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CASTIGLIANO'S THEOREM IN THE STUDY OF NONLINEAR ELASTIC STRUCTURAL ELEMENTS WITH POWER-LAW CONSTITUTIVE MODEL

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Abstract: *The strong development of the industry in the last period has led to the strong diversification of the materials used in engineering applications and also to the increase of the forces and speeds that appear in these systems. As a result, numerous researches were undertaken that took into account the non-linear behavior of materials during the operation of mechanical systems. Constituent laws can no longer be considered linear laws. Numerous forms of constitutive laws have been proposed to cover the existing non-linearities. Among them, the power law type laws were noted, in which the dependence is of an exponential type. In the work, results were obtained regarding strain and stress appearing in standard elements used in engineering. Also, the possibility of using Castigliano's theorem was studied in the case of application to materials with non-linear behavior, with a power law constitutive law. The obtained results can be useful to engineers because they offer quick results and with good precision.*

Key words: nonlinear constitutive equation; power law; Castigliano's theorem; tension; compression; bending; shear; torsion

1. INTRODUCTION

The theory of elasticity, in its classic form, uses as the main hypothesis in a good part of its developments, the hypothesis of linearity that exists between stress and strain. Steel and other materials frequently used in engineering have, in most applications, this behavior and the results obtained using this hypothesis are generally good. There are still materials such as cast iron, copper, brass, lead, rubber, certain plastics and composite materials frequently used in modern technology, but for which this linear law no longer works. One of the most used models for the study of nonlinear materials is the power law. Non-linear constitutive models have started to be used more often recently due to the numerous and varied engineering applications in which such materials appear.

Traditional results from continuum media mechanics are used to present the main problems involved in the use of nonlinear materials are

presented in [1]. Some types of materials with nonlinear behavior such as boron/aluminum fiber composite with metal matrix are described in [2,3]. In the paper [4], probabilistic models are used for the analysis of nonlinear systems. Micromechanical models remain very powerful models for the analysis of porous materials with phases with nonlinear behavior. In [6,7], a technique is developed that automatically adjusts the Jacobian based on the nonlinear constitutive law. In [8], nonlinear isotropic materials are studied. The paper [9] presents a method for the elastodynamic analysis of a one-dimensional nonlinear system. In [10], a useful study for post-buckling analysis is provided. The finite element method also offers in this case an important support for solving non-linear problems [11]. The technique is taken over by other authors in different versions [12-16]. Fiber-reinforced polymer composites, used more and more in real-world applications, present nonlinearities studied in [17,18].

Models for hyperelastic materials are presented in [19,20].

More complex phenomena involving nonlinear materials are studied in [21]. Different types of constitutive laws are analyzed in [22,23]. Large, non-linear deformations are presented in [24]. Constitutive laws represent the starting point in the study of such problems [25-27]. The use of FEM has been used by numerous researchers [28-35].

Biomechanics problems where living materials are strongly nonlinear [36-40]. Numerous engineering applications involving nonlinear materials have been studied and the results communicated in interesting papers [41-50], while theoretical advances in the field of nonlinear materials are presented in papers [51-60].

In this paper, the authors aim to make a contribution to the development of methods for the study of non-linear materials, whose dependence on stress can be conveniently described with a constitutive law of the power law type.

2. POWER LAW STRAIN-STRESS MODEL

The description of the power law type constitutive law is made through the relations: [61]:

$$\varepsilon = \left(\frac{\sigma}{\sigma_0} \right)^n, \quad (1)$$

respectively,

$$\gamma = \left(\frac{\tau}{\tau_0} \right)^n. \quad (2)$$

In these relationships, n is denoted by a real number that represents the exponent of the power-law, ε is the normal strain, γ is shear strain, σ is denoted the normal stress, τ represents the shear stress. The constants of material σ_0 and τ_0 have the same Young's modulus physical dimension [21]. These constants and the exponent n can be determined experimentally for every material considered. Hooke's law is a particular case when $n = 1$. When $n = 0$ the material is totally rigid whereas when $n = \infty$ we are in the plastic domain.

