TECHNICAL UNIVERSITY OF CLUJ-NAPOCA

ACTA TECHNICA NAPOCENSIS

Series: Applied Mathematics, Mechanics, and Engineering
Vol.. 57, Issue II, June, 2014

FORMULATIONS IN ADVANCED SYSTEMS DYNAMICS

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Abstract: *On the basis of new formulations, in this paper, a few important dynamics notions, such as acceleration energy of first and second order, as well as differential principles from analytical mechanics will be analyzed. These have been applied for multibody mechanical systems, such as robots. Initially, in this paper the applications of the matrix exponentials in the forward kinematics have been shortly presented. Based on D'Alembert-Lagrange principle, written in the generalized form, a few formulations on the third order differential equations of motion are described. In the final part of this paper the formulations have been applied on a 2TR-type mechanical robot structure.*

Key words: *matrix exponentials; dynamics; energies of higher order; multibody systems; robotics*

1. INTRODUCTION

The main objective of the paper consists in a few formulations about differential principle used to determine the differential equations of motion for any multibody system (abbreviated MBS). From the beginning, it is important to determine the geometry and the kinematical equations as well as the mass distribution parameters. So, the following section is devoted to presenting a few considerations regarding the forward kinematics and mass properties. The kinematical analysis is performed based on matrix exponential functions.

2. ADVANCED SYSTEMS KINEMATICS

2.1 Forward in Advanced Kinematics

Based on new formulations regarding matrix exponentials, according to [1] and [2], in the following, are defined the forward geometry and the kinematics equations, by considering aspects from variation principles, applied in advanced dynamics of the mechanical system.

Therefore, the matrix exponentials and their associated transformations are included in the Algorithm of Matrix Exponential in Forward Kinematics (abbreviated MEK), whose main steps are described in the following section.

2.2 The Algorithm of Matrix Exponentials

The matrix of the nominal geometry, corresponding to the initial configuration of MBS with the screw parameters, also named homogeneous coordinates, is completed as:

$$M_{vn}^{(0)**} \underset{[(n+1)\times 9]}{=} Matrix \left\{ \left[\bar{p}_i^{(0)T} \quad k_i^{(0)T} \quad \bar{v}_i^{(0)} \right] i = 1 \to n+1 \right\}^T \quad (1)$$

where $\left\{ \bar{p}_i^{(0)T} = [x_i \quad y_i \quad z_i]^T \right\}$ is the position vector of the origin of the system $\{i\}$ with respect to $\{0\}$ frame; $\bar{k}_i^{(0)}$ represents the unit vector corresponding to each driving axis [1] and [2], $i = 1 \to n$, while:

$$\bar{v}_i^{(0)} = \left\{ \bar{p}_i^{(0)} \times \right\} \bar{k}_i^{(0)} \cdot \Delta_i + (1 - \Delta_i) \cdot \bar{k}_i^{(0)} \quad (2)$$

where $\Delta_i = \{(1, if\ i = R); (0, if\ i = T)\}$ is an operator which marks out the type of joint: (R-rotation; T-prismatic joint), according to Figure 1.

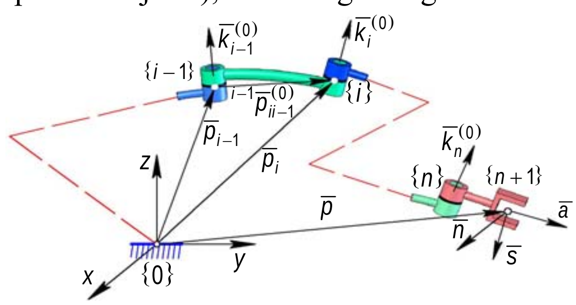

Fig. 1 Sequence from a MBS

The matrix of the screw parameters A_i is unchangeable for any MBS configuration. This property is an important advantage in the kinematical study of a multibody structure (MBS). The expression for this matrix is:

$$A_i = \left[\begin{array}{c|c} \left\{\overline{k}_i^{(0)} \times\right\} \Delta_i & \overline{v}_i^{(0)} \\ \hline 0\ 0\ 0 & 0 \end{array}\right]; \tag{3}$$

Throughout the paper, the following notations are implemented:

$$q_{i,j,k,m} \cdot \Delta_{i,j,k,m} = q^*_{i,j,k,m},$$
$$c\left(q_i^*\right) \equiv \cos\left(q_i^*\right); s\left(q_i^*\right) \equiv \sin\left(q_i^*\right) \tag{4}$$

and $\overline{\theta} = \left(q_i, for\ i = 1 \to n\right)^T$, which defines the column vector of the generalized coordinates, expressing the configuration space, according to analytical mechanics.

