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USING MACHINE LEARNING APPLICATIONS TO IMPROVE MATHEMATICAL OPTIMIZATION

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Abstract: *In this paper we want to demonstrate how an application of Machine Learning (regression) can be used to improve an optimization model (linear programming problem). We calculate the expected optimal number of enterprises engaged in e-sales in each country for 22 European countries using a linear programming problem, knowing their population, minimum wage on month and e-sales. We improve the optimization model by using the predicted e-sales obtained by regression.*
Keywords: *machine learning, regression, prediction, optimization, linear programming.*

1. INTRODUCTION: BACKGROUND AND LITERATURE REVIEW

The increasing integration of Machine Learning (ML) techniques into Mathematical Optimization (MO) has generated a rapidly expanding and highly interdisciplinary body of scientific literature. ML is no longer employed solely as a standalone predictive tool, rather it is increasingly embedded within optimization frameworks to enhance algorithmic efficiency, scalability and adaptability when addressing complex mathematical problems. In light of these developments, a structured review of the existing literature is essential to clarify how ML has been utilized in mathematical optimization and to identify the main directions through which researchers frame and advance this topic in scientific discourse.

To support this objective, this section adopts a bibliometric approach grounded in a systematic analysis of the Web of Science (WoS) Core Collection. The literature dataset was constructed using a topic-based query combining “machine learning” with key concepts related to mathematical optimization, including mathematical optimization, optimization, optimal control, combinatorial optimization, and related formulations. The search was further refined using explicit criteria to ensure both relevance and quality of the retrieved publications. Specifically, the analysis focused on documents published between 2005 and 2025, written in English, and classified as articles, review papers or proceedings papers, within the WoS subject categories of Applied Mathematics, Statistics & Probability and Mathematical Physics. These selection criteria were designed to capture the evolution of ML-based optimization research while maintaining a strong mathematical and methodological focus.

The resulting bibliographic dataset of 2,421 publications provides a comprehensive overview of how ML techniques have been employed across different optimization paradigms. The retrieved records were exported and analyzed using the R software, employing the Bibliometrix package and its interactive graphical interface, Biblioshiny. Through this bibliometric analysis, the literature review moves beyond a purely narrative synthesis and offers quantitative insights into the intellectual structure of the field, highlighting both established research streams and emerging directions in the application of ML to mathematical optimization.

The Keywords Plus word cloud emphasizes the strong methodological orientation of the literature at the intersection of ML and mathematical optimization. The prominence of terms such as optimization, algorithm,

convergence, and algorithms indicates that the scientific discourse is primarily focused on the development, analysis, and improvement of optimization methods rather than on purely application-driven outcomes. Several key research directions emerge from this visualization.



Fig. 1. Thematic Emphasis of the Literature on Machine Learning in Mathematical Optimization: Keywords Plus Analysis (Source: Authors' own processing)

First, the frequent occurrence of regression, classification, approximation, and prediction highlights the role of ML as a modeling and function-approximation tool within optimization frameworks. These concepts reflect the use of ML to construct surrogate models, improve objective function representations, and enhance decision-making in complex optimization problems. Second, terms such as convergence, complexity, regularization, shrinkage, and bounds point to a strong emphasis on theoretical properties and performance guarantees. This suggests that researchers not only apply ML techniques to optimization problems but also rigorously analyze their stability, efficiency, and convergence behavior within a mathematical framework. Third, the presence of keywords like neural networks, particle swarm optimization, global optimization, and variable selection indicates the integration of ML models with heuristic, metaheuristic, and hybrid optimization strategies. These approaches are often employed to address non-convex, high-dimensional, or computationally challenging problems.

The co-occurrence network of abstract bigrams highlights the central role of machine learning as a unifying concept connecting multiple research directions within mathematical optimization. The network structure reveals a dense core centered on "machine learning", indicating its pervasive integration across optimization-related studies rather than its treatment as an isolated methodology.

