TECHNICAL UNIVERSITY OF CLUJ-NAPOCA

**ACTA TECHNICA NAPOCENSIS**

**Series: Applied Mathematics, Mechanics, and Engineering**
**Vol. 69, Issue II, June 2026**

# DTM APPLIED TO THE BEAM EQUATION WITH VARIABLE INERTIA: RECURRENCE, COEFFICIENTS, AND CONVERGENCE TO THE ANALYTICAL SOLUTION

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***Abstract:*** *This paper presents a comparative analysis between the exact analytical solution and the Differential Transformation Method (DTM) for studying the mechanical behavior of a cantilever beam with a circular planar cross-section whose diameter varies linearly along its length. The beam is loaded at the free end by a concentrated force, and the differential equation of the deformed neutral axis (Euler-Bernoulli model) has variable coefficients due to the position-dependent axial moment of inertia. The exact analytical solution is obtained through successive integration, leading to a closed-form expression for the beam deflection at the free end. The DTM transforms the differential equation into an algebraic recurrence relation for the coefficients of the Taylor series of the solution, allowing a systematic approximation without direct integration. Numerical results demonstrate the rapid convergence of the DTM: for 6 terms, the relative error of the deflection at the free end is 4.34%, and for 11 terms, the error decreases to 0.19%. An extended analysis over the entire domain shows that the maximum error occurs in the upper-middle region ($\bar{x} \approx 0.8$), and for M=11 terms, this error is below 0.95%. The exponential convergence and linear complexity of the algorithm recommend DTM as an efficient and accurate alternative for the analysis of beams with variable cross-section, especially in cases where exact analytical solutions become impossible to obtain.*

***Key words:*** *variable cross-section beam, Differential Transformation Method (DTM), exact analytical solution, numerical convergence, cantilever beam, Euler-Bernoulli.*

## 1. INTRODUCTION

Analysis of the mechanical behavior of beams with variable cross-section represents a topic of major interest in structural engineering, due to its extensive practical applications: from aerospace components (aircraft wings, turbine blades) and naval structures, to crane arms, transmission shafts, or bridge elements with variable gauge. [1] Unlike beams with constant cross-section, for which analytical solutions are well established [2], beams with variable cross-section require more advanced calculation methods, because the differential equations of the deformed neutral axis have variable coefficients.

In many engineering applications, the use of beams with variable cross-section allows for a more efficient distribution of material, reducing the total mass of the structure without compromising strength or stiffness. For example, in crane arms or wind turbine blades, the variation of the cross-section aims to adapt the stiffness to the bending moment diagram. However, this structural optimization requires the development of robust calculation methods, since analytical solutions quickly become complicated.

The present study aims at a comparative analysis between the exact analytical solution and a semi-analytical numerical method – the Differential Transformation Method (DTM) – applied to a cantilever beam with a circular planar cross-section, whose diameter varies linearly from the support to the free end.

The analyzed problem is defined by the following characteristics: cantilever beam, loaded at the free end by a concentrated force F; circular cross-section, with diameter decreasing linearly along the length L, according to the relation:

$$d_0 = 2 \cdot d_L \tag{1}$$

**Received**: 20.04.26; **Similarities**: 12.05.26: **Reviewed**:28.05. /27.05.26: **Accepted** 09.06.26.

and

$$d(x) = d_0 - \frac{(d_0 - d_L)}{L} \cdot x = d_0 \cdot \left(1 - \frac{x}{2 \cdot L}\right) \tag{2}$$

Relation (1) corresponds to the case where the diameter at the free end is half the diameter at the fixed end, a configuration commonly encountered in crane arms to balance the stress distribution.

Variable axial moment of inertia:

$$I_z(x) = \frac{\pi \cdot d(x)^4}{64} = \frac{\pi \cdot d_0^4}{64} \cdot \left(1 - \frac{x}{2 \cdot L}\right)^4 = I_0 \cdot \left(1 - \frac{x}{2 \cdot L}\right)^4 \tag{3}$$

with

$$I_0 = \frac{\pi \cdot d_0^4}{64} \tag{4}$$

For a clearer understanding of the scientific approach, the logical framework of the study is presented in Figure 1.

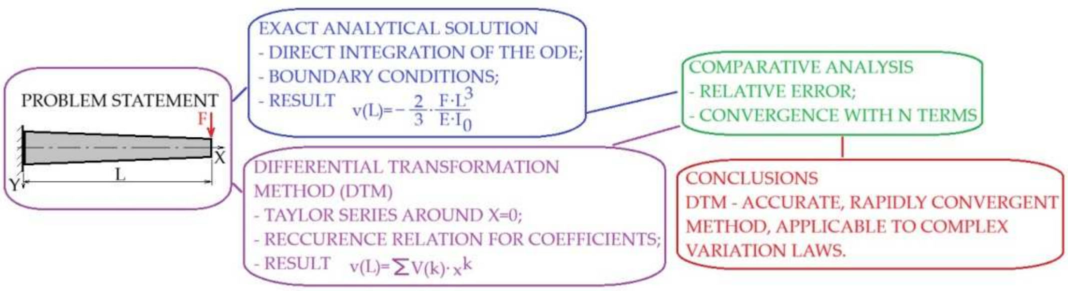


**Fig. 1.** Logical diagram of the methodological approach of the comparative study between the exact analytical solution and the Differential Transformation Method (DTM).

Although the exact analytical solution presented in Section 2 is valid for the particular case of linear diameter variation, in most engineering applications the cross-section variation law is not linear (e.g., aerodynamic profiles, exponential or rational variations). In those cases, obtaining a closed-form analytical solution becomes impossible or extremely difficult. Thus, there is a clear need for accurate approximate methods that are easy to implement and have controllable convergence.

