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TEMPERATURE DISTRIBUTIONS IN FUNCTIONALLY GRADED DISKS BY PSEUDOSPECTRAL CHEBYSHEV METHOD

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Abstract: *In this study, we investigate the nonlinear heat conduction in a functionally graded disk subjected to various thermal loads, employing the pseudospectral Chebyshev method. The thermal material properties of the disk are assumed to vary radially according to the Voigt homogenization scheme. Due to the temperature sensitivity of the material, the thermal conductivity is modeled as temperature-dependent. The inner surface of the disk is subjected to a constant base temperature, while the outer surface is governed by both Dirichlet and Neumann boundary conditions. Additionally, the influence of the convection coefficient is considered to simulate different operating conditions. These scenarios result in nonlinear differential equations, which are challenging for conventional methods to solve. The nonlinear temperature distributions in the functionally graded disk under thermo-mechanical loads are determined and graphically illustrated.*

Key words: *Nonlinear temperature distribution, Pseudospectral Chebyshev method, Voigt homogenization scheme, Functionally graded materials, Disk.*

1. INTRODUCTION

Functionally graded materials (FGMs) are composite materials characterized by a gradual change in composition of its constituents, such as metallic and ceramic, from one surface to another, leading to a continuous variation in material properties. These materials are intentionally engineered to exhibit tailored properties suitable for specific applications. One of the most common application areas of FGMs is disks. As indicated in the literature [1], disks are utilized in various practical engineering applications, including steam and gas turbine rotors, turbo generators, internal combustion engines, ship propellers, turbojet engines, and centrifugal compressors. The expanding adoption of functionally graded structural elements demands a thorough understanding of heat conduction, particularly in nonlinear contexts.

Kosedag, analysed stress and displacement of a rotating FGM annular disk. In the analysis the temperature and thickness kept constant, material properties are considered varying along the radial coordinate. The effect of material properties on stress, strain and displacement distribution is examined for different angular speed. Also, the effect of the inhomogeneity parameter on the critical angular velocity at which yielding started is obtained [2]. Korkkheili and Naghdabadi [3] introduce a semi-analytical solution for thermoelasticity, addressing both hollow and solid rotating axisymmetric disks made from functionally graded materials. Continuity between adjacent sub-domains ensured by using virtual sub-domains and result are presented in terms of stress, strain, and displacement components along the radius, influenced by centrifugal force and thermal loading. They compared their solution with finite element analysis from existing literature. Furthermore, they delve into the impact of radial gradation in constitutive components on stress, strain, and displacement for both centrifugal force and uniform thermal loadings in functionally graded disks.

Chiba derived the mean and variance of temperature within a functionally graded annular disc featuring spatially random heat transfer

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coefficients (HTCs) on its upper and lower surfaces. The stochastic temperature field within the disc is analyzed by first-order perturbation method and Vodicka's method [4]. Analytical stress analysis on annular rotating discs composed of functionally graded materials (FGMs) under uniform thermal loads at various temperature obtained by Turkmen [5]. In the analayses the elasticity modulus and thermal expansion coefficient were presumed to vary radially following power law functions and Poisson's ratio kept constant. An analytical thermoelastic investigation is done by Callioglu [6] for FG disks by utilizing infinitesimal deformation theory of elasticity and power law distribution for functional gradation. The study delves into examining the influence of radial gradation on stress and displacement components within the FG disc. Various loading conditions, including internal pressure, external pressure, centrifugal force, and steady-state temperature, are considered in the analysis.

The theoretical analysis of a transient thermoelastic problem of FG solid circular disk is studied by Noda et al. [7]. It is assumed that the disk is subjected to uniform heat supply from its outer surface while being cooled from both the upper and lower flat surfaces. To ensure the consistency of material position dependency, the functionally graded disk is composed of multiple thin circular layers. Tutuncu and Temel [8] solved a variable coefficient differential equation to obtain thermoelastic behavior of FG rotating disks under nonuniform changes in temperature. These equations stem from spatial variations in thermomechanical properties, thickness, and temperature changes. Closed-form solutions for such equations are only achievable for specific forms of grading functions. They aimed to point up the versatility of the complementary functions method as a unified meshless approach and demonstrate its applicability to functionally graded disks with arbitrary thermomechanical properties and variable thickness. Elastic stress analysis of a variable thickness hollow disk fabricated from FGMs experiencing a linearly increasing temperature distribution is handled by Kursun and Topcu [9].

