TECHNICAL UNIVERSITY OF CLUJ-NAPOCA

ACTA TECHNICA NAPOCENSIS

Series: Applied Mathematics, Mechanics, and Engineering

Vol. 68, Issue II, June, 2025

INTRODUCING NONUNIFORMITY COEFFICIENTS IN THE STUDY OF VIBRATIONS OF MULTILAYER COMPOSITE BARS

Sabin RIZESCU, Alexandru BOLCU, Dumitru BOLCU, Marius Marinel STANESCU

Abstract: *Sandwich composite bars are largely used in mechanical structures containing subparts made of composite materials. Due to fabrication issues, quite often, a large majority of sandwich bar exhibits a certain number of nonuniformities across its transversal section. Nonuniformities are to be assessed and quantified while building up mathematical model for the vibrations of such a bar. It was introduced three nonuniformity coefficients in writing the equations of motion for a vibrating multilayer sandwich bar. Only the case of a bar with geometric and mass symmetry was considered. Even though this case might be considered a very particular one, however the achievement here is that all considerations that have been made are in full respect with all requirements of PSDBT theory.*

Key words: *composite materials, sandwich multilayer bars, nonuniformity coefficients, mass and stiffness parameters, vibrations.*

1. INTRODUCTION

A special topic in the study of the dynamics of composite materials consists by sandwich bars made of multiple superimposed layers, with constant thickness. Most studies focus on sandwich bars made of three layers, with the middle layer exhibiting viscoelastic behavior, and the lower and upper layers exhibiting superior elastic and strength properties. The vast majority of studies have been based on the following assumptions regarding the behavior of sandwich laminates:

- there is continuity of displacements and stresses on the separation surfaces between layers;
- transverse inertia forces are predominant, with longitudinal inertia and rotational inertia of the bar section are small enough to be neglected;
- there are no deformations along the thickness of the bar, thus transverse deformations are uniform across the entire the bar transverse section;
- the core exhibits elastic or viscoelastic behavior, so, absorbing shear (tangential) stresses;
- the outer layers exhibit elastic behavior, so, being subjected to pure bending.

Higher-order deformation theories take into consideration the deformation of transverse sections and satisfy the conditions of nullifying tangential stresses on the outer surfaces, without a shear correction factor. In the specialized literature, various higher-order shear theories that meet these conditions are proposed by several researchers. These theories differ primarily in the functions that evaluate the variation of tangential stresses across the section of the bar. The most well-known higher-order deformation theories are based on the following types of functions:

- Parabolic, symbolized as PSDBT (Parabolic Shear Deformation Beam Theory), introduced by [1];
- Trigonometric, symbolized as TSDBT (Trigonometric Shear Deformation Beam Theory), introduced by [2];
- Hyperbolic, symbolized as HSDBT (Hyperbolic Shear Deformation Beam Theory), introduced by [3];
- Exponential, symbolized as ESDBT (Exponential Shear Deformation Beam Theory), introduced by [4].

A new higher-order deformation theory, symbolized as ASDBT, was proposed more

Received: 07.04.25; **Similarities**: 29.05.25: **Reviewed**: 07.04./ 07.04.25: **Accepted**:17.06.25.

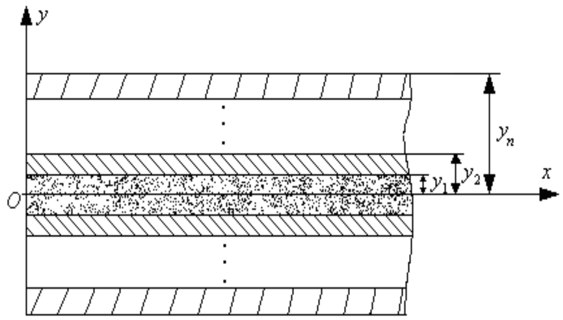


Fig. 1. Beam with geometric and mass symmetry relative to the middle layer.

recently by [5], and a comparison of the results of these theories was conducted by [6] and, later, by [7].

It is evident that higher-order deformation theories describe the composite's behavior with greater accuracy compared to linear deformation theories. At first glance, higher-order deformation theories appear to be free of drawbacks. However, there are cases—by no means few—where drawbacks may arise. In this context, Savithri and Varadan in [8] note significant drawbacks of higher-order deformation theories when composite laminates (plates or beams) are subjected to concentrated forces.