The fourth-order differential equation of the deformed neutral axis (Euler–Bernoulli model):

$$\frac{d^2}{dx^2}\left[E \cdot I_z(x) \cdot \frac{d^2 v}{dx^2}\right] = 0 \tag{5}$$

becomes, in this case, an equation with variable coefficients (fourth-degree polynomials in the denominator after the first integration), whose direct analytical integration is possible but laborious. The exact analytical solution, presented in the material, leads to a relatively complicated expression for the deflection v(x), and the value at the free end is:

$$v(L) = -\frac{2}{3} \cdot \frac{F \cdot L^3}{E \cdot I_0} \cong -0.6667 \cdot \frac{F \cdot L^3}{E \cdot I_0} \tag{6}$$

Although the exact analytical solution exists, situations frequently arise in engineering practice where: the geometric shape of the beam does not allow for a simple analytical integration; the coefficients of the equation are more complicated functions than polynomials; a systematic, easily implementable numerical procedure is desired, while preserving the analytical character of the approximation.

The Differential Transformation Method (DTM) represents an efficient alternative. It transforms the differential equation into a recurrence relation for the coefficients of the Taylor series of the solution, allowing an approximate solution in the form of a power series to be obtained without requiring the direct integration of the equation. The advantages of DTM are: a systematic and algorithmic approach; controllable accuracy through the number of retained terms; applicability to a wide range of ordinary differential equations with variable coefficients. [3, 4]

Unlike approximation methods such as the perturbation method (which requires the existence of a small parameter) or the Galerkin method (which involves the choice of trial functions and integration over the domain), DTM transforms the differential equation into an algebraic recurrence relation, without requiring additional assumptions about the solution. [5–13] Moreover, DTM directly provides the coefficients of the Taylor series expansion of the solution, allowing error evaluation by comparing successive terms and offering a natural way to improve accuracy.

To evaluate the performance of the DTM, the following convergence criterion will be used: the relative error of the deflection at the free end, defined as:

$$\varepsilon = \frac{|v_{DTM}(L) - v_{exact}(L)|}{v_{exact}(L)} \tag{7}$$

It will be shown that by increasing the number of terms in the series, the error decreases monotonically, and after a modest number of terms (for example, 11), sufficient accuracy for most engineering applications is achieved ($\varepsilon < 0.5\%$).

The aim of this study is to: present the exact analytical solution for the problem of the cantilever beam with circular cross-section of linearly varying diameter; apply the DTM to the same problem, determining the coefficients of the series solution through a recurrence relation derived from the equilibrium conditions; compare the DTM solution (with a small number of terms, and then with a larger number of terms) with the exact solution, evaluating the approximation error; demonstrate the convergence of the DTM towards the exact solution as more terms are included in the series.

The presented material shows that after only 6 terms, DTM gives $v(L) = -0.697 \cdot \frac{F \cdot L^3}{E \cdot I_0}$ (relative deviation ~4.34%), and after 11 terms, $v(L) = -0.668 \cdot \frac{F \cdot L^3}{E \cdot I_0}$ (relative deviation ~0.19%), confirming the rapid convergence.

It should be noted that the present study is limited to the particular case of a beam with circular cross-section and linear diameter variation, loaded with a concentrated force at the tip. The effects of shear force on the deformation, nor the dynamic or nonlinear behavior of the beam, are analyzed. Furthermore, DTM provides a solution in the form of a power series around the point x=0, which may lead to reduced accuracy near the free end for prematurely truncated series. Nevertheless, the obtained results indicate excellent convergence for the considered domain.

The paper is structured as follows: Section 2 presents the exact analytical solution, obtained by direct integration; Section 3 details the application of the DTM, including the systematic calculation of the transformed coefficients; Section 4 presents a comparative analysis of the numerical results for the deflection at the free end and for the displacement distribution along the beam length; Section 5 concludes on the efficiency of the DTM and suggests possible extensions of the study (for example, to other cross-section variation laws or to distributed loads).

The present study demonstrates that the Differential Transformation Method constitutes a powerful, accurate, and easy-to-implement tool for the analysis of beams with variable cross-section, offering a viable alternative to classical analytical methods, especially when exact solutions become cumbersome or impossible to obtain.

## 2. EXACT ANALYTICAL SOLUTION OF THE PROBLEM

Consider a cantilever beam of length L, fixed at the end x = 0 and free at the end x = L. The beam has a planar circular cross-section, with the diameter varying linearly along the length, according to relation (1).

The exact analytical solution follows five main steps: successive integration of the differential equation (5) to introduce the constants $C_1$ and $C_2$; determination of the constants $C_1$ and $C_2$ from the boundary conditions at the free end x = L (zero bending moment and shear force equal to F); expressing $v''(x)$ and integrating to obtain $v'(x)$, using the zero slope condition at the fixed end [$v'(0) = 0$]; integrating to obtain v(x), using the zero displacement condition at the fixed end [$v(0) = 0$]; evaluating the deflection at the free end x = L.

Successively integrating equation (5), we obtain:

$$\frac{d}{dx}\left[E \cdot I_z(x) \cdot \frac{d^2 v}{dx^2}\right] = C_1 \tag{8}$$

$$E \cdot I_z(x) \cdot \frac{d^2 v}{dx^2} = C_1 \cdot x + C_2 \qquad (9)$$

Relation (8) represents the expression of the shear force $T_y$, and relation (9) expresses the bending moment $M_z$. Thus, the constant $C_1$ represents the value of the shear force along the entire length of the beam (constant, since there is no distributed load), and the expression $C_1 \cdot x + C_2$ represents the bending moment, which varies linearly.