Various cases of thickness profiles are investigated including linear, constant, and convergent-divergent hyperbolic types. By examining these different scenarios, they aim to gain insights into the effects of thickness profiles on the thermal and mechanical responses of FG hollow disks. Gonczi and Ecsedi investigated a thermoelastic boundary value problem concerning a hollow circular FG disk with an arbitrary gradient. The steady-state temperature distribution is modeled as a function of the radial coordinate, with prescribed temperatures set at the inner and outer cylindrical boundary surfaces. To solve this two-point boundary value problem efficiently, it is transformed into an initial value problem. Furthermore, to ensure the accuracy of the numerical results, they are validated against an analytical solution derived from the same two-point boundary value problem [10]. Using metal matrix composites with carbon nanotubes (CNTs) and ceramics in a functionally graded manner Daviran et al. obtained FG disk and analysed performance of its heat conduction property. They stated that the combination of these materials likely offers unique thermal properties suitable for applications where thermal shock resistance is crucial. Also, it is indicated in study that implementing the analysis through the differential quadrature method in Matlab suggests a sophisticated approach [11].

In the present study, the nonlinear temperature distribution in a FG disk have been investigated. The resulting temperature distribution is estimated by solving the conductive-convective heat equation. Conductivity is varying along with the radial direction in compliance with Voigt rule and temperature for both constituents in grading. The nonlinear heat conduction problem is transformed into a system of nonlinear equations using the pseudospectral Chebyshev method. This non-linear system of equations can also be easily solved by any iterative method.

2. GENERAL DEFINITIONS OF THE PROBLEM

A functionally graded disk with inner radius (r_i) and outer diameter (r_o) is considered. The disk's inner radius remains at a base temperature, T_b , while its outer part is surrounded by an environment with temperature T_a. Given that material properties change according to Voigt's sheme at any location along the disk's radius. Among the properties thermal conduction coefficient alter with temperature. The assumption is made that the inner wall consists of pure ceramic (ZrO_2), the outer wall is pure metal ($Ti-6Al-4V$), and the region between these two walls is comprised of varying volumes of ceramic and metal. The temperature dependent heat conduction coefficient is taken from experimental data for ceramic and metal constituents as below [12]:

$$ZrO_2:\ k_C = 2.072 - 3.656x10^{-4}\,T + 4.347x10^{-7}\,T^2\ \ W/(mK) \quad (1)$$

$$Ti-6Al-4V:\ \ k_m = 1.1 + 0.017\,T\ \ \ W/(mK) \quad (2)$$

Grading in the radial direction is handled with the help of the Voigt homogenization scheme. Accordingly, k determined as below:

$$k(r,T) = k_c\,V_c + k_m\,V_m \quad (3)$$

with

$$V_m = \left(\frac{r-r_i}{r_o-r_i}\right)^n,\ \ V_c = 1-\mathrm{V_m} \quad (4)$$

Here, V_c and V_m volume fraction rates for ceramic and metal, respectively and n is inhomegeneity parameter. The exponent n serves as a parameter for regulating the density of material within both the inner and outer walls of the functionally graded disk. The configuration of the FG disk is given in Figure 1.

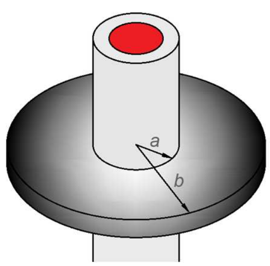


Fig. 1. Illustration of the functionally graded disk

Considering temperature-dependent heat conduction, the energy balance equation for functionally graded disk can be written as [13],

$$t\frac{d}{dr}\left[k(r,T)r\frac{dT}{dr}\right] - 2hr(T-T_a) = 0 \quad (5)$$

t refers the thickness of the disk. In the study, two temperature boundary conditions are considered. The initial condition pertains to the Neumann boundary, concerning temperature change rate, where the heat flux is specified at the body's boundary [13],

$$T = T_b\ \text{ at }\ r = r_i\ \text{ and }\ \frac{dT}{dr} = 0 \text{ at } r = r_o. \quad (6)$$