The exponential of the rotation matrix is defined, according to [3], by means of the following expression:

$$\left\{\begin{array}{l} R\left(\overline{k}_i\ ;\ q_i^*\right) = \exp\left\{\left\{\overline{k}_i^{(0)} \times\right\} \cdot q_i^*\right\} = I_3 \cdot c\left(q_i^*\right) + \\ +\left\{\overline{k}_i^{(0)} \times\right\} s\left(q_i^*\right) + \overline{k}_i^{(0)} \cdot \overline{k}_i^{(0)T}\left[1 - c\left(q_i^*\right)\right] \end{array}\right\} \tag{5}$$

where $\left\{\overline{k}_i^{(0)} \times\right\}$ is the skew-symmetric matrix associated to the unit vector belonging to every kinematical axis. In the position study based on matrix exponentials, a new column vector is established, according to [1] and [2]:

$$\begin{array}{l} \overline{b}_i = \left\{I_3 \cdot q_i + \left\{\overline{k}_i^{(0)} \times\right\}\left[1 - c\left(q_i^*\right)\right] + \right. \\ \left. + \overline{k}_i^{(0)} \cdot \overline{k}_i^{(0)T} \cdot \left[q_i - s\left(q_i^*\right)\right]\right\} \cdot \overline{v}_i^{(0)} \end{array} \tag{6}$$

where, I_3 represents the unit matrix.

Among the expressions based on matrix exponentials for defining the locating transformation are mentioned the following:

$$e^{A_i \cdot q_i} = \begin{bmatrix} \exp\left\{\left\{\overline{k}_i^{(0)} \times\right\} q_i^*\right\} & \overline{b}_i \\ 0\quad 0\quad 0 & 1 \end{bmatrix} \tag{7}$$

$$\exp\left\{\sum_{j=0}^{i} A_j \cdot q_j\right\} = \begin{bmatrix} \exp\{R\} & \exp\{p\} \\ 0\ 0\ 0 & 1 \end{bmatrix} \tag{8}$$

where $\exp\{R\} = \prod_{j=0}^{i} \exp\left\{\left\{\overline{k}_j^{(0)} \times\right\} q_j^*\right\}$

$$\exp\{p\} = \sum_{j=0}^{i}\left\{\prod_{k=0}^{i} \exp\left\{\left\{\overline{k}_k^{(0)} \times\right\} q_k^*\right\}\right\} \cdot \overline{b}_{j+1} \tag{9}$$

The exponentials for the locating matrices (homogeneous transformation matrix), which define the position and the orientation of the frames $\{n\}$ and $\{n+1\}$ with respect to fixed frame $\{0\}$, are:

$$T_{x0} = \prod_{i=1}^{n}\left(e^{A_i \cdot q_i}\right) \cdot T_{x0}^{(0)} = \exp\left\{\sum_{i=1}^{n} A_i \cdot q_i\right\} \cdot T_{x0}^{(0)} \tag{10}$$

$$where \quad x = \{n; n+1\}$$

The previous results are further used to determine the forward kinematic equations for any robot structure. In the following, a few expressions from the matrix exponentials algorithm in kinematics (MEK) are presented.

First, an external loop $(i = 1 \to n)$ is opened, this yielding to:

$$\underset{(3\times 3)}{ME}\left(V_{i1}\right) = \exp\left\{\sum_{j=0}^{i-1}\left\{\overline{k}_j^{(0)} \times\right\} \cdot q_j^*\right\} \tag{11}$$

$$\underset{(3\times 6)}{ME}\left(V_{i2}\right) = \left[\begin{array}{c|c} I_3 & \Delta_i \cdot \left\{\overline{k}_i^{(0)} \times\right\} \end{array}\right] \tag{12}$$

$$\underset{\{6\times[9+3\cdot(n-i)]\}}{ME}\left(V_{i3}\right) \equiv \left[ME\left(V_{i31}^*\right)\quad ME\left(V_{i32}^*\right)\quad ME\left(V_{i33}^*\right)\right] \tag{13}$$