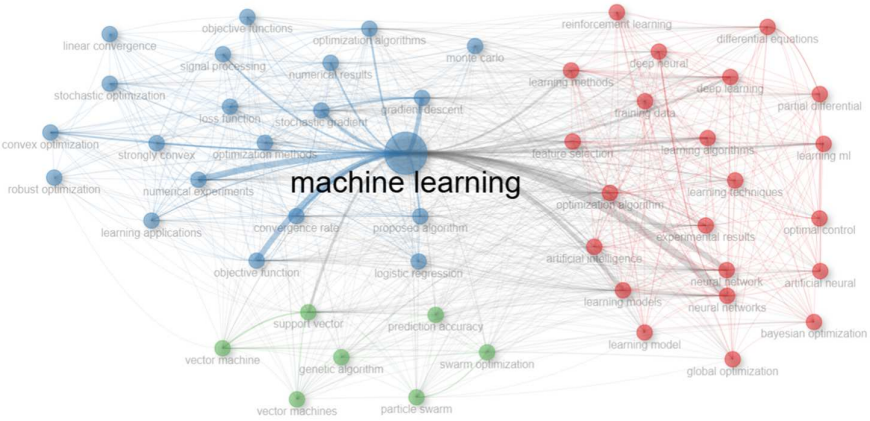


Fig. 2. Co-occurrence Network of Abstract Bigrams in Machine Learning and Mathematical Optimization Research (Source: Authors' own processing)

Three main thematic directions can be identified. The first cluster emphasizes classical mathematical optimization and learning-enhanced algorithms, with frequent co-occurrences such as stochastic gradient, gradient descent, convex optimization, loss function, and convergence rate. This reflects a strong research stream in which ML techniques are embedded within optimization methods to improve convergence properties, robustness, and numerical performance, often supported by rigorous mathematical analysis. A second prominent direction focuses on advanced learning paradigms and model-driven optimization, characterized by terms such as deep learning, neural networks, reinforcement learning, learning algorithms, and optimal control. This cluster illustrates how researchers position ML as a decision-making or control mechanism, particularly for high-dimensional, nonlinear, or dynamic optimization problems governed by differential equations. The third, smaller cluster highlights metaheuristic and hybrid approaches, including genetic algorithms, particle swarm, swarm optimization, and support vector machines. This indicates an application-oriented discourse in which ML is combined with evolutionary and population-based optimization techniques to address global or non-convex problems.

Recent research at the intersection of ML and mathematical optimization shows a shift from using ML merely as an external predictor toward embedding learning components directly within optimization pipelines.

A first visible stream is “learning to optimize”, where learning mechanisms are explicitly trained to accelerate or guide optimization procedures for continuous and mixed-integer problems. Recent tutorials consolidate this paradigm and clarify how learning can be used to design optimizers, improve search, and support decision-making in complex formulations [1]. Concrete problem-driven evidence of this direction is provided by work that learns strategies to accelerate the tightening of convex relaxations in challenging nonconvex optimization settings (e.g., AC optimal power flow), demonstrating how ML can reduce computational burden while preserving the optimization structure [2].

A second dominant approach is surrogate modeling and data-driven approximation for optimization. Here, ML models (often regression or deep models) approximate expensive objective functions, constraints, or simulation-based evaluators, enabling faster search and making optimization feasible in settings where exact evaluations are costly. This line is exemplified by machine learning–based surrogate models designed explicitly to support optimization loops [3], as well as broader surrogate modeling frameworks that emphasize algorithmic integration (training, validation, uncertainty handling, and solver coupling) [4]. Related work also studies surrogate-driven procedures for tuning and optimizing complex models, illustrating how surrogates can be used to reduce the cost of iterative improvement in high-dimensional spaces [5].

A third major direction is Bayesian optimization (BO) and sequential decision strategies for black-box and expensive optimization. BO is widely adopted when objective evaluations are expensive, noisy, or derivative-free, and the literature emphasizes its role in sample-efficient optimization and automated design [6, 7]. This includes algorithmic developments and practical implementations for global black-box optimization [8], constrained and process-oriented optimization settings and computational frameworks that support BO-driven exploration (often in scientific computing and simulation workflows) [6].

A fourth line of work emphasizes learning-guided discrete optimization and solver control, particularly through reinforcement learning. Reinforcement learning has been applied to branch-and-bound optimization, where learned policies exploit retrospective trajectories to improve solver decisions and reduce search effort [9]. More broadly, ML methods are discussed as tools for solving combinatorial problems and guiding search strategies [10]. In parallel, learning-based formulations for binary optimization highlight how stochastic learning can be integrated with experimental design

objectives, illustrating ML's role in navigating discrete design spaces [11]. Work connecting RL with stochastic optimization also reinforces the view that learning can act as an adaptive controller over optimization dynamics [11].