The subsequent condition, the Dirichlet boundary, sets the temperature directly at the problem domain's boundary [13],

$$T = T_b\ \text{ at }\ r = r_i \text{ and } T = T_{tip}\ \text{ at }\ r = r_o \quad (7)$$

where T_{tip} denotes the temperature at the outer radius of the disk. Since the outer radius of the disk is exposed to the ambient environment, T_{tip} equals T_a. The study has utilized the following non-dimensional parameters due to their advantageous role in simplifying the problem [13].

$$\theta = \frac{T}{T_b},\ \theta_a = \frac{T_a}{T_b},\ \theta_{tip} = \frac{T_{tip}}{T_b},\ \eta = \frac{r}{r_i},\ R = \frac{r_o}{r_i} \quad (8)$$

By incorporating the nondimensional terms into the heat transfer equation (6), we obtain:

$$\frac{d^2\theta}{d\eta^2} + \left[\frac{1}{k}\frac{dk}{d\eta} + \frac{1}{\eta}\right]\frac{d\theta}{d\eta} - \frac{N}{k}\theta = -\frac{N}{k}\theta_a \quad (9)$$

here, $N = 2hr_i^2/t$.

3. PSEUDOSPECTRAL CHEBYSHEV METHOD

The nonlinear temperature distribution for disk is obtained by the help of pseudospectral Chebyshev method, which is based on the first type Chebyshev polynomial approximation. The method utilizes Chebyshev-Gauss-Lobatto scheme to distribute points for related coordinate. The scheme offers more frequent points near the border and relatively sparser points near the border that increases accuracy of results. The points are distributed in the radial direction by using the function below [14].

$$r_j = \cos(j\pi/N),\ \ j = 0,1,\dots,N \quad (10)$$

Chebyshev-Gauss-Lobatto achieves this characteristic by evenly spacing the points along both the semicircle and their projections onto the horizontal axis. Differential Matrix (D_N) of size $(N+1)\times(N+1)$ in the range $[-1,1]$ in the Lagrange interpolation polynomial is created by using these determined points. More explanation can be found in Trefethen [14] for creating the Chebyshev differential matrix. Once the matrix is created, it is quite practical to take its derivatives. So that, multiplying the displacement vector (u) by the differential matrix from the left once yields the first derivative ($u' = D_N u$), twice yields the second derivative ($u'' = D_N^2\, u$), thrice yields the third derivative ($u''' = D_N^3\, u$). Even though the Chebyshev polynomials are defined in the range $[-1,1]$, it can be transformed into an arbitrary range $[a, b]$ by simple variable substitution.

Using the Chebyshev differential matrix, we can readily transform the nonlinear energy balance equation (Eq. (5)) into a nonlinear system in the following manner.

$$S_\theta \theta_{i+1} = R_\theta(\eta) \quad (11)$$

here,

$$S_\theta = D^2 + \left[\frac{1}{k}\frac{dk}{d\eta} + \frac{1}{\eta}\right] D - \frac{N}{k}, \quad R_\theta(\eta) = -\frac{N}{k}\theta_a \quad (12)$$

The temperature values of the current iteration (θ_{i+1}) and the previous iteration (θ_i) are utilized in obtaining non-trivial solutions for system (11) by imposing boundary conditions (6-7) into the stiffness matrix (S_θ) with corresponding values in the right-hand side function (R_θ). Subsequently, the iterative solution of this nonlinear system (11) can be achieved any iteration method.

4. NUMERICAL RESULTS

The pseudospectral Chebyshev solution for nonlinear temperature distribution of FGM disk is obtained. The disk experiences two different thermal boundary conditions, specifically referred to as the Neumann and Dirichlet conditions. The heat conduction coefficient of the functionally graded disk varies with temperature and follows the Voigt homogenization method throughout the disk's radius. The interior of the disk consists entirely of pure ceramic material (ZrO_2), while the exterior of is composed of metal ($Ti - 6Al - 4V$), with its composition determined by specific volume ratios of metals and ceramics. It is assumed that the annular ratio of the disk is $R = 3$ and its thickness is $t = 0.004$ m. The computations are carried out for radius of $r_i = 0.02\, m, r_o = 0.06\, m,$ the ambiant temperature of $T_a = 300\, K$, the temperature at the inner radius $T_b = 573\, K$, the temperature of the tip of disk. $T_{tip} = 373\, K$. Pseudospectral simulations are performed by taking $N = 12$ collocation points. Prior to delving into non-linear temperature distributions, it's beneficial to examine how temperature impacts the heat conduction coefficient and the difference between temperature dependent – temperature independent solutions. To this end, the variation of dimensionless k with temperature and radius are plotted, as depicted in Figures 2 for Neumann boundary condition, and Figure 3 for Dirichlet boundary condition with various inhomogeneity parameters ($m = 0.5, 2, 5$). The inhomogeneity parameter determines the change of the heat conduction coefficient in the radial direction. In accordance with the selected material pair, the heat conduction coeffient decreases in the radial direction from the inside to the outside as it is seen in Figure 2-3(a). It is observed that increasing n values result in lower k values (Figure 2a-3a) and faster change due to temperature change (Figure 2-3(b)).