The reality is that, generally, the shape and type of loading of a composite, which in turn suggest how it is subjected to stress, directly leads to certain assumptions about deformation, with the aim of formulating a priori a displacement hypothesis. Changing the loading or the support conditions for the same composite leads to a change in its stress state, and thus to a potential need to adopt a different deformation hypothesis. The conclusion is that, for a non-homogeneous and anisotropic material, a constitutive equation based on a particular loading and support hypothesis may not be suitable when the loading or support conditions change. Dong and Chun in [9] formulate a constitutive equation for laminated plates subjected to shear using a method based on a priori assumptions of deformation hypotheses and the development of an original concept for generalizing the notion of shear plane. Results are provided for the calculation of shear stiffness for various cases of laminated and sandwich profiles.

For a simply supported sandwich beam, Xavier Chew and Lee in [10], applying Hamilton's principle, developed a mathematical model for the vibrations of the beam. The mathematical model is based on the HSDBT theory. The study is conducted for different ratios between the thicknesses of the outer layers and between these thicknesses and that of the middle layer. Additionally, various scenarios for the elastic properties of all the three layers are considered. The numerical results analysis showed that the natural frequencies and damping factors obtained using HSDBT theory are some how lower than those obtained using FSDBT theory.

2. THEORETICAL CONSIDERATIONS

A sandwich beam of width b, having geometric and mass symmetry with respect to the middle layer, is considered (see Figure 1).

Let $y_k, k = \overline{1, n-1}$ denote the abscissa of the separation surface between layers k and k+1 and let y_n denote the abscissa of the outer surface of layer n. Due to symmetry, it is highlighted the base deformations for y≥0.

Let:

$$B = \begin{pmatrix} 1 \\ -\frac{1}{3y_n^2} \end{pmatrix} \tag{1}$$

and let:

$$B_k = A_{k,k+1} \cdot B_{k+1}, \quad k = \overline{1, n} \tag{2}$$

where $A_{k,k+1}$ is the compatibility matrix between the stresses and strains of layer k and the stresses and strains of layer k+1. It has the form:

$$A_{k,k+1} = \begin{pmatrix} \frac{3G_k - G_{k+1}}{2G_k} & \frac{3G_k - 3G_{k+1}}{2G_k} \cdot y_k^2 \\ \frac{G_{k+1} - G_k}{2G_k y_k^2} & \frac{3G_{k+1} - G_k}{2G_k} \end{pmatrix} \tag{3}$$

where G_k is the shear modulus of the material of layer k.

The inertia and stiffness parameters for the bar's section are:

$$\langle EI_1 \rangle = 2b \sum_{k=1}^{n} E_k \left(\frac{y_k^3 - y_{k-1}^3}{3} \quad \frac{y_k^5 - y_{k-1}^5}{5} \right) B_k \tag{4}$$

$$\langle EI_2 \rangle = 2b \sum_{k=1}^{n} E_k \frac{y_k^3 - y_{k-1}^3}{3} \tag{5}$$

$$\langle GA \rangle = 2b \sum_{k=1}^{n} G_k (y_k - y_{k-1} \quad y_k^3 - y_{k-1}^3) B_k \tag{6}$$

$$\langle \rho A \rangle = 2b \sum_{k=1}^{n} \rho_k (y_k - y_{k-1}) \tag{7}$$

$$\langle \rho I_1 \rangle = 2b \sum_{k=1}^{n} \rho_k \left(\frac{y_k^3 - y_{k-1}^3}{3} \quad \frac{y_k^5 - y_{k-1}^5}{5} \right) B_k \tag{8}$$

$$\langle \rho I_2 \rangle = 2b \sum_{k=1}^{n} \rho_k \frac{y_k^3 - y_{k-1}^3}{3} \tag{9}$$

where ρ_k is the volume density of layer k.

Let us denote:

- cross-sectional area:

$$A = bh \tag{10}$$

- geometric moment of inertia:

$$I = \frac{bh^3}{12} \tag{11}$$

- medium average density:

$$\rho = \frac{\langle \rho A \rangle}{A} = \frac{2}{h} \sum_{k=1}^{n} \rho_k (y_k - y_{k-1}) \tag{12}$$

- medium average rotation:

$$\theta = -\frac{1}{\langle EI_2 \rangle} \iint_{(S)} yE(x,y) u_x(x,y,t) dS \tag{13}$$