In the classical theory of elasticity, it is considered, for most of the developments made, that the strain-stress relationship is a linear relationship. In reality, the linearity hypothesis is an approximation that describes very well most materials and their behavior within reasonable loading limits. However, in reality, materials are generally non-linear. Certain materials are strongly non-linear and for this reason it was found that special non-linear models must be developed for some of them. Among the multitude of models that were used in the study of these unusual materials, the constitutive law of the power-law type is counted. Although there are some studies on the behavior of these materials [21], an organized and systematic study has not been done until now. We mention the contribution [61] that completes the analysis of such systems. In the current work, we will deal with the possibility of using Castigliano's theorem to solve such problems.

3. STANDARD LOADING CASES

In previous papers and book the basic loading cases are studied and basic results are obtained considering the hypothesis of linear dependence strain-stress (Hook's law) [61]. In the following we will adapt the main results to the case when we work with a power law dependency. We present these results [61].

3.1. Tension. Compression

The simplest load and frequently encountered in practice is tension (pulling)/compression (squashing). Some real-world scenarios frequently encountered in engineering practice will be presented next, form an isotropic, homogeneous, with constant section beam.

a) In the study will be used the Bernoulli's hypothesis. This means that $\varepsilon = ct$ for a single, arbitrary section at each point of the middle average fiber. As a consequence, taking into account the Eq.(1), the stress is constant for all of the section's points:

$$\sigma = \sigma_0 (ct)^{\frac{1}{n}}. \quad (3)$$

If is written the equilibrium equation of the stresses acting on the section, it obtains:

$$N = \int_A \sigma dA = \sigma A . \quad (4)$$

N is normal load in the section. So, the stress can be determined with the relation:

$$\sigma = \frac{N}{A} , \quad (5)$$

and the beam's lengthening/shortening is:

$$\Delta l = \varepsilon \cdot l = \left(\frac{\sigma}{\sigma_0} \right)^n l = \left(\frac{N}{\sigma_0 A} \right)^n l ; \quad (6)$$

where l is the length of the beam and A the area of its section.

3.2. Torsion

The precedent hypothesis for the material are the same, that is a beam with a circular section and the length l . We denote with R the section's radius and ρ the radius of the elementary element dA , on which the stress τ is applied due to the torsional moment M_r . The equilibrium equation offer us:

$$M_r = \int_A \tau \rho dA . \quad (7)$$

The sliding angle is:

$$\gamma = \rho \theta , \quad (8)$$

It results:

$$\tau = \tau_0 \gamma^{\frac{1}{n}} . \quad (9)$$

Using Eqs. (8) and (9) in Eq. (7), the moment becomes:

$$M_r = \tau_0 \theta^{\frac{1}{n}} \int_A \rho^{\frac{n+1}{n}} dA = \tau_0 \theta^{\frac{1}{n}} I_r , \quad (10)$$

It is denoted:

$$I_r = \int_A \rho^{\frac{n+1}{n}} dA , \quad (11)$$

the cross-section's generalized polar moment. So, the stress is:

$$\tau = \frac{M_r \rho^{\frac{1}{n}}}{I_r} , \quad (12)$$

and the rotation angle:

$$\varphi = l \theta = \left(\frac{M_r}{\tau_0 I_r} \right)^n l , \quad (13)$$

3.3. Bending

The basics of the bending in Strength of Materials in the hypothesis of the linear elasticity are considered known. Using the Bernoulli's assumption, the elongation of a fiber situated at a distance y from the neutral axis can be expressed by:

$$\varepsilon = \frac{y}{\rho} , \quad (14)$$

It is denoted with ρ the average fiber's radius of curvature. The expression of stress is (see Eq.(1)):

$$\sigma = \sigma_0 \left(\frac{y}{\rho} \right)^{\frac{1}{n}} . \quad (15)$$

Considering equilibrium equation it results:

$$M_i = \int_A \sigma y dA = \frac{\sigma_0}{\rho^{\frac{1}{n}}} \int_A y^{\frac{n+1}{n}} dA = \frac{\sigma_0 I_z}{\rho^{\frac{1}{n}}} , \quad (16)$$

It is denoted:

$$I_z = \int_A y^{\frac{n+1}{n}} dA . \quad (17)$$

the generalized axial moment of the cross section. Using Eq.(16) it results:

$$\frac{1}{\rho} = \left(\frac{M_i}{\sigma_0 I_z} \right)^n . \quad (18)$$

From Eq.(15) and Eq. (18) it is obtained the stress:

$$\sigma = \frac{M_i y^{\frac{1}{n}}}{I_z} . \quad (19)$$

or:

$$\sigma = \frac{M_i}{W_z}, \quad (20)$$

In Eq.(20) it is noted:

$$W_z = \frac{I_z}{y^n}. \quad (21)$$

(the generalized axial strength modulus).