Inside expression (13), the terms have the following meaning:

$$\left\{\begin{array}{c} ME\left(V_{i31}^*\right) = \begin{bmatrix} I_3 \\ [0]_{3\times 3} \end{bmatrix} \\ ME\left(V_{i33}^*\right) = \begin{bmatrix} [0]_{3\times 3} \\ \exp\left\{\sum_{k=i}^{n}\left\{\overline{k}_k^{(0)} \times\right\} q_k^*\right\} \end{bmatrix} \\ ME\left(V_{i32}^*\right) = \begin{bmatrix} [0]_{3\times 3} \\ \exp\left\{\sum_{m=i-1}^{k-1}\left\{\overline{k}_m^{(0)} \times\right\} q_m^* \cdot \delta_m\right\} \end{bmatrix} \end{array}\right\} \tag{14}$$

Applying a series of matrix transformations, in expressions (11) - (13), the following exponential expressions are obtained:

$$\underset{(6\times 6)}{ME}\left\{J_{i1}\right\} = \begin{bmatrix} ME\{V_{i1}\} & [0] \\ [0] & ME\{V_{i1}\} \end{bmatrix}; \underset{(6\times 9)}{ME}\left\{J_{i2}\right\} = \begin{bmatrix} ME\{V_{i2}\} & [0] \\ [0] & I_3 \end{bmatrix}$$
$$\underset{\{9\times[12+3\cdot(n-i)]\}}{ME}\left\{J_{i3}\right\} = \begin{bmatrix} ME\{V_{i3}\} & [0] \\ [0] & I_3 \end{bmatrix} \tag{15}$$

$$M_{iv\omega} = \begin{Bmatrix} M_{iv}^{T} = \left(\left[\bar{v}_i^{(0)T}\ \bar{b}_k{}^{T}\ \bar{p}_n^{(0)T}\right]\right) \\ \Delta_i \cdot \bar{k}_i^{(0)T} \end{Bmatrix} \tag{16}$$

Equation (16) contains the column vector of screw parameters, as well as the position and orientation parameters of the robot's end-effector. They are included [2] in the Jacobian matrix, according to the following expressions:

$$\underset{(6\times n)}{{}^{0}J}\left[\bar{\theta}(t)\right] = \left\{{}^{0}J_i\left[\bar{\theta}_i(t)\right],\ i = 1 \rightarrow n\right\} \tag{17}$$

where every column is defined as follows:

$${}^{0}J_i\left[\bar{\theta}_i(t)\right] = \left[ME\{J_{i1}\} \cdot \begin{bmatrix} ME(V_{i2})ME(V_{i3}) & [0] \\ [0] & I_3 \end{bmatrix} \{M_{iv\omega}\}\right]$$

Remark: The Jacobian matrix defined with (17), is also known as the transfer velocity matrix.

The absolute values for angular and linear velocities and accelerations, corresponding to any kinetic link $(i = 1 \rightarrow n)$ from a MBS projected on $\{0\}$ frame, are determined with:

$${}^{0}\bar{\omega}_i = \sum_{j=1}^{i}\left\{\exp\left\{\sum_{k=1}^{j-1}\left\{\bar{k}_k^{(0)}\times\right\}\cdot q_k \cdot \Delta_k\right\}\right\} \cdot \bar{k}_j^{(0)} \cdot \dot{q}_j \cdot \Delta_j \tag{18}$$

$${}^{0}\dot{\bar{\omega}}_i = \sum_{j=1}^{i}\left\{ME\left\{V_{j1}\right\}\cdot \ddot{q}_j + ME\left\{\dot{V}_{j1}\right\}\cdot \dot{q}_j\right\}\cdot \bar{k}_j^{(0)} \cdot \Delta_j \tag{19}$$

$$\begin{aligned} {}^{0}\ddot{\bar{\omega}}_i &= \sum_{j=1}^{i}\left\{ME\left\{V_{j1}\right\}\cdot \dddot{q}_j + ME\left\{\dot{V}_{j1}\right\}\cdot \ddot{q}_j\right\}\cdot \bar{k}_j^{(0)} \cdot \Delta_j \\ &+ \sum_{j=1}^{i}\left\{ME\left\{\dot{V}_{j1}\right\}\cdot \ddot{q}_j + ME\left\{\ddot{V}_{j1}\right\}\cdot \dot{q}_j\right\}\cdot \bar{k}_j^{(0)} \cdot \Delta_j \end{aligned} \tag{20}$$