Finally, ML has become deeply connected to differential-equation–driven optimization, optimal control, and scientific machine learning, where optimization is central either to training or to solving the underlying mathematical problem. Neural-network approaches have been used to address high-dimensional Hamilton–Jacobi–Bellman equations, illustrating ML's role in overcoming the curse of dimensionality in dynamic optimization and control contexts [12]. Physics-informed neural networks (PINNs) and scientific ML frameworks show how deep models can encode physical constraints and solve inverse/forward problems that often reduce to constrained optimization tasks [13,14].

Analyzing these sources it indicates that researchers commonly position ML in mathematical optimization in four complementary roles: learning-enhanced optimizers and acceleration mechanisms, surrogate modeling for expensive objectives/constraints, Bayesian optimization for probabilistic global search under uncertainty, and learning-guided decision policies embedded in discrete solvers and control-oriented optimization, alongside scientific ML approaches that reframe physics-based problems as constrained learning/optimization systems.

This paper uses ML model (regression) to improve an optimization model (Linear Programming).

2. MODEL FORMULATION

Considering the scientific concerns regarding the use of mathematical optimization as highlighted in the conducted literature review the present paper focuses on the importance and the applicability of ML regression used in specific scenario contributing to the development of this topic in scientific discourse.

A multiple linear regression (MLR) model is a statistical technique used to predict a single dependent variable using two or more independent (predictor) variables, modeling their linear relationship to understand their combined influence and make better predictions than simple regression. The formula is:

$$y=\beta_0+\beta_1 x_1+\beta_2 x_2+...+\beta_n x_n+\varepsilon, \qquad (1)$$

where y is the dependent (predicted) variable, x_i are the predictor variables, β_i are the coefficients (slopes), β_0 is the intercept and ε is the error term.

If relationships are not linear, a logarithmic transformation can be applied:

$$\log(y)=\beta_0+\beta_1\log(x_1)+\beta_2\log(x_2)+...+\beta_n\log(x_n)+\varepsilon, \qquad (2)$$

The general form of a Linear Programing (LP) problem is:

$$\min(\max)\ z=c_1x_1+c_2x_2+...+c_nx_n$$
$$\begin{cases} a_{11}x_1+a_{12}x_2+...+a_{1n}x_n<b_1 \\ a_{21}x_1+a_{22}x_2+...+a_{2n}x_n<b_2 \\ ..... \\ a_{m1}x_1+a_{m2}x_2+...+a_{mn}x_n<b_m \end{cases}, \qquad (3)$$
$$x_1+x_2+...+x_n>0$$

where x are the decision variables, c are the contribution coefficient, z is the unknown function, b are the resource constraints, a represents the amount of the resource required to produce one unit of x.

We want to calculate the predicted e-sale in 22 European countries, knowing their population, monthly minimum wage and e-sales (percentage of enterprises) in 2024 (Table 1) and use these predictions as input in an LP problem that determines the optimal number of enterprises engaged in e-sales in each country.

We use the second form of a MLR (2), in our case we have the following regression equation:

$$\log(E-sales)=\beta_0+\beta_1\log(\text{Population})+\beta_2\log(\text{Minimum Wage})+\varepsilon \qquad (4)$$

We transform the values of the population, wage and e-sales using the natural logarithm (*ln*) function to obtain *log(E-sales)*, *log(Population)* and *log(Minimum Wage)* (Table 2).

Table 1

The data for 22 European countries (Source: Eurostat)

Country	Population	Minimum wages per month (€)	e-sales (percentage of enterprises)
Belgium	11817096,00	1994,00	29,74
Bulgaria	6445481,00	477,00	13,95
Czechia	10900555,00	764,00	21,18
Germany	83456045,00	2054,00	20,93
Estonia	1374687,00	820,00	21,70
Ireland	5351681,00	2146,00	36,91
Greece	10400720,00	910,00	23,57
Spain	48619695,00	1323,00	30,40
France	68467362,00	1767,00	16,09
Croatia	3861967,00	840,00	30,55
Cyprus	966365,00	1000,00	21,02
Latvia	1875316,00	700,00	17,19
Lithuania	2885891,00	924,00	39,73
Luxembourg	672050,00	2571,00	9,91
Hungary	9584627,00	697,00	20,94
Malta	563443,00	925,00	32,10
Netherlands	17942942,00	2070,00	28,91
Poland	36620970,00	978,00	15,93
Portugal	10639726,00	957,00	19,47
Romania	19067576,00	663,00	13,53
Slovenia	2123949,00	1358,00	23,57
Slovakia	5424687,00	750,00	14,36