Secondly, temperature-dependent (TD) and temperature-independent (TID) solutions are compared depending on the heat convection coefficient. Figure 4 belongs to the Neumann boundary condition, and Figure 5 belongs to the Dirichlet boundary condition results, respectively.

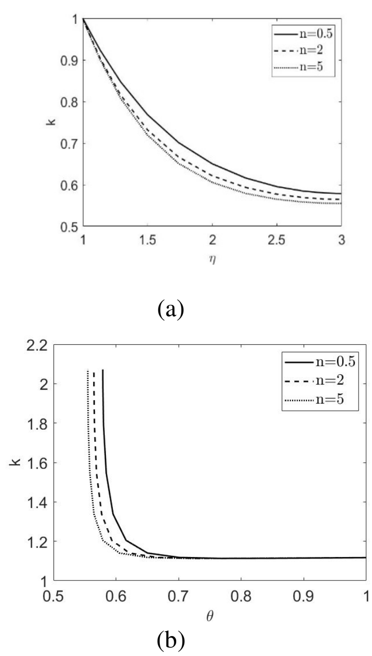
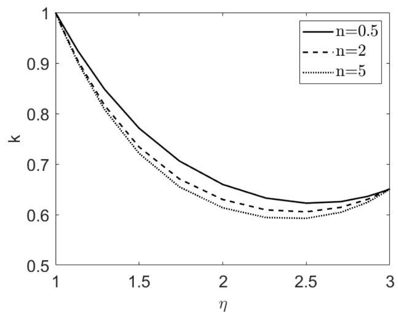
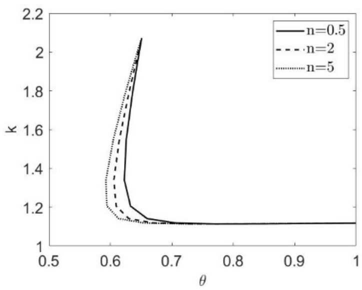


Fig. 2 Variation of heat conduction coefficient depending on radius (a) and temperature (b) for Neumann boundary condition

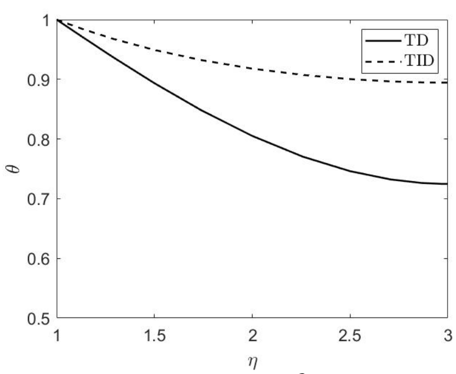


(a)

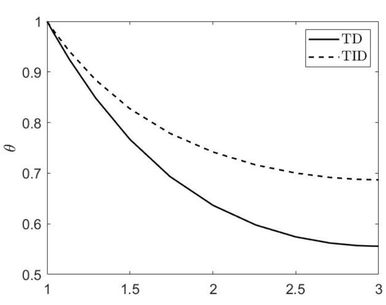


(b)

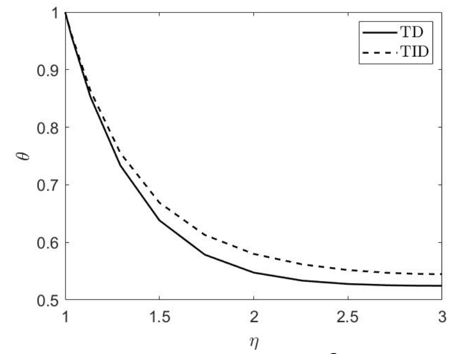
Fig. 3 Variation of heat conduction coefficient depending on radius (a) and temperature (b) for Dirichlet boundary condition