$${}^{0}\bar{v}_i = \sum_{j=1}^{i}\left\{\prod_{k=1}^{3} ME\left\{J_{jk}\right\}\right\}\cdot M_{jv}\cdot \dot{q}_j\ ; \tag{21}$$

$$\begin{aligned} {}^{0}\dot{\bar{v}}_i &= \sum_{j=1}^{i}\left\{\left\{\prod_{k=1}^{3} ME\left\{J_{jk}\right\}\right\}\cdot M_{jv}\cdot \ddot{q}_j\right\} + \\ &+ \sum_{j=1}^{i}\left\{\frac{d}{dt}\left\{\left\{\prod_{k=1}^{3} ME\left\{J_{jk}\right\}\right\}\cdot M_{jv}\right\}\cdot \dot{q}_j\right\} \end{aligned} \tag{22}$$

$$\begin{aligned} {}^{0}\ddot{\bar{v}}_i &= \sum_{j=1}^{i}\left\{\left\{\prod_{k=1}^{3} ME\left\{J_{jk}\right\}\right\}\cdot M_{jv}\cdot \dddot{q}_j\right\} + \\ &+ \sum_{j=1}^{i}\left\{\frac{d}{dt}\left\{\left\{\prod_{k=1}^{3} ME\left\{J_{jk}\right\}\right\}\cdot M_{jv}\right\}\cdot \ddot{q}_j\right\} + \\ &+ \sum_{j=1}^{i}\left\{\frac{d^2}{dt^2}\left\{\left\{\prod_{k=1}^{3} ME\left\{J_{jk}\right\}\right\}\cdot M_{jv}\right\}\cdot \dot{q}_j\right\} \end{aligned} \tag{23}$$

For the transfer of the above kinematical parameters from $\{0\}$ in $\{i\}$ moving frame, the next matrix exponential expression is applied:

$${}^{i}\bar{V}_i = ME\left({}^{i}R\right)\cdot {}^{0}\bar{V}_i\ ;\ where\ \bar{V} = \left\{\bar{\omega};\dot{\bar{\omega}};\ddot{\bar{\omega}};\bar{v};\dot{\bar{v}};\ddot{\bar{v}}\right\} \tag{24}$$

$$ME\left({}^{i}R\right) = \left[R_{i0}^{(0)}\right]^{-1}\cdot \prod_{j=i}^{1}\exp\left[-\left(\bar{k}_j^{(0)}\times\right)\cdot q_j \cdot \Delta_j\right] \tag{25}$$

The significance of the symbols from these equations is shown by means of (11) – (16).

2.3 The Rotation Motion

The analysis performed in [2] reveals the fact that the absolute rotation of a mobile reference system $\{n\}$ attached to a rigid body (S_n), belonging to a MBS, is defined from kinematical point of view by means of the rotation matrix, the orientation vector, and by the angular velocity and angular acceleration, respectively.

According to [2] and [4], the orientation vector, denoted with $\bar{\Omega}(t)$, is defined as follows:

$$\bar{\theta}_{ii-1} = \left(q_j,\ j = \left(N_{i-1}+1\right)\rightarrow N_i\right)^T,\ \ i = 1\rightarrow n \tag{26}$$

$$\bar{\theta}_i = \left[\bar{\theta}_{jj-1}^{T},\ \ j = 1\rightarrow i\right]^T$$

where, according to (4), q_j defines the generalized coordinates that can be either linear and/or angular and which are included in the $\bar{\theta}_{ii-1}$ and $\bar{\theta}_i$ represents the column vector of the generalized coordinates which characterize the degrees of freedom of the rigid body (S_i), with respect to a fixed reference system $\{0\}$. As a result, the column vector $\bar{\Omega}(t)$ is written by means of the orientation angle set, thus:

$$\bar{\Omega}(t) = \begin{pmatrix}\alpha_A(t)\\ \beta_B(t)\\ \gamma_C(t)\end{pmatrix} = \begin{pmatrix} f_4\left(q_i(t)\cdot\Delta_i;\ i = 1\rightarrow k\right)\\ f_5\left(q_i(t)\cdot\Delta_i;\ i = 1\rightarrow k\right)\\ f_6\left(q_i(t)\cdot\Delta_i;\ i = 1\rightarrow k\right)\end{pmatrix}. \tag{27}$$