Table 2

Calculation of the log variables

Country	log_Population	log_Minimum Wage	log_E-sales
Belgium	16.28505785	7.597898	3.39249294
Bulgaria	15.67888982	6.167516	2.63547951
Czechia	16.20432426	6.638568	3.05305734
Germany	18.23983064	7.627544	3.04118354
Estonia	14.13373663	6.709304	3.07731226
Ireland	15.49292128	7.671361	3.60848252
Greece	16.15738559	6.813445	3.15997472
Spain	17.69953925	7.187657	3.41444261
France	18.04186772	7.477038	2.77819796
Croatia	15.1666872	6.733402	3.41936469
Cyprus	13.78129689	6.907755	3.04547437
Latvia	14.44428774	6.55108	2.84432782
Lithuania	14.87534425	6.828712	3.68210657
Luxembourg	13.41808802	7.85205	2.29354435
Hungary	16.07567102	6.546785	3.04166121
Malta	13.24182145	6.829794	3.46885603
Netherlands	16.70270739	7.635304	3.36418756
Poland	17.41613159	6.88551	2.76820412
Portugal	16.18010529	6.863803	2.96887482
Romania	16.76349986	6.496775	2.60490944
Slovenia	14.56878765	7.213768	3.15997472
Slovakia	15.50647076	6.620073	2.66444656

Using the *regression* function in STATA we estimate the values of the constant β_0 and the coefficients β_1 and β_2:

Table 3

The STATA regression values

log_esales_	Coefficient
log_population	-0.016187
log_minimumwage_	0.1497737
constant	2.274774

Using the values of the *Table 3* and formula (4) we calculate the predicted e-sales values for each country :

Table 4

The predicted e-sales values

Country	Predicted e-sales
Belgium	23,23188919
Bulgaria	18,95556811
Czechia	20,16356801
Germany	22,61620815
Estonia	21,0636027
Ireland	23,78716233
Greece	20,71141662
Spain	21,36530382
France	22,18483281
Croatia	20,79286474
Cyprus	21,81842478
Latvia	20,47085907
Lithuania	21,18879683
Luxembourg	25,26111034
Hungary	19,93068073
Malta	21,75340123
Netherlands	23,20614144
Poland	20,51759551
Portugal	20,85982417
Romania	19,56619781
Slovenia	22,55035651
Slovakia	20,33385794

We construct the LP problem:

$$\max z = \sum_{i=1}^{n} E_i x_i \qquad (5)$$

$$\begin{cases} \sum_{i=1}^{n} x_i \le B \\ x_i \le p_i \ , \ \forall i \\ x_i \ge 0,2 \ , \ \forall i \end{cases} , \qquad (6)$$

where:

- E_i are the predicted e-sales per enterprise in country *i* (*Table 4*);
- x_i the number of enterprises targeted/supported in country $i, i = \overline{1,n}$ (n=22 countries, the unit of measure is millions);
- the objective function z maximizes the total predicted e-sales generated by the supported enterprises;
- the first restriction is that the total number of supported enterprises cannot exceed *B*, where *B* represents the total number of enterprises that can (or are intended to) be supported. We assume $B = 10\%$ of the total population (used as a proxy for enterprises). In our case the total population is 359062831,00 so *B*=35906283,10 (the unit of measure being millions the value of *B* will be 35,91);
- the second restriction is that the number of supported enterprises in each country cannot exceed the total number of enterprises (proxied by population) in that country;
- the last restriction is a minimum coverage constraint that forces every country to receive at least some targeting. We set a fixed minimum of 0.2 million enterprises per country.