3.4. Shear

We consider that Juravski's [62,63] assumption are used (Figure 4). The section is considered divide in two parts. The part between sections I and II and the section across a plane parallel to the neutral plane is separated by the distance y , and the equilibrium equation is:

$$\int_{A_y} (\sigma + d\sigma) dA - \int_{A_{y1}} \sigma dA - \tau_{xy} dx = 0. \quad (22)$$

Considering the expression of stresses:

$$\sigma = \frac{M_i y^n}{I_z}. \quad (23)$$

and:

$$\sigma + d\sigma = \frac{(M_i + dM_i) y^n}{I_z}. \quad (24)$$

it obtains:

$$\int_{A_y} \frac{dM_i y^n}{I_z} dA = \tau_{xy} b dx. \quad (25)$$

With the notation:

$$S_z = \int_{A_y} y^n dA, \quad (26)$$

it obtains:

$$\tau_{xy} = \frac{TS_z}{bI_z}. \quad (27)$$

4. CASTIGLIANO'S THEOREM

4.1. Deformation Energy

a) **Internal energy.** According to the nature of the request, the internal energy is given by the expressions:

$$L_i = \int_V dV \int_0^\sigma \sigma d\varepsilon, \quad (28)$$

or:

$$L_i = \int_V dV \int_0^\tau \tau d\gamma. \quad (29)$$

Considering the constitutive power law it obtains:

$$d\varepsilon = n \frac{\sigma^{n-1}}{\sigma_0} d\sigma, \quad (30)$$

and:

$$d\gamma = n \frac{\tau^{n-1}}{\tau_0} d\tau, \quad (31)$$

Using Eq.(30) and (31), the expressions of the internal energy are obtained:

$$L_i = \frac{n}{(n+1)\sigma_0^n} \int_V \sigma^{n+1} dV, \quad (32)$$

and:

$$L_i = \frac{n}{(n+1)\tau_0^n} \int_V \tau^{n+1} dV, \quad (33)$$

For tension/compression, if Eq. (5) is taken into account, we obtain:

$$L_i = \frac{n}{(n+1)\sigma_0^n A^n} \int_0^l N^{n+1} dl, \quad (34)$$

where, if we consider constant the external tension/compression force N , we get:

$$L_i = \frac{nN^{n+1}l}{(n+1)\sigma_0^n A^n}, \quad (35)$$

For pure bending it results:

$$L_i = \frac{n}{(n+1)\sigma_0^n I_z^n} \int_0^l M_i^{n+1} dl, \quad (36)$$

and, if the bending moment is considered constant:

$$L_i = \frac{nM_i^{n+1}l}{(n+1)\sigma_0^n I_z^n}, \quad (37)$$

Finally, for the torsion of beams with a circular section, we obtain:

$$L_i = \frac{n}{(n+1)\tau_0^n I_r^n} \int_0^l M_r^{n+1} dl, \quad (38)$$

and, if the twisting moment is constant:

$$L_i = \frac{nM_r^{n+1}l}{(n+1)\tau_0^n I_r^n}, \quad (39)$$

b) **Complementary energy.** Analogously, if the nature of the loads is taken into account, the relationships are obtained, respectively:

$$L_c = \int_V dV \int_0^\sigma \varepsilon d\sigma, \quad (40)$$

$$L_c = \int_V dV \int_0^\tau \chi d\tau, \quad (41)$$

If Eq.(1) and Eq.(2) are taken into account, the relationships are obtained: P_j ($j \neq k$).