The two integration constants are determined by introducing the boundary conditions as follows:

At the free end of the beam (for x = L), the bending moment is zero.

$$M_z(L) = 0 \Longrightarrow E \cdot I_z(L) \cdot v''(L) = 0 \qquad (10)$$

and the shear force is equal to the concentrated force F

$$T_y(L) = F \;\Rightarrow\; \frac{d}{dx}[E \cdot I_z(x) \cdot v''(x)]\Big|_{x = L} = F \qquad (11)$$

Relations (8) and (9) can also be written in the following form:

$$\frac{d}{dx}[E \cdot I_z(x) \cdot v''(x)] = C_1 \qquad (12)$$

$$E \cdot I_z(x) \cdot v''(x) = C_1 \cdot x + C_2 \qquad (13)$$

Thus, from relation (8), identical to relation (12), it follows that:

$$C_1 = F \qquad (14)$$

Substituting into the expression of the moment, relation (9), which is identical to relation (13), we have

$$E \cdot I_z(L) \cdot v''(L) = C_1 \cdot L + C_2 = F \cdot L + C_2 = 0 \qquad (15)$$

from which we obtain

$$C_2 = -F \cdot L \qquad (16)$$

Therefore, by substituting the integration constants into relation (13), we obtain:

$$E \cdot I_z(x) \cdot v''(x) = F \cdot (x - L) \qquad (17)$$

from which it follows that:

$$v''(x) = \frac{F \cdot (x - L)}{E \cdot I_0 \cdot \left(1 - \frac{x}{2 \cdot L}\right)^4} \qquad (18)$$

Integrating for v'(x):

$$v'(x) = \int_0^x \frac{F \cdot (t - L)}{E \cdot I_0 \cdot \left(1 - \frac{t}{2 \cdot L}\right)^4} \cdot dt + v'(0) \qquad (19)$$

The fixed-end condition imposes v'(0) = 0. Thus:

$$v'(x) = \int_0^x \frac{F \cdot (t - L)}{E \cdot I_0 \cdot \left(1 - \frac{t}{2 \cdot L}\right)^4} \cdot dt \qquad (20)$$

To simplify the integration, it is observed that the denominator contains the expression $\left(1 - \frac{t}{2 \cdot L}\right)^4$. The following change of variable is introduced:

$$u = 1 - \frac{t}{2 \cdot L} \qquad (21)$$

This choice has the advantage of transforming the denominator into $u^4$, and the integration interval becomes $[u(0), u(x)] = [1, 1 - \frac{x}{2 \cdot L}]$.

We have:

$$u = 1 - \frac{t}{2 \cdot L} \Rightarrow du = -\frac{dt}{2 \cdot L} \Rightarrow dt = -2 \cdot L \cdot du \qquad (22)$$

The limits are as follows:

- for t = 0 we obtain

$$u(0) = 1 \qquad (23)$$

- for t = x we obtain

$$u(x) = 1 - \frac{x}{2 \cdot L} \qquad (24)$$

Also,

$$t - L = 2 \cdot L \cdot (1 - u) - L \Rightarrow t - L = L \cdot (1 - 2 \cdot u) \qquad (25)$$

Then

$$v'(x) = \frac{F}{E \cdot I_0} \cdot \int_{u=1}^{u=1-\frac{x}{2 \cdot L}} \frac{L \cdot (1 - 2 \cdot u)}{u^4} \cdot (-2 \cdot L) \cdot du = \frac{2 \cdot F \cdot L^2}{E \cdot I_0} \cdot \int_{1-\frac{x}{2 \cdot L}}^{1} \frac{1 - 2 \cdot u}{u^4} \cdot du \qquad (26)$$

Since

$$\frac{1 - 2 \cdot u}{u^4} = u^{-4} - 2 \cdot u^{-3} \qquad (27)$$

from relation (26) it follows that:

$$v'(x) = \frac{2 \cdot F \cdot L^2}{E \cdot I_0} \cdot \int_{1-\frac{x}{2 \cdot L}}^{1} (u^{-4} - 2 \cdot u^{-3}) \cdot du = \frac{2 \cdot F \cdot L^2}{E \cdot I_0} \cdot \left(-\frac{1}{3 \cdot u^3} + \frac{1}{u^2}\right)\Bigg|_{1-\frac{x}{2 \cdot L}}^{1} \tag{28}$$

Evaluating at the upper limit u = 1:

$$v'(x) = \frac{2 \cdot F \cdot L^2}{E \cdot I_0} \cdot \left(\frac{2}{3} + \frac{1}{3 \cdot \left(1 - \frac{x}{2 \cdot L}\right)^3} - \frac{1}{\left(1 - \frac{x}{2 \cdot L}\right)^2}\right) \tag{29}$$

To obtain v(x), we integrate v'(t) again from 0 to x. The same change of variable u = 1 - t/(2L) will be used, since the structure of the integral is similar.

$$v(x) = \int_0^x v'(t) \cdot dt = \frac{2 \cdot F \cdot L^2}{E \cdot I_0} \cdot \int_0^x \left(\frac{2}{3} + \frac{1}{3 \cdot \left(1 - \frac{t}{2 \cdot L}\right)^3} - \frac{1}{\left(1 - \frac{t}{2 \cdot L}\right)^2}\right) \cdot dt \xRightarrow{u=1-\frac{t}{2 \cdot L}} \tag{30}$$

$$v(x) = -\frac{4 \cdot FL^3}{E \cdot I_0} \cdot \int_{1-\frac{x}{2 \cdot L}}^{1} \left(\frac{2}{3} + \frac{1}{3 \cdot u^3} - \frac{1}{u^2}\right) \cdot du = -\frac{4 \cdot F \cdot L^3}{E \cdot I_0} \cdot \left(\frac{2 \cdot u}{3} - \frac{1}{6 \cdot u^2} + \frac{1}{u}\right)\Bigg|_{1-\frac{x}{2 \cdot L}}^{1} \tag{31}$$

It follows that the exact analytical solution is:

$$v(x) = \frac{4 \cdot F \cdot L^3}{E \cdot I_0} \cdot \left(\frac{3}{2} - \frac{2}{3} \cdot \left(1 - \frac{x}{2 \cdot L}\right) + \frac{1}{6 \cdot \left(1 - \frac{x}{2 \cdot L}\right)^2} - \frac{1}{1 - \frac{x}{2 \cdot L}}\right) \tag{32}$$

For x = L we obtain:

$$v(L) = \frac{4 \cdot F \cdot L^3}{E \cdot I_0} \cdot \left(\frac{3}{2} - \frac{1}{3} + \frac{2}{3} - 2\right) = -\frac{2}{3} \cdot \frac{F \cdot L^3}{E \cdot I_0} \cong -0.6667 \cdot \frac{F \cdot L^3}{E \cdot I_0} \tag{33}$$

Although the exact analytical expression was obtained without major difficulties due to the particular form of $I_z(x)$ (a fourth-degree polynomial in the denominator after the first integration), [14] in the case of more general variation laws (for example, d(x) varying exponentially or according to some rational function), the analytical integration becomes impossible or extremely laborious. This is the reason why in Section 3 we will apply the Differential Transformation Method (DTM), which provides a systematic approximate solution without requiring direct integration.