So, in keeping with the same [2] and [4], it is obvious that the resultant rotation is defined (from kinematical point of view) by any of the twelve sets of orientation angles, this leading to the twelve sets of resultant matrices. Thus, the general form of the resultant orientation matrix for any multibody system can be expressed by:

$$\begin{Bmatrix} {}^{0}_{n}[R] = R\left[\alpha_A(t) - \beta_B(t) - \gamma_C(t)\right] = \\ = R\left[\bar{A};\alpha_A(t)\right]\cdot R\left[\bar{B};\beta_B(t)\right]\cdot R\left[\bar{C};\gamma_C(t)\right]\end{Bmatrix}. \tag{28}$$

In expression (28), there are included the unit vectors of the axes around which simple rotations (27) are performed. The mathematical significance of the unit vectors is shown below:

$$\left\{\begin{matrix} {}^{(n)0}\bar{A} = \left\{ {}^{(n)0}\bar{x}; {}^{(n)0}\bar{y}; {}^{(n)0}\bar{z} \right\}; \\ \bar{A} = \left\{ \bar{x} = \begin{pmatrix}1\\0\\0\end{pmatrix};\ \bar{y} = \begin{pmatrix}0\\1\\0\end{pmatrix};\ \bar{z} = \begin{pmatrix}0\\0\\1\end{pmatrix} \right\} \end{matrix}\right\} \quad (29)$$

$$\left\{\begin{matrix} {}^{(n)0}\bar{B} = \left\{ {}^{(n)0}\bar{y}; {}^{(n)0}\bar{z}; {}^{(n)0}\bar{x} \right\} \\ \bar{B} = \left\{ \bar{y} = \begin{pmatrix}0\\1\\0\end{pmatrix};\ \bar{z} = \begin{pmatrix}0\\0\\1\end{pmatrix};\ \bar{x} = \begin{pmatrix}1\\0\\0\end{pmatrix} \right\} \neq \bar{A} \end{matrix}\right\} \quad (30)$$

$$\left\{\begin{matrix} {}^{(n)0}\bar{C} = \left\{ {}^{(n)0}\bar{z}; {}^{(n)0}\bar{x}; {}^{(n)0}\bar{y} \right\} \\ \bar{C} = \left\{ \bar{z} = \begin{pmatrix}0\\0\\1\end{pmatrix};\ \bar{x} = \begin{pmatrix}1\\0\\0\end{pmatrix};\ \bar{y} = \begin{pmatrix}0\\1\\0\end{pmatrix} \right\} \neq \bar{B} \end{matrix}\right\} \quad (31)$$

Applying some properties, according to [2] and [4], the components of the simple rotation matrices, included in (29), are determined as:

$$\left\{\begin{matrix} R\left[\bar{A};\alpha_A(t)\right] = \exp\left[(\bar{A}\times)\cdot\alpha_A(t)\right] = \\ = I_3\cdot c\left[\alpha_A(t)\right] + \bar{A}\times s\left[\alpha_A(t)\right] + \\ + \bar{A}\cdot\bar{A}^T\cdot\left[1 - c\left[\alpha_A(t)\right]\right] = \\ = I_3 + \bar{A}\times s\left[\alpha_A(t)\right] + (\bar{A}\times)^2\cdot\left[1 - c\left[\alpha_A(t)\right]\right] \end{matrix}\right\} \quad (32)$$

Similar to (32), there can be obtained the same expressions for $R\left[\bar{B};\beta_B(t)\right]$ and $R\left[\bar{C};\gamma_C(t)\right]$ by substituting the versor $\bar{A}$ with $\bar{B}$ and $\bar{C}$, respectively $\alpha_A(t)$ with the corresponding rotation angles $\beta_B(t)$ and $\gamma_C(t)$. Substituting the obtained rotation matrices in (28) it results an exponential form of the expressions, which characterizes the resultant rotation matrix.

$$\left\{\begin{matrix} {}^0_n[R] = R\left[\alpha_A(t) - \beta_B(t) - \gamma_C(t)\right] = \\ = \exp\left[\sum_{i=1}^{n}\left(\bar{k}_i^{(0)}\times\right)q_i^*\right]\cdot R_{n0}^{(0)} = \\ = \exp\left[(\bar{A}\times)\cdot\alpha_A(t) + (\bar{B}\times)\cdot\beta_B(t) + (\bar{C}\times)\cdot\gamma_C(t)\right] \end{matrix}\right\} \quad (33)$$

where $R_{n0}^{(0)}$ represents the resultant rotation matrix, for the initial configuration, which assumes that $q_i = 0,\ i = 1 \to n$.