- medium average shear modulus:

$$G = \frac{1}{A} \iint_{(S)} G(x,y) dS = \frac{2}{h} \sum_{k=1}^{n} G_k (y_k - y_{k-1}) \tag{14}$$

- medium average elastic modulus:

$$E = \frac{\langle EI_2 \rangle}{I} \tag{15}$$

With these notations, the equations of motion characterizing the transverse vibrations of the beam are:

$$\rho A \ddot{w} - KGA \left(\frac{\partial^2 w}{\partial x^2} - \frac{\partial \theta}{\partial x} \right) = p_y \tag{16}$$

respectively

$$\rho I \left(K_1 \frac{\partial \ddot{w}}{\partial x} - K_2 \ddot{\theta} \right) + KGA \left(\frac{\partial w}{\partial x} - \theta \right) + \\ + EI \frac{\partial^2 \theta}{\partial x^2} = 0 \tag{17}$$

where:

$$K = \frac{\langle GA \rangle}{GA} \cdot \frac{\langle EI_2 \rangle}{\langle EI_1 \rangle} \tag{18}$$

$$K_1 = \frac{\langle \rho I_1 \rangle \langle EI_2 \rangle - \langle \rho I_2 \rangle \langle EI_1 \rangle}{\rho I \langle EI_1 \rangle} \tag{19}$$

$$K_2 = \frac{\langle \rho I_1 \rangle \langle EI_2 \rangle}{\rho I \langle EI_1 \rangle} \tag{20}$$

are coefficients that account for the nonuniformities of stresses within the section. In the case of a homogeneous beam:

$$K = \frac{5}{6}, \; K_1 = 0, \; K_2 = 1 \tag{21}$$

and the equations of motion are identical to those in the classical Timoshenko beam theory.

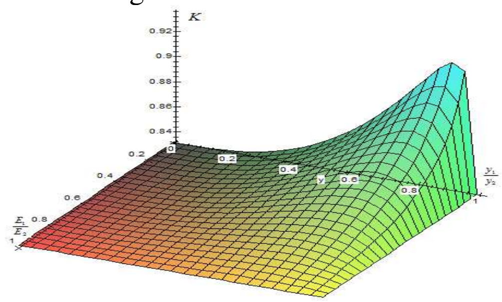
It should be noted that in the classical Timoshenko beam theory, a one and single nonuniformity coefficient for stresses is used. In the presented model, three coefficients are introduced to characterize the nonuniformities of stresses within the beam's section.

If coefficient K accounts for the variations in tangential stresses, coefficients K_1 and K_2 assess the variations in normal stresses. By specifying these coefficients, the math. models composite beams presented in [11] and [12] are obtained.

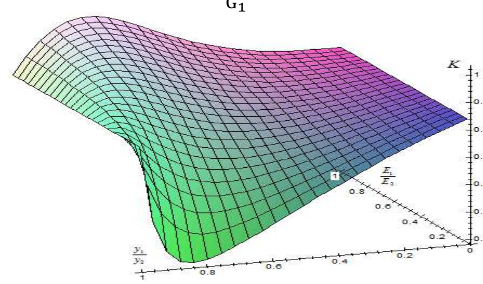
3. THE SANDWICH BEAM CASE

Consider a beam consisting of three layers: a middle layer and two outer layers, arranged symmetrically.

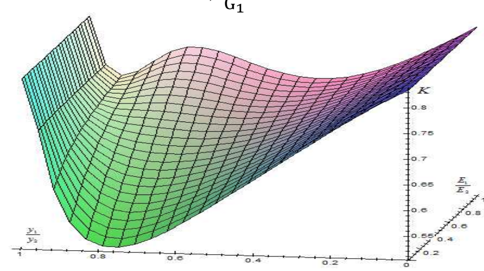
The variation of the nonuniformity coefficient K, as a function of the ratios of the elastic moduli and the thicknesses of the core and the faces, for different values of the ratio $\frac{G_2}{G_1}$ is shown in Figure 2.