$$L_c = \frac{1}{(n+1)\sigma_0^n} \int_V \sigma dV, \quad (42)$$

and:

$$L_i = \frac{1}{(n+1)\tau_0^n} \int_V \tau dV. \quad (43)$$

Thus, if we are dealing with tension/compression, it can be written:

$$L_c = \frac{1}{(n+1)\sigma_0^n A^n} \int_0^l N^{n+1} dl, \quad (44)$$

where, if we consider constant the external tension/compression force N , we get:

$$L_c = \frac{N^{n+1}l}{(n+1)\sigma_0^n A^n}, \quad (45)$$

If we are dealing with pure bending, then:

$$L_c = \frac{1}{(n+1)\sigma_0^n I_z^n} \int_0^l M_i^{n+1} dl, \quad (46)$$

and, if the bending moment is considered constant:

$$L_c = \frac{M_i^{n+1}l}{(n+1)\sigma_0^n I_z^n}, \quad (47)$$

When twisting the beams with a circular section, you get:

$$L_c = \frac{1}{(n+1)\tau_0^n I_r^n} \int_0^l M_r^{n+1} dl, \quad (48)$$

and, if the twisting moment is constant:

$$L_c = \frac{M_r^{n+1}l}{(n+1)\tau_0^n I_r^n}, \quad (49)$$

In the case of compound loads, the sum of the energies is made.

4.2. Generalized Castigliano's Theorem

It is considered a non-linear elastic system actuated by independent forces P_1, P_2, \dots, P_m . By giving to a force P_k an elementary increase dP_k , keeping the other forces constant, the complementary energy increases by:

$$dL_c = \frac{\partial L_c}{\partial P_k} dP_k. \quad (50)$$

The complementary energy increases only due to the displacement of the P_i force, being equal to:

$$dL_c = \delta_k dP_k. \quad (51)$$

The other forces P_j ($j \neq k$) by their displacement do not produce complementary mechanical work. Castigliano's generalized theorem results:

$$\delta_k = \frac{\partial L_c}{\partial P_k}. \quad (52)$$

Therefore, if the complementary energy of a certain elastic system (therefore also

nonlinear) is expressed as a function of external (independent) forces, then the derivative of this energy in relation to one of the forces is equal to the displacement projected on the force at the point of its application .

The theorem is demonstrated analogously in the case of the rotations that are produced by an externally applied couple. The theorem applies to:

- calculation of displacements of nonlinear and linear systems;
- raising the indeterminacy of systems in the form of:

$$\frac{\partial L_c}{\partial X} = 0 . \quad (53)$$

where X is the statically undeterminate reaction.

5. RESULTS

5.1. The Deformation to Basic Load

a) For tension/compression it can be written:

$$\Delta l = \frac{\partial L_c}{\partial P} = \frac{\partial}{\partial P} \left[\frac{1}{(n+1)\sigma_0^n A^n} \int_0^l N^{n+1} dl \right] \quad (54)$$

$$= \frac{1}{(n+1)\sigma_0^n A^n} \int_0^l N^n \frac{\partial N}{\partial P} dl .$$

For $N = P = ct$, it results:

$$\Delta l = \frac{N^n l}{\sigma_0^n A^n} . \quad (55)$$

For bending, if the bending moment $M = M_i = ct$ is considered, we obtain, in the same way:

$$f = \frac{\partial L_c}{\partial P}$$

$$= \frac{1}{\sigma_0^n I_z^n} \int_0^l M_i^n \frac{\partial M_i}{\partial P} dl = \left(\frac{M_i}{\sigma_0 I} \right)^n \frac{l^2}{2} . \quad (56)$$

and:

$$\varphi = \frac{\partial L_c}{\partial M}$$

$$= \frac{1}{\sigma_0^n I_z^n} \int_0^l M_i^n \frac{\partial M_i}{\partial M} dl = \left(\frac{M_i}{\sigma_0 I} \right)^n l . \quad (57)$$

For the torsional stress, it is obtained, analogously:

$$\varphi = \frac{\partial L_c}{\partial M}$$

$$= \frac{1}{\tau_0^n I_r^n} \int_0^l M_r^n \frac{\partial M_r}{\partial M} dl = \left(\frac{M_r}{\tau_0 I_r} \right)^n l . \quad (58)$$