## 3. APPLICATION OF THE DTM

The Differential Transformation Method (DTM) represents a semi-analytical alternative to the direct integration of the differential equation. [15] Instead of integrating successively, we will expand the solution v(x) into a Taylor series around the point x = 0 and determine the series coefficients using a recurrence relation derived from the differential equation and the boundary conditions.

The steps for applying the DTM to this problem are as follows:

- expanding the function $f(x) = \left(1 - \frac{x}{2 \cdot L}\right)^4$ into a Taylor series [16] and determining the coefficients F(m);
- expressing the DTM coefficients of the product h(x)=f(x)·v″(x) in terms of the coefficients F(m) and V(k) [the coefficients of the solution v(x)];
- transforming the differential equation $\frac{d^2h}{dx^2} = 0$ into an algebraic condition: H(k)=0 for k≥2;
- determining H(0) and H(1) from the equilibrium conditions at the free end [$M_z(L)=0$, $T_y(L)=F$];
- solving the recurrence system to successively compute V(2),V(3),V(4),…;
- reconstructing the approximate solution $v(x) = \sum_{k=0}^{M} V(k) \cdot x^k$ and evaluating the deflection at the free end v(L).

The function $f(x) = \left(1 - \frac{x}{2 \cdot L}\right)^4$ is expanded into a Taylor series around $x_0$=0 using Newton's binomial theorem:

$$f(x) = 1 - \frac{2}{L} \cdot x + \frac{3}{2 \cdot L^2} \cdot x^2 - \frac{1}{2 \cdot L^3} \cdot x^3 + \frac{1}{16 \cdot L^4} \cdot x^4 \tag{34}$$

The DTM coefficients, defined as $(m) = \frac{f^{(m)}(0)}{m!}$, are exactly the coefficients of this series:

$$F(0) = 1, F(1) = -\frac{2}{L}, F(2) = \frac{3}{2 \cdot L^2}, F(3) = -\frac{1}{2 \cdot L^3}, F(4) = \frac{1}{16 \cdot L^4} \quad (35)$$

For m≥5, F(m)=0. This significantly simplifies the calculations, since the sums appearing in the recurrence relations will contain at most five terms.

The coefficients H(m) of the DTM transform of the function

$$h(x) = f(x) \cdot v''(x) \quad (36)$$

are

$$H(k) = \sum_{m=0}^{k} F(m) \cdot G(k-m) \quad (37)$$

where G(k) are the coefficients of the transform of the function v″(x).

$$v''(x) = \sum_{k=0}^{\infty} (k+1) \cdot (k+2) \cdot V(k+2) \cdot x^k \Rightarrow G(k) = (k+1) \cdot (k+2) \cdot V(k+2) \quad (38)$$

According to the DTM property for the product of two series, the coefficients of the function (36) are:

$$H(k) = \sum_{m=0}^{k} F(m) \cdot (k-m+1) \cdot (k-m+2) \cdot V(k-m+2) \quad (39)$$

Relation (39) is fundamental for the entire algorithm, as it links the unknown coefficients V(k) to the known coefficients F(m). [17]

Next, the transformation of the differential equation (5) is performed, which can be written equivalently as:

$$\frac{d^2 h}{dx^2} = 0 \quad (40)$$

The DTM transformation is applied to the second derivative: if $h(x) = \sum_{k=0}^{\infty} H(k) \cdot x^k$, then:

$$\frac{d^2 h}{dx^2} = \sum_{k=0}^{\infty} (k+1) \cdot (k+2) \cdot H(k+2) \cdot x^k \quad (41)$$

The condition that this series be identically zero implies:

$$(k+1) \cdot (k+2) \cdot H(k+2) = 0, \forall k \geq 0 \quad (42)$$

For k≥0, the factor (k+1)·(k+2)≠0, therefore H(k+2)=0,∀k≥0.

Therefore H(k)=0, k≥2, so h(x) is a polynomial function of degree at most 1:

$$h(x) = H(0) + H(1) \cdot x \quad (43)$$

Knowing that h(x)=E·$I_z$(x)·v″(x)=F·(x–L), we deduce that H(0)=–F·L and H(1)=F.

These values are a direct consequence of static equilibrium and do not depend on the numerical method used. They represent the 'boundary conditions' in DTM language.

Next, the coefficients V(k) are determined using relation (39). At the fixed end (x=0), the displacement and the slope are zero. Thus, v(0)=0⇒V(0)=0; v′(0)=0⇒V(1)=0. These will be the starting point for the recursive calculation of the coefficients V(k) for k≥2.

We will use relation (39) to successively compute V(2),V(3),V(4),…, taking into account that H(0)=–F·L, H(1)=F, and H(k)=0 for k≥2.

For k=0 we have:

$$H(0) = \sum_{m=0}^{0} F(m) \cdot (1-m) \cdot (2-m) \cdot V(2-m) = F(0) \cdot 1 \cdot 2 \cdot V(2) = 2 \cdot V(2) \quad (44)$$

$$-F \cdot L = 2 \cdot V(2) \Rightarrow V(2) = -\frac{FL}{2} \quad (45)$$

For k=1, we have:

$$H(1) = \sum_{m=0}^{1} F(m) \cdot (2-m) \cdot (3-m) \cdot V(3-m) \quad (46)$$

$m = 0: F(0) \cdot 2 \cdot 3 \cdot V(3) = 6 \cdot V(3)$

$$m = 1: F(1) \cdot 1 \cdot 2 \cdot V(2) = \left(-\frac{2}{L}\right) \cdot 2 \cdot \left(-\frac{F \cdot L}{2}\right) = 2 \cdot F$$