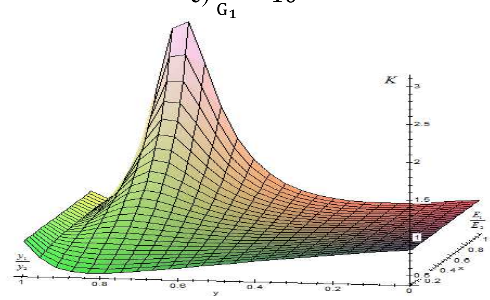
a) $\frac{G_2}{G_1} = 1$



b) $\frac{G_2}{G_1} = 5$

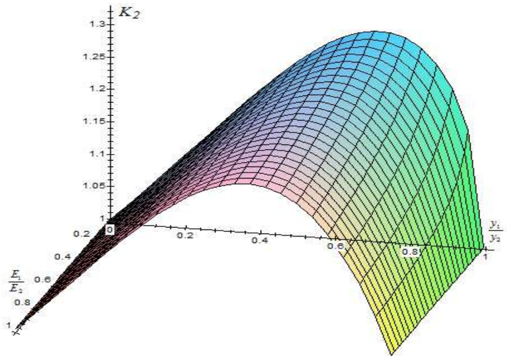


c) $\frac{G_2}{G_1} = 10$

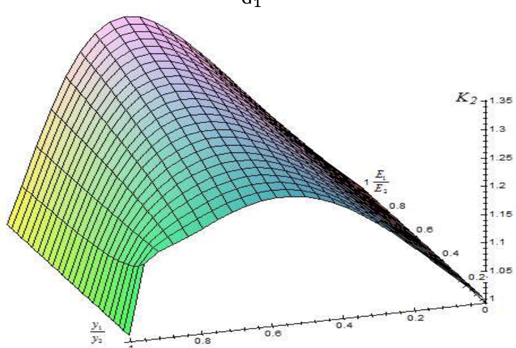


d) $\frac{G_2}{G_1} = 20$

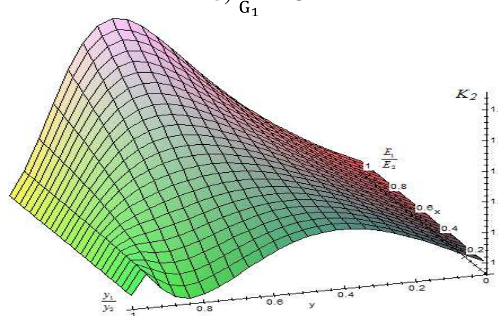
Fig. 2. Variation of the nonuniformity coefficient K for different values of the ratio $\frac{G_2}{G_1}$.



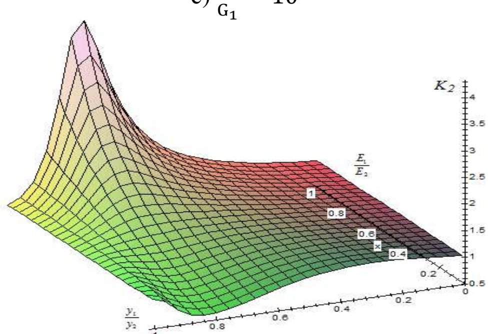
a) $\frac{G_2}{G_1} = 1$



b) $\frac{G_2}{G_1} = 5$



c) $\frac{G_2}{G_1} = 10$

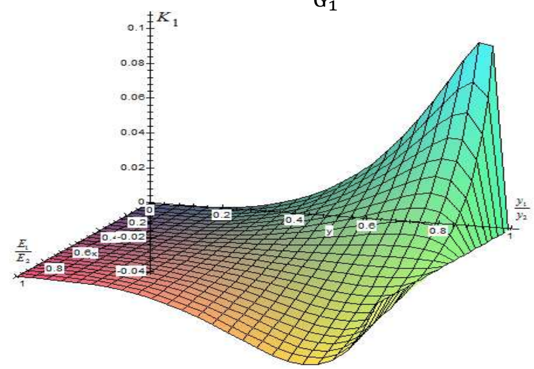


d) $\frac{G_2}{G_1} = 20$

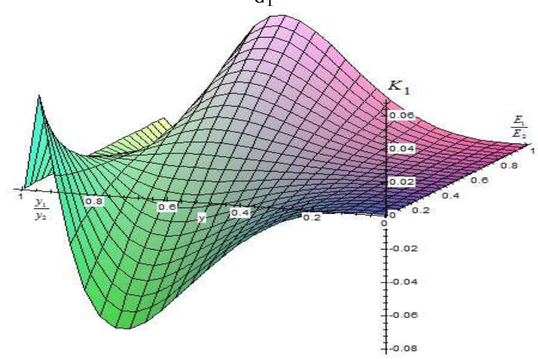
Fig. 3. Variation of the nonuniformity coefficient K_2 for different values of the ratio $\frac{G_2}{G_1}$.