(a) $h = 5\ W/m^2\,K$



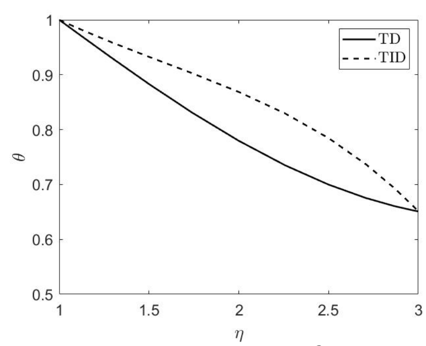
(b) $h = 25\ W/m^2\,K$



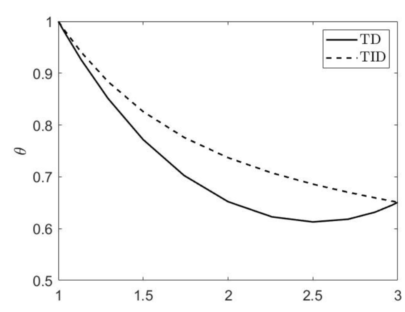
(c) $h = 100\ W/m^2\,K$

Fig. 4 Comparition of temperature dependent and independet solutions to temperature distribution for (a) $h = 5\ W/m^2\,K$ (b) $h = 25\ W/m^2\,K$ and (c) $h = 100\ W/m^2\,K$ for Neumann boundary condition

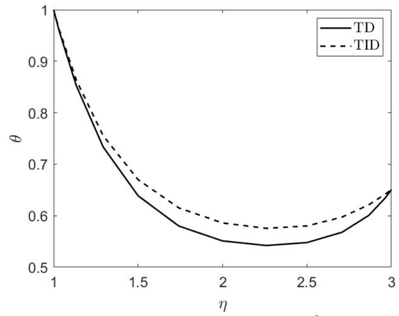
In the case of low convection coefficient ($h = 5\ W/m^2\,K$), it is observed that there is a significant difference between the results for both boundary conditions in Figure 4-5. Accordingly, it is important to make temperature-dependent solutions in low convection environments. At sufficiently high convection values ($h = 100\ W/m^2\,K$), the difference between the results decreases considerably.



(a) $h = 5\ W/m^2\,K$



(b) $h = 5\ W/m^2\,K$

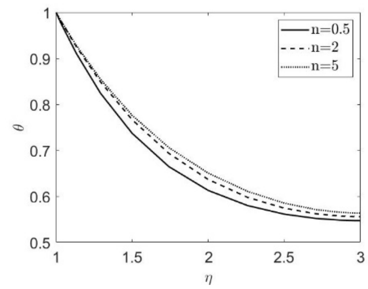


(c) $h = 100\ W/m^2\,K$

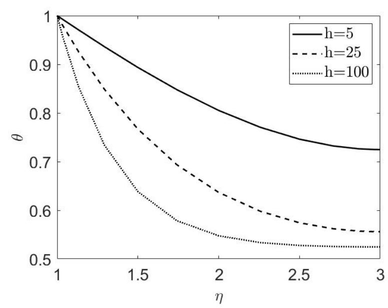
Fig. 5 Comparition of temperature dependent and independet solutions to temperature distribution for (a) $h = 5\ W/m^2\,K$ (b) $h = 25\ W/m^2\,K$ and (c) $h = 100\ W/m^2\,K$ for Dirichlet boundary condition

Figure 6,7 illustrates the effect of the inhomogeneity parameter and heat convection coefficient on the nonlinear temperature distribution considering Neumann and Dirichlet boundary conditions. In both boundary conditions, the temperature decreases from the inner wall to the outer wall. According to the results obtained in Fig. 1-2(A), heat conduction worsens as the heat conduction coefficient decreases with increasing n, and therefore the temperature (θ) value increases with increasing n in Fig. 6(a) and 7(a). As required by the derived expression in the Neumann boundary condition, the temperature value on the outer wall is different for each n. In the Dirichlet boundary condition, temperatures are the Same on the outer wall for each n value. The lowest temperature values along the disk wall occur when $n = 0.5$. This shows that lower temperatures are achieved in the ceramic-rich disc.