5.2. Deformation of a Beam Loaded with a Concentrated Force

Using the presented method, the displacement in the middle of the beam will be calculated:

$$f = \frac{\partial L_c}{\partial P} = \frac{2}{\sigma_0^n I_z^n} \int_0^{l/2} M_i^n \frac{\partial M_i}{\partial P} dl . \quad (59)$$

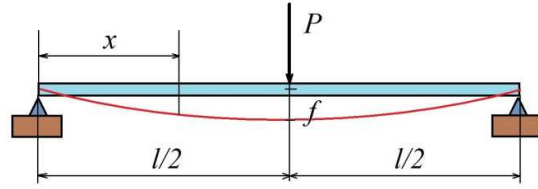


Figure 1. Beam loaded with a concentrated force

Because:

$$M_i = \frac{P}{2} x . \quad (60)$$

it results:

$$\frac{\partial M_i}{\partial P} = \frac{x}{2} . \quad (61)$$

and:

$$f = \frac{2}{\sigma_0^n I_z^n} \int_0^{l/2} \left(\frac{Px}{2} \right)^n \frac{x}{2} dx$$

$$= \frac{P^n l^{n+2} x}{(n+2)\sigma_0^n I_z^n 2^{n+2}} . \quad (62)$$

If we consider $n = 1$, we find the well-known relationship from the Strength of Materials course:

$$f = \frac{Pl^3}{48EI_z} . \quad (63)$$

5.3. Deformation of an angled beam

The vertical (v) and horizontal (u) displacement of the free end is calculated.

$$v = \frac{\partial L_c}{\partial P} \quad (64)$$

where:

$$L_c = \frac{1}{(n+1)\sigma_0^n I_z^n} \left[\int_0^l (Px)^{n+1} dx + \int_0^h (Pl)^{n+1} dy \right] \quad (65)$$

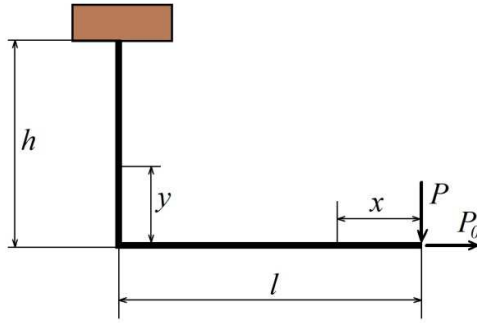


Figure 2. Angled beam

It obtains:

$$v = \frac{P^n}{\sigma_0^n I_z^n} \left(\frac{l^{n+2}}{n+2} + l^{n+2} h \right) \quad (66)$$

$$u = \frac{\partial L_c}{\partial P_0} \quad (67)$$

where:

$$L_c = \frac{1}{(n+1)\sigma_0^n I_z^n} \left[\int_0^l (Px)^{n+1} dx + \int_0^h (P_0 y - Pl)^{n+1} dy \right] \quad (68)$$

After performing the calculations, you will get:

$$L_c = \frac{1}{(n+1)\sigma_0^n I_z^n} P^{n+1} l^{n+2} + \frac{1}{(n+1)\sigma_0^n I_z^n P_0} \left[(P_0 h - Pl)^{n+2} - (-Pl)^{n+2} \right] \quad (69)$$

Then it results for u :

$$u = \frac{[-(P_0 h - Pl)^{n+2} + (-Pl)^{n+2}]}{(n+1)(n+2)\sigma_0^n I_z^n P_0^2} + \frac{[P_0(n+2)(P_0 h - Pl)^{n+2} h]}{(n+1)\sigma_0^n I_z^n P_0^2} \quad (70)$$

If $P_0 = 0$, Eq. (70) is undetermined. Using l'Hospitals' rule, we get:

$$v = -\frac{P^n l^n h^2}{2\sigma_0^n I_z^n} \quad (71)$$

5.4 the undetermination of a beam

Let X be an undeterminate static reaction (Fig.3). The bending moments in the two regions of the bar are:

$$M_1 = Xx \quad ; \quad M_2 = X\left(\frac{l}{2} + x\right) - \frac{Px}{4} \quad (72)$$