Figure 3 shows the variation of the nonuniformity coefficient K_2 for $\frac{\rho_2}{\rho_1} = 2$ and different values of the ratio $\frac{G_2}{G_1}$.

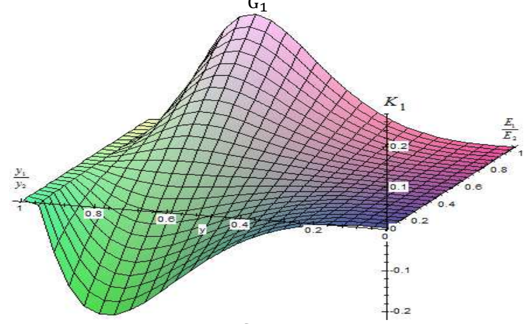
Figure 4 shows the variation of the nonuniformity coefficient K_2 for $\frac{\rho_2}{\rho_1} = 2$ and different values of the ratio $\frac{G_2}{G_1}$.



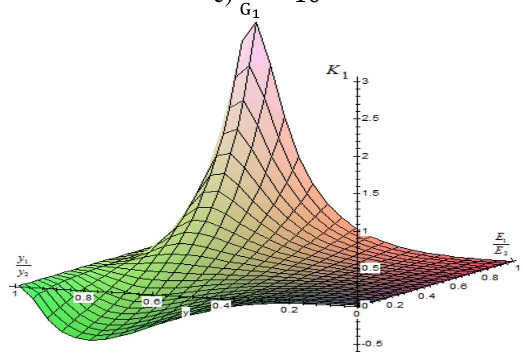
a) $\frac{G_2}{G_1} = 1$



b) $\frac{G_2}{G_1} = 5$



c) $\frac{G_2}{G_1} = 10$



d) $\frac{G_2}{G_1} = 20$

Fig. 4. Variation of the nonuniformity coefficient K_1 for different values of the ratio $\frac{G_2}{G_1}$.

The nonuniformity coefficient K depends only on the elastic properties and dimensions of the constituent layers, whereas the coefficients K_1 and K_2 also depend on the material volume densities. Coefficients K and K_2 have strictly positive values, while the coefficient K_1 can take negative values or be zero.

To ensure the rigidity of sandwich-type composite materials, the core material does not have to exhibit too much weak mechanical properties. That's why a maximum value of the ratio $\frac{G_2}{G_1} = 20$ comes as big enough for the analysis of real composite materials. Even further, watching Figures 2, 3, and 4, one can observe that the values of coefficients K, K_1, and K_2 corresponding to $\frac{G_2}{G_1}$ ratios of 1, 5, and 10 do not show significant variations. However, for a ratio of $\frac{G_2}{G_1} = 20$, the values of the three non-uniformity coefficients increase significantly. Thus, a $\frac{G_2}{G_1}$ ratio greater than 20 leads to excessively high non-uniformity coefficients and this kind of situation is not advisable in manufacturing sandwich composite materials.

4. CONCLUSIONS

The proposed mathematical model for studying the vibrations of multilayer composite beams generalizes the Timoshenko model by accounting for both nonuniformities in the section of tangential stresses and nonuniformities in normal stresses. Tangential and normal stresses are considered in all layers, satisfying all continuity conditions on the separation surfaces between layers, for both stress tensor components and strain tensor components. It is also considered that the stresses on the outer surfaces are zero. While the Timoshenko theory introduces a coefficient to account for nonuniformities in tangential stresses, the proposed mathematical model introduces three such coefficients, which also account for nonuniformities in normal stresses.

The proposed model, by varying the stresses and strains within the section, can be incorporated into theories such as PSDBT (Parabolic Shear Deformation Beam Theory).

The analysis of the graphical representations for the three nonuniformity coefficients and the ratios between the mass and stiffness characteristics of the beam, in the case of the sandwich beam, shows significant variations both with respect to the mass and elastic properties of the constituent layers and with respect to their relative thicknesses.

If the mass and elastic properties of the layer materials are very similar, the values of the nonuniformity coefficients correspond to those for homogeneous materials. For $K = \frac{5}{6}$, $K_1 = 0$, $K_2 = 1$, the mathematical model coincides with the classical Timoshenko model. Similar results are obtained when the ratio of the layer thicknesses $\frac{y_1}{y_2}$ approaches zero or is very close to one. The largest variations for both the nonuniformity coefficients and the ratios between the mass and stiffness characteristics of the beam occur for values of the ratio $\frac{y_1}{y_2}$ within the range 0.8 to 0.9.