Effects of exposure to ambient convective conditions that can be airstream or fluid sourced are also examined for the FG disk. To illustrate the impact of the physical environment, the body will be placed in three different conditions: still air $(h = 5\ W/m^2\,K)$, forced convective air $(h = 25\ W/m^2\,K)$, and immersed in fluid $(h = 100\ W/m^2\,K)$. During these computations, the inhomogeneity coefficient is kept constant as $n = 2$. In the case of results obtained using both Neumann and Dirichlet boundary conditions (Figures 6(b), 7(b)), considerable change in the values of temperature from free to forced convection is observed. While the temperature distribution is almost linear for $h = 5\ W/m^2\,K$, very different temperature profiles are observed with increasing h. The lowest temperature along the radial direction of the FG disc is reached when $h = 100\ W/m^2\,K$. This also means that there is the largest temperature difference between the inner and outer walls. Accordingly, while the lowest temperature was obtained as $\theta = 0.52$ at $r = 3$ m in the Neumann boundary condition, it is obtained as $\theta = 0.54$ around $r = 2.25$ m in the Dirichlet boundary condition. It can be deduced that the convection coefficient of the environment can be increased to keep the tip temperature of the disk low.

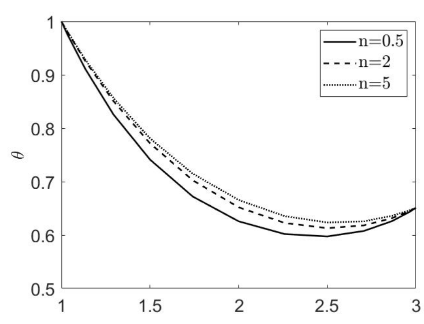


(a)

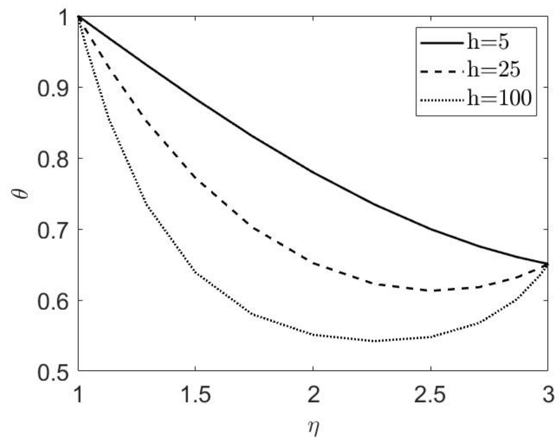


(b)

Fig. 6 Variation of dimensionless nonlinear temperature distribution depending on (a) inhohmegeneity coefficient and (b) heat convection coefficient for Neumann boundary condition



(a)



(b)

Fig. 7 Variation of dimensionless nonlinear temperature distribution depending on (a) inhohmegeneity coefficient and (b) heat convection coefficient for Dirichlet boundary condition

5. CONCLUSION

A numerical solution for the nonlinear temperature distribution analysis of a functionally graded disk has been obtained by efficiently solving the problem using the pseudospectral Chebyshev method. It is assumed that the disk consists of a ceramic/metal mixture in a Voigt homogenization scheme with varying ratio through the radial direction. To make a more physically realistic analysis, the heat conduction coefficient is also taken as variable depending on the temperature. The temperature distribution is obtained using Neumann and Dirichlet boundary conditions separately. A combination of the Pseudospectral Chebyshev Method (PCM) and the fixed-point iteration method is utilized for performing numerical analysis.

A parametric study was undertaken, and the following significant observations can be outlined:

- By choosing a practical graded material pair and homogenization scheme (like Voigt), one could regulate simultaneously the heat conduction coefficient with temperature and in the radial direction.
- In case of low convection coefficient, it is important to make a temperature-dependent solution.
- A variation in the inhomogeneity parameter (n) has a significant effect on the heat conduction coefficient value and thus temperature distribution. When the Dirichlet boundary condition is applied, higher temperatures are achieved for identical values of n, especially at the tip of the disk.
- The nonlinear temperature field in the FG disk due to the thermal loading has a strong dependency on the convective properties of the environment. At higher convection coefficients (e.g., $h = 100\,W/m^2\,K$), the temperatures on the disk decrease relatively faster as more heat is absorbed into the environment.