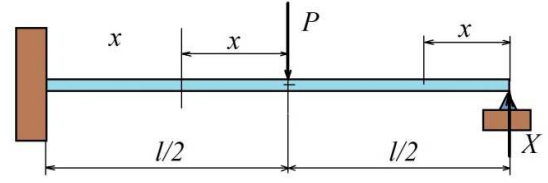


Fig. 3. Undetermined beam

Write the zero displacement condition in support 2:

$$\frac{1}{\sigma_0^n I_z^n} \int_0^{l/2} (Xx)^n x dx + \frac{1}{\sigma_0^n I_z^n} \int_0^{l/2} \left[X\left(\frac{l}{2} + x\right) - \frac{Px}{4} \right]^n \left(\frac{l}{2} + x\right) dx = 0 \quad (73)$$

If the integration is done, it results after elementary transformations:

$$\left(\frac{X}{P}\right)^n \left(\frac{X}{P} - 1\right)^2 + \left(2\frac{X}{P} - 1\right)^{n+1} \left(2\frac{X}{P} - \frac{2n+3}{n+1}\right) - \left(\frac{X}{P}\right)^{n+1} \left(\frac{X}{P} - \frac{n+2}{n+1}\right) = 0 \quad (74)$$

Introducing the new variable:

$$t = \frac{X}{P} \quad (75)$$

the following equation is obtained:

$$t^n (t-1)^2 + (2t-1)^{n+1} \left(2t - \frac{2n+3}{n+1}\right) - t^{n+1} \left(t - \frac{n+2}{n+1}\right) = 0 \quad (76)$$

which will give the unknown force X , after solving the algebraic equation using a numerical method.

6. DISCUSSION AND CONCLUSIONS

In the classic theory of elasticity, a basic hypothesis is the linearity of the dependence between strain and stress, the hypothesis on which a large part of this theory is built.

Of course, in the real world this hypothesis is only a very convenient approximation for the development of the theory. In reality, practically all materials are non-linear, but for materials mainly used in engineering applications, the hypothesis of small deformations and Hooke's law work excellently.

However, there are also materials that are strongly non-linear (cooper) or that are subjected to loads high enough for the non-linearity to develop significant negative effects. The need to use these materials, for various reasons, as well as the development of the industry through the appearance of structures of high complexity or with large dimensions and loaded with great forces, led to the need to study such non-linear materials.

Different nonlinearity models were considered and used. Among them, the models that use a constitutive power law type law stand out. Some applications using this law are shown in the references used. In the current work, we were concerned with the possibility of using Castigliano's theorem for materials whose constitutive law is of the power law type.

Some examples illustrate the possibility of using this theorem in the specific conditions given by leaving the linear law and replacing it with an exponential law. Some differences are highlighted that appear by comparison with the classical calculation when the strain-stress dependence is linear.

The cases presented in this paper can provide a starting point for future research on the behavior of such materials. The results can be extended to other types of constitutive laws, in the case of some engineering applications that claim this.

These applications become extremely necessary in the contemporary context of the development of new and composite materials. The obtained results can be easily used in future applications, being able to offer quick and simple solutions, desired by any designer.

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Teorema Castigliano in studiu al elementelor structurale elastice neliniare cu model constitutiv al legii puterii

Rezumat. Dezvoltarea puternică a industriei în ultima perioadă a dus la diversificarea puternică a materialelor utilizate în aplicații ingineresti și, de asemenea, la creșterea forțelor și vitezelor care apar în aceste sisteme. Ca urmare s-au întreprins numeroase cercetări care au ținut seama de comportarea neliniară a materialelor în timpul funcționării sistemelor mecanice. Legile constitutive nu mai pot fi considerate legi lineare. S-au propus numeroase forme de legi coconstitutive care să acopere neliniariitățile existente. Printre acestea s-au remarcat legile de tipul power law, în care dependența este de tip exponențial. În cadrul lucrării s-au obținut rezultate privind strain și stress aparute în elemente standard utilizate în inginerie. De asemenea a fost studiată posibilitatea utilizării teoremei lui Castigliano în cazul aplicării la materiale cu comportare neliniară, cu lege constitutivă de tip power law. Rezultatele obținute pot fi utile inginerilor căci oferă rezultate rapide și cu o bună precizie.

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