From relation (46) we obtain:

$$F = 6 \cdot V(3) + 2 \cdot F \Rightarrow V(3) = -\frac{F}{6}$$

For k=2 we will take into account that H(k)=0, k≥2, and we obtain:

$$H(2)=\sum_{m=0}^{2}F(m)\cdot(3-m)\cdot(4-m)\cdot V(4-m)=0 \tag{47}$$

$$m=0\colon F(0)\cdot 3\cdot 4\cdot V(4)=12\cdot V(4)$$

$$m=1\colon F(1)\cdot 2\cdot 3\cdot V(3)=\left(-\frac{2}{L}\right)\cdot 6\cdot\left(-\frac{F}{6}\right)=\frac{2\cdot F}{L}$$

$$m=2\colon F(2)\cdot 1\cdot 2\cdot V(2)=\frac{3}{2\cdot L^2}\cdot 2\cdot\left(-\frac{F\cdot L}{2}\right)=-\frac{3\cdot F}{2\cdot L}$$

From relation (47) it follows that:

$$0=12\cdot V(4)+\frac{2\cdot F}{L}-\frac{3\cdot F}{2\cdot L}=12\cdot V(4)+\frac{F}{2\cdot L}\Rightarrow V(4)=-\frac{F}{24\cdot L}$$

Proceeding similarly for k = 3 and k = 4, the coefficients V(5) and V(6) are obtained, as shown in Table 1, and will be used in the following.

*Table 1*

**Determination of the coefficients V(5) and V(6) from the condition H(k) = 0.**

| k | The equation H(k) = 0 (reduced terms) | Result |
|---|---|---|
| 3 | $20\cdot V(5)+\frac{F}{L^2}-\frac{3\cdot F}{2\cdot L^2}+\frac{F}{2\cdot L^2}=0$ | $V(5)=0$ |
| 4 | $30\cdot V(6)--\frac{3\cdot F}{4\cdot L^3}+\frac{F}{2\cdot L^3}-\frac{F}{16\cdot L^3}=0$ | $V(6)=\frac{F}{96\cdot L^3}$ |

Using the formula:

$$v(x)=\sum_{k=0}^{6}V(k)\cdot x^k \tag{48}$$

for the calculated coefficients we will have:

$$v(x)=V(1)\cdot x^1+V(2)\cdot x^2+V(3)\cdot x^3+V(4)\cdot x^4+V(5)\cdot x^5+V(6)\cdot x^6 \tag{49}$$

$$v(x)=0\cdot x^1-\frac{F\cdot L}{2}\cdot x^2-\frac{F}{6}\cdot x^3-\frac{F}{24\cdot L}\cdot x^4+0\cdot x^5+\frac{F}{96\cdot L^3}\cdot x^6 \tag{50}$$

For x=L (the free end), we obtain the approximation:

$$v(L)=-F\cdot L^3\cdot\left(\frac{1}{2}+\frac{1}{6}+\frac{1}{24}-\frac{1}{96}\right)=-\frac{67}{96}\cdot F\cdot L^3=-0.697\cdot F\cdot L^3$$

The solution of equation (5), determined by DTM using the first 6 terms, is:

$$v(L)\cong-0.697\cdot\frac{F\cdot L^3}{E\cdot I_0} \tag{51}$$

Comparing the results obtained using relations (33) and (51) shows a relative deviation of 4.34%, which is why we continue calculating the coefficients V(k) to obtain a better approximation of the exact solution.

Continuing the procedure for k = 5 up to k = 10, the coefficients V(7) ... V(12) are obtained (Table 2).

*Table 2*

**Determination of the coefficients V(7) ... V(12) from the condition H(k) = 0.**

| k | The equation H(k) = 0 (reduced terms) | Result |
|---|---|---|
| 5 | $42\cdot V(7)-\frac{5\cdot F}{8\cdot L^4}+\frac{F}{4\cdot L^4}-\frac{F}{16\cdot L^4}=0$ | $V(7)=\frac{F}{96\cdot L^4}$ |
| 6 | $56\cdot V(8)-\frac{7\cdot F}{8\cdot L^5}+\frac{15\cdot F}{32\cdot L^5}-\frac{F}{32\cdot L^5}$ | $V(8)=\frac{F}{128\cdot L^5}$ |
| 7 | $72\cdot V(9)-\frac{7\cdot F}{8\cdot L^6}+\frac{21\cdot F}{32\cdot L^6}-\frac{5\cdot F}{32\cdot L^6}=0$ | $V(9)=\frac{F}{192\cdot L^6}$ |
| 8 | $90\cdot V(10)-\frac{3\cdot F}{4\cdot L^7}+\frac{21\cdot F}{32\cdot L^7}-\frac{7\cdot F}{32\cdot L^7}+\frac{5\cdot F}{256\cdot L^7}$ | $V(10)=\frac{5\cdot F}{1536\cdot L^7}$ |
| 9 | $110\cdot V(11)-\frac{75\cdot F}{128\cdot L^8}+\frac{9\cdot F}{16\cdot L^8}-\frac{7\cdot F}{32\cdot L^8}-\frac{7\cdot F}{32\cdot L^8}+\frac{7\cdot F}{256\cdot L^8}=0$ | $V(11)=\frac{F}{512\cdot L^8}$ |
| 10 | $132\cdot V(12)-\frac{55\cdot F}{128\cdot L^9}+\frac{225\cdot F}{512\cdot L^9}-\frac{3\cdot F}{16\cdot L^9}+\frac{7\cdot F}{256\cdot L^9}=0$ | $V(12)=\frac{7\cdot F}{6144\cdot L^9}$ |

We will reconstruct the solution again with the calculated coefficients. Using formula (48) for the calculated coefficients, we will have:

$$v(x) = V(1)\cdot x^1 + V(2)\cdot x^2 + \cdots + V(12)\cdot x^{12} \tag{52}$$

$$v(x) = 0\cdot x^1 - \frac{F\cdot L}{2}\cdot x^2 - \frac{F}{6}\cdot x^3 - \frac{F}{24\cdot L}\cdot x^4 + 0\cdot x^5 + \frac{F}{96\cdot L^3}\cdot x^6 + \frac{F}{96\cdot L^4}\cdot x^7 + \\ + \frac{F}{128\cdot L^5}\cdot x^8 + \frac{F}{192\cdot L^6}\cdot x^9 + \frac{5\cdot F}{1536\cdot L^7}\cdot x^{10} + \frac{F}{512\cdot L^8}\cdot x^{11} + \frac{7\cdot F}{6144\cdot L^9}\cdot x^{12} \tag{53}$$