So our LP problem will be:

$$\begin{aligned} \max z \ = {} & 22.23x_1 + 18.95x_2 + 20.16x_3 \\ & + 22.61x_4 + 21.06x_5 + 23.78x_6 \\ & + 20.71x_7 + 21.36x_8 + 21.75x_9 \\ & + 20.79x_{10} + 21.81x_{11} + 20.47x_{12} \\ & + 21.18x_{13} + 25.26x_{14} + 19.93x_{15} \\ & + 21.75x_{16} + 23.20x_{17} + 20.51x_{18} \\ & + 20.85x_{19} + 19.56x_{20} + 22.55x_{21} + 20.33x_{22} \end{aligned} \qquad (7)$$

$$\begin{cases} x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 + x_9 \\ + x_{10} + x_{11} + x_{12} + x_{13} + x_{14} + x_{15} + x_{16} \\ + x_{17} + x_{18} + x_{19} + x_{20} + x_{21} + x_{22} <= 35.91 \end{cases} \qquad (8)$$

$x_1 <= 11.81$ (9)
$x_2 <= 6.44$ (10)
$x_3 <= 10.90$ (11)
$x_4 <= 83.45$ (12)
$x_5 <= 1.37$ (13)
$x_6 <= 5.35$ (14)
$x_7 <= 10.40$ (15)
$x_8 <= 48.61$ (16)
$x_9 <= 68.46$ (17)
$x_{10} <= 3.86$ (18)
$x_{11} <= 0.96$ (19)
$x_{12} <= 1.87$ (20)
$x_{13} <= 2.88$ (21)

$$x_{14} <= 0.67 \quad (22)$$
$$x_{15} <= 9.58 \quad (23)$$
$$x_{16} <= 0.56 \quad (24)$$
$$x_{17} <= 17.94 \quad (25)$$
$$x_{18} <= 36.62 \quad (26)$$
$$x_{19} <= 10.63 \quad (27)$$
$$x_{20} <= 19.06 \quad (28)$$
$$x_{21} <= 2.12 \quad (29)$$
$$x_{22} <= 5.42 \quad (30)$$
$$x_i >= 0.2,\ i = \overline{1,22} \quad (31)$$

Using LINGO software to solve this LP problem we obtain the following values for objective function and for variables value (millions) (Table 5).

So, we target 8.35 million enterprises in Germany, 5.35 million enterprises in Ireland, 0.67 million enterprises in Luxembourg and 17.94 million enterprises in Netherlands. In the rest of the countries the minimum value of 0.2 million enterprises is allocated.

The objective value is 824.3387 which implies that the weighted average predicted e-sales rate among the supported enterprises is approximately $\frac{824.3387}{35.9} = 22.95\%$.

Table 5

The LINGO values

Objective function	824.3387
x_1	0.20
x_2	0.20
x_3	0.20
x_4	8.35
x_5	0.20
x_6	5.35
x_7	0.20
x_8	0.20
x_9	0.20
x_{10}	0.20
x_{11}	0.20
x_{12}	0.20
x_{13}	0.20
x_{14}	0.67
x_{15}	0.20
x_{16}	0.20
x_{17}	17.94
x_{18}	0.20
x_{19}	0.20
x_{20}	0.20
x_{21}	0.20
x_{22}	0.20

So, on average, the optimization targets enterprises with a predicted e-sales participation rate of 22.95%.

3. CONCLUSION

ML applications such as regression play an important role in improving the optimization models. Using regression we estimate predicted (expected) values that can be incorporated into optimization problems. In our optimization model (LP problem) we use the predicted e-sales in our objective function instead of the actual observed e-sales because optimization should be based on expected future performance rather than past realized outcomes.

Optimization is about the future, not the past. For example the Luxembourg actual e-sales are 9,91, which is relatively low while the predicted e-sales are 25.26, which indicates strong structural potential. If we had used actual e-sales we would have almost ignored Luxembourg, but by using precited e-sales we identify its structural potential and allocate resources more rationally.

Predicted e-sales are used in the optimization problem because they represent expected, systematic performance based on economic fundamentals, whereas observed e-sales include random shocks and country-specific noise that would distort optimal allocation decisions.

This paper shows how ML can make optimization more forward-looking and demonstrates a framework suitable for students and interdisciplinary audiences.

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Utilizarea aplicațiilor de învățare automată pentru îmbunătățirea Optimizării matematice

Rezumat: În această lucrare dorim să demonstrăm cum o aplicație a învățării automate (regresie) poate fi utilizată pentru a îmbunătăți un model de optimizare (problemă de programare liniară). Calculăm numărul optim așteptat de întreprinderi implicate în vânzări online în fiecare țară pentru 22 de țări europene folosind o problemă de programare liniară, cunoscând populația, salariul minim lunar și vânzările online ale acestora. Îmbunătățim modelul de optimizare utilizând vânzările online prezise obținute prin regresie.

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