ_2 prove the correctness of the theoretical formulas derived from the perturbation theory by which the errors of the detached and recoil escapements are determined [10].

This paper can be important mostly for education in the field of the theory of mechanisms, non-linear dynamics and mathematics, 3D modeling and simulation. In particular, this work can be useful for better understanding the operational principles of clock and watch mechanism, as well as their design, construction and maintenance. Moreover, the method of perturbation exposed in this paper, can be applied for solving many other problems of non-linear mechanics, especially non-linear oscillation and celestial mechanics.

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Dinamica neliniara a mecanismelor de escapare

Mecanismele de ceas aparțin domeniului mecanicii de precizie, care măsoară trecerea timpului printr-un grad ridicat de uniformitate a mișcărilor lor. Subansamblul cheie al mecanismului de ceas este un oscilator cuplat cu un mecanism de evacuare care introduce perturbări în procesul oscilator astfel încât oscilațiile nu mai sunt libere, ci forțate și neliniare cu o frecvență care este supusă modificării. Această lucrare este dedicată analizei dinamicii mecanismului de ceas folosind teoria calculului perturbației. Rezolvarea aproximativă a ecuațiilor diferențiale neliniare care descriu mișcarea oscilatorului ceasului sunt determinate prin metoda perturbației la scară de două timpi și reprezintă formule aplicabile universal pentru erorile de scăpare. Aceste formule sunt verificate prin simulările computerizate ale funcționării mecanismului de evacuare.

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