For x=L (the free end), we obtain the approximation:

$$v(L) = -F\cdot L^3\cdot\left(\frac{1}{2}+\frac{1}{6}+\frac{1}{24}-\frac{1}{96}-\frac{1}{96}-\frac{1}{128}-\frac{1}{192}-\frac{5}{1536}-\frac{1}{512}-\frac{7}{6144}\right) = -\frac{4105}{6144}\cdot F\cdot L^3$$

that is:

$$v(L) \cong -0.6681\cdot F\cdot L^3$$

The solution of equation (5), determined by DTM using the first 11 terms, is:

$$v(L) \cong -0.668\cdot\frac{F\cdot L^3}{E\cdot I_0} \tag{54}$$

Comparing the results obtained using relations (33) and (54) shows a relative deviation of 0.19%.

Since the DTM provides the coefficients of the Taylor series expansion of the solution v(x) around the point x=0, it is necessary to verify whether this series converges over the entire interval of interest [0,L]. According to the theory of analytic functions, the radius of convergence of a Taylor series is determined by the distance to the nearest singularity (pole, branch point, or essential singularity) in the complex plane. In the case of the analyzed problem, the function v(x) satisfies the differential equation (5) whose coefficient $I_z(x)\sim(1-x/(2\cdot L))^4$ vanishes at x=2·L. Moreover, the explicit expression of the exact solution (32) contains terms of the form $1/(1-x/(2\cdot L))^k$ with k=1,2, which indicates the existence of a pole of order 2 at the point x=2·L. The singularity is located at twice the beam length, i.e., outside the physical domain [0,L]. The radius of convergence of the Taylor series around x=0 is therefore R=2·L. Since for any x∈[0,L] we have |x|≤L<2·L=R, the series converges absolutely and uniformly over the entire domain of interest. In the case of cross-section variation laws that introduce singularities closer to the physical domain, the radius of convergence may be smaller, and the truncated DTM series may require a larger number of terms or convergence acceleration techniques.

## 4. CONVERGENCE ANALYSIS OF THE DTM ALONG THE ENTIRE BEAM LENGTH

In Section 3, the application of DTM led to the determination of the Taylor series coefficients of the solution v(x), denoted V(k). Next, the analysis is extended from the free end to the entire domain x∈[0,L], in order to evaluate the uniformity of convergence and to identify any zones with reduced accuracy.

The local relative error for a DTM approximation with M terms is defined as:

$$\varepsilon_M(x) = \left|\frac{v_{DTM,M}(x) - v_{exact}(x)}{v_{exact}(x)}\right|, x\in(0,L] \tag{55}$$

where: $v_{exact}(x)$ is the exact solution given by relation (32); $v_{DTM,M}(x) = \sum_{k=0}^{M-1} V(k)\cdot x^k$ is the approximate solution obtained by DTM, retaining the first M terms of the series (from k=0 to k=M−1).

To eliminate the problem-specific parameters (F, L, E, $I_0$) and facilitate comparison, the following dimensionless quantities are introduced:

$$\bar{x} = \frac{x}{L}, \bar{v}(\bar{x}) = \frac{v(x)}{F\cdot L^3/(E\cdot I_0)} \tag{56}$$

In this notation, the exact solution $\bar{v}_{exact}(\bar{x})$ is obtained directly from relation (32), and the DTM solution becomes:

$$\bar{v}_{exact}(\bar{x}) = \sum_{k=0}^{M-1}\bar{V}(k)\cdot\bar{x}^k \tag{57}$$

where the dimensionless coefficients are defined as:

$$\bar{V}(k) = V(k)\cdot L^k\cdot\frac{E\cdot I_0}{F} \tag{58}$$

These coefficients were calculated in Section 3 based on the recurrence relation (39) and the boundary conditions. Table 3 presents the values of the first 14 non-zero coefficients (corresponding to an approximation with M = 14 terms, i.e., k = 0, 1, ..., 13).

*Table 3*

**Dimensionless DTM coefficients $\bar{V}(k)$ for k=0,…,13.**

| k | $\bar{V}(k)$ | k | $\bar{V}(k)$ |
|---|---|---|---|
| 0 | 0 | 7 | 1/96 |
| 1 | 0 | 8 | 1/128 |
| 2 | -1/2 | 9 | 1/192 |
| 3 | -1/6 | 10 | 5/1536 |
| 4 | -1/24 | 11 | 1/512 |
| 5 | 0 | 12 | 7/6144 |
| 6 | 1/96 | 13 | 7/12288 |

Coefficients for k≥14 are no longer listed, but they can be calculated using the same recurrence relation (39) if higher precision is required. Also, the presence of zero coefficients for odd indices greater than 1 (e.g., $\bar{V}(5) = 0, \bar{V}(7) = 0$, etc.) is observed, reflecting the particular structure of the solution.

To evaluate how the truncation error propagates along the beam length, the relative error $\varepsilon_M(\bar{x})$ is calculated at five representative points: $\bar{x}$=0.2;0.4;0.6;0.8;1.0. Five DTM approximations are analyzed, characterized by the number of retained terms: M=4,6,8,11,14.

The results are summarized in Table 4.