5. REFERENCES

[1] Reddy, J.N., *A simple higher-order theory for laminated composite plates*, ASME-Journal of Applied Mechanics, 51(4), pp. 745-752, 1984.

[2] Touratier, M., *An efficient standard plate theory*, International Journal of Engineering Science, 29(8), pp. 901-916, 1991.

[3] Soldatos, K.P., *A transverse shear deformation theory for homogeneous monoclinic plates*, Acta Mechanica, 94(3–4), pp. 195–220, 1992.

[4] Karama, M., Afaq, K.S., Mistou, S., *Mechanical behavior of laminated composite beam by the new multi-layered laminated composite structures model with transverse shear stress continuity*, International Journal of Solids and Structures, 40(6), pp. 1525-1546, 2003.

[5] Aydogdu, M., *A new shear deformation theory for laminated composite plates, Composite Structures*, 89, pp. 94-101, 2009.

[6] Simşek M., *Fundamental frequency analysis of functionally graded beams by using different higher-order beam theories*, Nuclear Engineering and Design, 240, pp. 697-705, 2010.

[7] Guenfound, H., Ziou, H., Himeur M., Guenfound M., *Analyses of a composite functionally graded material beam with a new transverse shear deformation function*, J. Appl. Eng. Sci. Technol., 2(2), pp. 105-113, 2016.

[8] Savithri, S., Varadan, T.K., *Laminated Plates Under Uniformely Distributed and Concentrated Louds*, ASME Jour. Appl. Mech., pp. 211-214, 1992.

[9] Dong, S.B., Chun, C.K., *Shear Constitutive Relations for Laminated Anisotropic Shells and Plates (Part I+II)*, ASME Jour. Appl. Mech., part I, pp. 372-379, 1992.

[10] Xavier, B.P., Chew, C.H., Lee, K.H., *An improved zig-zag model for the variation of soft-cared unsymmetric sandwich beams*, Compos. Engng., 4(5), pp. 549-563, 1994.

[11] Demir, E., Callioglu, H., Sayer, M., *Free vibration of symmetric FG sandwich Timoshenko beam with simply supported edges*, Indian Journal of Engineering & Material Science, 20, pp. 515-521, 2013.

[12] Cunedioglu, Y., *Free vibration analysis of edge cracked symmetric functionally graded sandwich beams*, Structural Engineering and Mechanics, 56(6), pp. 1003-1020, 2015.

Introducerea coeficienților de neuniformitate la studiul vibrațiilor barelor compozite multistrat

Barele compozite de tip sandwich sunt utilizate pe scară largă în structuri mecanice care conțin subansambluri realizate din materiale compozite. Este un fapt verificat că orice tip de bară sandwich prezintă un anumit număr de neuniformități pe secțiunea sa transversală. Neuniformitățile trebuie evaluate și cuantificate la construirea modelului matematic pentru vibrațiile unei astfel de bare. Introducem trei coeficienți de neuniformitate în formularea ecuațiilor de mişcare pentru o bară sandwich cu mai multe straturi care vibrează. A fost considerat doar cazul unei bare cu simetrie geometrică şi masică. Deşi acest caz ar putea fi considerat unul foarte particular, realizarea este că toate considerațiile pe care le-am făcut respectă pe deplin și se potrivesc perfect cu toate cerințele teoriei PSDBT.

Sabin RIZESCU, PhD, Associate Professor, University of Craiova, Faculty of Mechanics, Department of Applied Mechanics and Civil Engineering, sabin.rizescu@edu.ucv.ro, 107, Calea Bucuresti, Craiova Romania.

Alexandru BOLCU, PhD, Assistant Professor, University of Craiova, Faculty of Mechanics, Department of Applied Mechanics and Civil Engineering, alexandru.bolcu@edu.ucv.ro, 107, Calea Bucuresti, Craiova, Romania.

Dumitru BOLCU, PhD, Professor, University of Craiova, Faculty of Mechanics, Department of Applied Mechanics and Civil Engineering, dumitru.bolcu@edu.ucv.ro, 107, Calea Bucuresti, Craiova, Romania.

Marius Marinel STANESCU, PhD, Professor, University of Craiova, Faculty of Mechanics, Department of Applied Mechanics and Civil Engineering, marius.stanescu@edu.ucv.ro, 107, Calea Bucuresti, Craiova, Romania.