6. REFERENCES

[1] M. Bayat, M. Saleem, B. B. Sahari, A. M. S. Hamouda, and E. Mahdi, *"Analysis of functionally graded rotating disks with variable thickness,"* Mechanics Research Communications, 35(5), pp. 283–309, 2008.

[2] A. Kösedağ, *"Fonksiyonel Derecelendirilmiş Malzemeden Yapılmış Dönen, İçi Boş Bir Diskin Gerilme ve Deplasman Dağılımının Belirlenmesi,"* M.S. thesis, Gazi University, Ankara, Turkey, 2007.

[3] S. A. H. Kordkheili and R. Naghdabadi, *"Thermoelastic analysis of a functionally graded rotating disk,"* Composite Structures, 79(4), 2007.

[4] R. Chiba, *"Stochastic heat conduction analysis of a functionally graded annular disc with spatially random heat transfer coefficients,"* Applied Mathematical Modelling, 33(1), 2009.

[5] D. Türkmen, *"Functionally graded discs subjected to uniform temperature loading,"* Science and Engineering of Composite Materials, 17(1), 2010.

[6] H. Çallioglu, *"Stress analysis in a functionally graded disc under mechanical loads and a steady state temperature distribution,"* Sadhana - Academy Proceedings in Engineering Sciences, 36(1), 2011.

[7] N. Noda, Y. Ootao, and Y. Tanigawa, *"Transient thermoelastic analysis for a functionally graded circular disk with*

piecewise power law," Journal of Theoretical and Applied Mechanics, 50(3), 2012.

[8] N. Tutuncu and B. Temel, *"An efficient unified method for thermoelastic analysis of functionally graded rotating disks of variable thickness,"* Mechanics of Advanced Materials and Structures, 20(1), 2013.

[9] A. Kurşun and M. Topçu, *"Thermal stress analysis of functionally graded disc with variable thickness due to linearly increasing temperature load,"* Arabian Journal for Science and Engineering, 38(12), 2013.

[10] D. Gönczi and I. Ecsedi, *"Thermoelastic analysis of functionally graded hollow circular disc,"* Archive of Mechanical Engineering, 62(1), 2015.

[11] S. Daviran, S. Mohasseb, A. Kasaeian, and O. Mahian, *"Differential quadrature method for thermal shock analysis of CNT reinforced metal-ceramic functionally graded disc,"* Composite Structures, 161, 2017.

[12] A. Yıldırım, D. Yarımpabuç, V. Arikan, M. Eker, and K. Celebi, *"Nonlinear thermal stress analysis of functionally graded spherical pressure vessels,"* International Journal of Pressure Vessels and Piping, 200, p. 104830, 2022.

[13] S. Deka, A. Mallick, P. P. Behera, and P. Thamburaja, *"Thermal stresses in a functionally graded rotating disk: An approximate closed form solution,"* Journal of Thermal Stresses, 44(1), 2020.

[14] L. N. Trefethen, *Spectral Methods in Matlab*, Philadelphia, PA, USA: SIAM, 2000.

DISTRIBUȚII DE TEMPERATURĂ ÎN DISCURI CLASIFICATE FUNCȚIONAL PRIN METODA CHEBYSHEV PSEUDOSPECTRALĂ

Rezumat: *În acest studiu, investigăm conducerea neliniară a căldurii într-un disc gradat funcțional supus diferitelor sarcini termice, folosind metoda pseudospectrală Chebyshev. Se presupune că proprietățile materialului termic ale discului variază radial în conformitate cu schema de omogenizare Voigt. Datorită sensibilității la temperatură a materialului, conductivitatea termică este modelată ca fiind dependentă de temperatură. Suprafața interioară a discului este supusă unei temperaturi de bază constante, în timp ce suprafața exterioară este guvernată de condițiile la limită Dirichlet și Neumann. În plus, influența coeficientului de convecție este considerată a simula diferite condiții de funcționare. Aceste scenarii au ca rezultat ecuații diferențiale neliniare, care sunt dificil de rezolvat pentru metodele convenționale. Distribuțiile neliniare ale temperaturii în discul gradat funcțional sub sarcini termomecanice sunt determinate și ilustrate grafic.*

Cuvinte cheie: *Distribuția neliniară a temperaturii, Metoda Chebyshev pseudospectrală, Schema de omogenizare Voigt, Materiale clasificate funcțional, Disc.*

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