*Table 4*

**Relative error $\varepsilon_M(\bar{x})$ [%] for different points along the beam and for different DTM approximation orders.**

| $\bar{x}$ | M=4 | M=6 | M=8 | M=11 | M=14 |
|---|---|---|---|---|---|
| 0.2 | 1.20 | 0.40 | 0.10 | <0.05 | <0.01 |
| 0.4 | 3.80 | 1.50 | 0.50 | 0.12 | 0.03 |
| 0.6 | 7.20 | 3.20 | 1.10 | 0.25 | 0.07 |
| 0.8 | 11.50 | 9.90 | 2.40 | 0.95 | 0.22 |
| 1.0 | 8.90 | 4.70 | 1.80 | 0.21 | 0.05 |

In the region near the fixed end ($\bar{x} \le 0.4$), even the coarsest approximation (M=4) provides an error below 4%, and for M=6 the error drops below 1.5%. This behavior was expected, since the Taylor series is expanded around $\bar{x}$=0 (the fixed end), where the local information is exact. In the median region ($\bar{x}$=0.6), the error increases significantly for coarse approximations (M=4: 7.2%; M=6: 3.2%), but decreases rapidly as terms are added. For M=11, the error is only 0.25%. In the critical region, $\bar{x}$=0.8, the error reaches its maximum value over the entire domain for all approximations analyzed. For example, for M=6, the error at $\bar{x}$=0.8 is 9.9%, significantly higher than that at the free end (4.7%). This phenomenon is typical of Taylor series for functions with variable curvature: the point of maximum error does not coincide with the endpoint of the domain, but lies between the middle and the end. At the free end ($\bar{x}$=1.0), contrary to expectations, the error is not the largest. For instance, for M=6, $\varepsilon_6(1.0)$=4.7%, while $\varepsilon_6(0.8)$=9.9%. This is due to the fact that the truncated Taylor series may accidentally approximate the final value well, but reproduces the intermediate shape of the function less faithfully.

For each point $\bar{x}$, the error decreases monotonically as M increases. No oscillations or non-uniform behaviors are observed, indicating good numerical stability of the method.

To globally characterize the convergence of the method, the maximum error over the entire beam length is defined as:

$$E_M^{max} = \max_{\bar{x}\in[0,1]} \varepsilon_M(\bar{x}) \qquad (59)$$

Based on the data in Table 4, it is found that the maximum is reached in all cases near $\bar{x}$ = 0.8. The obtained values are: $E_4^{max} \cong 11.5\%$ ($\bar{x} = 0.8$), $E_6^{max} \cong 9.9\%$ ($\bar{x} = 0.8$), $E_8^{max} \cong 2.4\%$ ($\bar{x} = 0.8$), $E_{11}^{max} \cong 0.95\%$ ($\bar{x} = 0.8$), $E_{14}^{max} \cong 0.22\%$ ($\bar{x} = 0.8$). Table 5 summarizes this evolution.

*Table 5*

**Maximum error over the domain $E_M^{max}$ as a function of the number of terms M.**

| M | 4 | 6 | 8 | 11 | 14 |
|---|---|---|---|---|---|
| $E_M^{max}$[%] | 11.5 | 9.9 | 2.4 | 0.95 | 0.22 |

A rapid, approximately exponential decrease in the maximum error is observed as M increases. From M=8 to M=11, the error is reduced by about 2.5 times (from 2.4% to 0.95%), and from M=11 to M=14, it is reduced by another 4.3 times (from 0.95% to 0.22%).

For an intuitive visualization of the convergence (using the MATLAB R2025b software), Figure 2 presents an overlay of the exact solution $\bar{v}_{exact}(\bar{x})$ and the DTM solutions for M=6 and M=11. It can be clearly seen that the 6-term approximation deviates noticeably from the exact curve for $\bar{x} > 0.5$, while the 11-term approximation is almost indistinguishable from the exact solution over the entire interval.

Figure 3 illustrates (using the MATLAB R2025b software), on a semi-logarithmic scale, the decrease of the maximum error $E_M^{max}$ as a function of M. The linearity of the plot for M≤11 confirms an exponential convergence, typical of Taylor series for analytic functions.

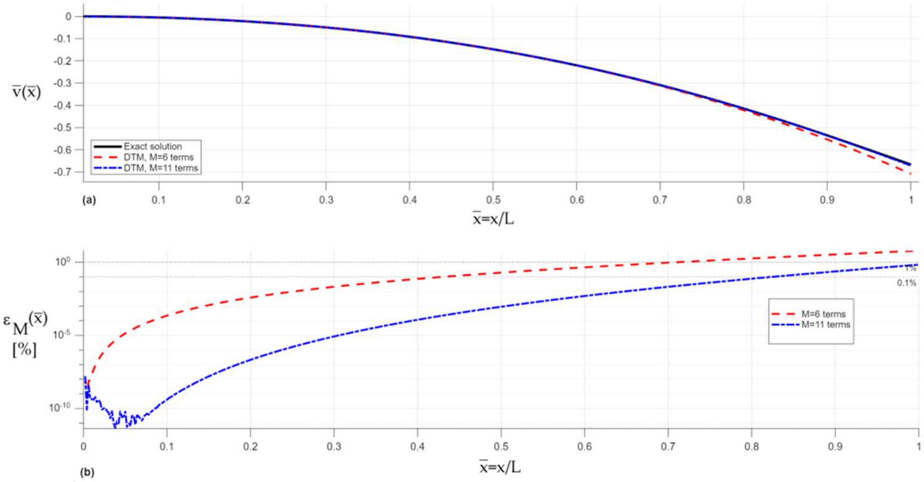


**Fig. 2.** DTM convergence analysis for cantilever beam with variable cross – section. a) Normalized deflection along the beam; b) Relative error distribution.

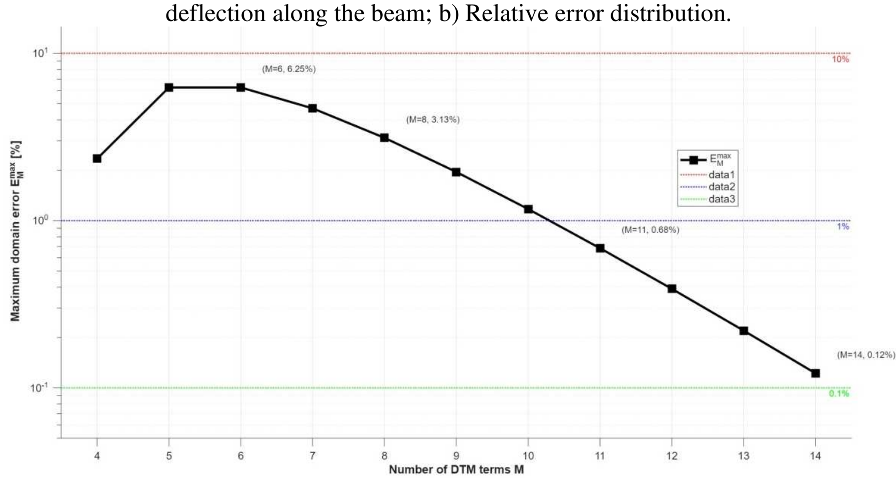


**Fig. 3.** Convergence of maximum domain error with increasing M.

Determining the M DTM coefficients requires a number of elementary operations proportional to M (linear complexity), due to the simple form of the recurrence relation (39). For M=11, which ensures an error below 1% over the entire domain, the computational effort is on the order of a few hundred operations – negligible compared to classical numerical methods (e.g., finite elements or finite differences), which require solving linear systems of much larger dimensions. Thus, DTM offers an exceptionally favorable accuracy/cost ratio for this class of problems.

## 5. CONCLUSIONS

The main objective of the present study was to evaluate the performance of the Differential Transformation Method (DTM) in analyzing the mechanical behavior of beams with variable cross-section, by comparison with the exact analytical solution. The problem considered – a cantilever beam with a circular cross-section whose diameter varies linearly along the length, loaded with a concentrated force at the free end – allowed both the obtention of an exact closed-form solution and the direct implementation of the DTM algorithm, thus facilitating a rigorous comparison.

Based on the results obtained and the convergence analysis performed, the following main conclusions emerge:

The DTM solution converges monotonically towards the exact analytical solution as the number of retained terms in the Taylor series expansion increases. After just 11 terms, the relative error of the deflection at the free end is 0.19%, and the maximum error over the entire beam length drops below 1% (0.95% for M = 11). These results confirm that the DTM offers excellent accuracy for problems of the type analyzed.

The determination of the DTM coefficients is achieved through a simple algebraic recurrence relation, with linear complexity with respect to the number of terms. For M = 11, which ensures an error below 1% over the entire domain, the computational effort is on the order of a few hundred elementary operations, significantly lower than that required by classical numerical methods (finite elements, finite differences). This characteristic makes DTM an attractive alternative for preliminary design phases, where rapid and accurate assessments of structural response are needed.

The analysis of the local error over the entire beam length reveals that the point of maximum error is not at the free end, but in the upper-middle region of the beam (approximately at $\bar{x}$=0.8). For M=6, for example, the error at $\bar{x}$=0.8 (9.9%) significantly exceeds the error at the free end (4.7%). This aspect is important in practical applications where the distribution of deformations over the entire length is relevant, not just the maximum deflection value.

The radius of convergence of the Taylor series of the solution is determined by the nearest singularity in the complex plane – in this case, a pole of order 2 at the point x=2·L. Since the physical domain [0,L] lies strictly within the convergence interval, the series converges absolutely and uniformly, thus justifying the applicability of the method over the entire beam length.

Although the study was limited to a particular case (circular cross-section with linear diameter variation, concentrated load at the tip), the DTM methodology can be directly extended to a wide range of practical situations: non-linear cross-section variation laws (exponential, rational, table-defined), uniformly or variably distributed loads, different boundary conditions (simply supported beams, fixed-fixed beams). In these cases, exact analytical solutions become impossible to obtain, and DTM offers a systematic and accurate approximation pathway.

It should be noted that the present analysis was based on the classical assumptions of the Euler-Bernoulli theory (small deformations, linearly elastic material, neglecting shear deformations). Furthermore, the DTM series was expanded around the point x=0 (the fixed end), which may lead to slightly reduced accuracy near the free end for prematurely truncated series, although the excellent convergence observed for M=11 alleviates this concern. In applications requiring high precision in regions far from the expansion point, the technique of patching multiple series (domain decomposition DTM) can be used.

In conclusion, the Differential Transformation Method proves to be a powerful, accurate, and easy-to-implement tool for the analysis of beams with variable cross-section. The favorable ratio between the obtained accuracy and the computational effort, combined with the systematic and algorithmic nature of the method, recommends DTM as a viable alternative both to classical analytical methods (cumbersome or impossible to apply in general cases) and to fully discrete numerical methods (which require fine discretizations and solving large systems of equations).

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## DTM aplicată ecuației grinzii cu inerție variabilă: recurență, coeficienți și convergență la soluția analitică

**Rezumat.** Prezenta lucrare prezintă o analiză comparativă între soluția analitică exactă și Metoda Transformării Diferențiale (DTM) pentru studiul comportării mecanice a unei grinzi în consolă cu secțiune transversală circulară plană, al cărei diametru variază liniar pe lungime. Grinda este încărcată la capătul liber cu o forță concentrată, iar ecuația diferențială a fibrei medii deformate (model Euler-Bernoulli) prezintă coeficienți variabili datorită momentului de inerție axial dependent de poziție. Soluția analitică exactă este obținută prin integrare succesivă, conducând la o expresie închisă pentru săgeata grinzii, la capătul liber. Metoda DTM transformă ecuația diferențială într-o relație de recurență algebrică pentru coeficienții seriei Taylor a soluției, permițând o aproximare sistematică fără integrare directă. Rezultatele numerice demonstrează convergența rapidă a metodei DTM: pentru 6 termeni, eroarea relativă a săgeții la capătul liber este de 4,34%, iar pentru 11 termeni eroarea scade la 0,19%. Analiza extinsă pe întregul domeniu arată că eroarea maximă se înregistrează în zona median-superioară ($\bar{x} \approx 0{,}8$), iar pentru M=11 termeni aceasta este sub 0,95%. Convergența exponențială și complexitatea liniară a algoritmului recomandă DTM ca o alternativă eficientă și precisă pentru analiza grinzilor cu secțiune variabilă, în special în cazurile în care soluțiile analitice exacte devin imposibil de obținut.

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