The expression for the absolute angular velocity, with projections on $\{0\}$ fixed reference system, in a generalized form, which takes into account the notations from (27) and (31), is defined by means of the following matrix:

$${}^{(n)0}\bar{\omega}_n = \dot{\alpha}_A(t)\cdot{}^{(n)0}\bar{A}(t) + \dot{\beta}_B(t)\cdot{}^{(n)0}\bar{B}(t) + \dot{\gamma}_C(t)\cdot{}^{(n)0}\bar{C}(t) \quad (34)$$

$${}^{(n)0}\bar{\omega}_n = \left[{}^{(n)0}\bar{A}(t)\ {}^{(n)0}\bar{B}(t)\ {}^{(n)0}\bar{C}(t)\right]\cdot\left[\dot{\alpha}_A(t)\ \dot{\beta}_B(t)\ \dot{\gamma}_C(t)\right]^T$$

Expression (34) can also be written in the following form:

$${}^{(n)0}\bar{\omega}_n = {}^{(n)0}J_\Omega(t)\cdot\left[\dot{\alpha}_A(t)\ \dot{\beta}_B(t)\ \dot{\gamma}_C(t)\right]^T \quad (35)$$

The angular transfer matrix from (35) is defined as follows:

$${}^0J_\Omega(t) = \left\{{}^0\bar{A} \ \vdots\ {}^0\bar{B} \ \vdots\ {}^0\bar{C}\right\} \quad (36)$$

The components of the transfer matrix (Jacobian Matrix), defined with (36), have the following significance:

$$\left\{\begin{matrix} {}^0\bar{B} = R\left[\bar{A};\alpha_A(t)\right]\cdot\bar{B} = \\ = \left\{\exp\left[(\bar{A}\times)\cdot\alpha_A(t)\right]\right\}\cdot\bar{B} \end{matrix}\right\} \quad (37)$$

$$\left\{\begin{matrix} {}^0\bar{C} = R\left[\bar{A};\alpha_A(t)\right]\cdot R\left[\bar{B};\beta_B(t)\right]\cdot\bar{C} = \\ = \left\{\exp\left[(\bar{A}\times)\cdot\alpha_A(t) + (\bar{B}\times)\cdot\beta_B(t)\right]\right\}\cdot\bar{C} \end{matrix}\right\} \quad (38)$$

Taking into account (36), the expression of the angular velocity, projected on $\{0\}$ fixed system, is defined by means of:

$${}^0\bar{\omega}_n = \left\{\begin{matrix} \bar{A} \\ R\left[\bar{A};\alpha_A(t)\right]\cdot\bar{B} \\ R\left[\bar{A};\alpha_A(t)\right]\cdot R\left[\bar{B};\beta_B(t)\right]\cdot\bar{C} \end{matrix}\right\}^T\cdot\begin{bmatrix}\dot{\alpha}_A(t)\\ \dot{\beta}_B(t)\\ \dot{\gamma}_C(t)\end{bmatrix}. \quad (39)$$

Applying the first time derivative on (35), it results the expression for the absolute angular acceleration:

$$\left\{\begin{matrix} {}^{(n)0}\dot{\bar{\omega}}_n = {}^{(n)0}J_\psi(t)\cdot\left[\ddot{\alpha}_A(t)\ \ddot{\beta}_B(t)\ \ddot{\gamma}_C(t)\right]^T \\ + {}^{(n)0}\dot{J}_\psi(t)\cdot\left[\dot{\alpha}_A(t)\ \dot{\beta}_B(t)\ \dot{\gamma}_C(t)\right]^T \end{matrix}\right\} \quad (40)$$

The time derivative of the angular transfer matrix is defined as:

$${}^0\dot{J}_\psi(t) = \left\{\dot{\bar{A}} \ \vdots\ {}^0\dot{\bar{B}} \ \vdots\ {}^0\dot{\bar{C}}\right\}; \quad (41)$$

The components of angular transfer matrix are determined as:

$$\dot{\bar{A}} = (0\quad 0\quad 0)^T; \quad (42)$$

$$\left\{\begin{matrix} {}^0\dot{\bar{B}} = \frac{d}{dt}\left\{R\left[\bar{A};\alpha_A(t)\right]\cdot\bar{B}\right\} = \\ = \frac{d}{dt}\left\{\left\{\exp\left[(\bar{A}\times)\cdot\alpha_A(t)\right]\right\}\cdot\bar{B}\right\} = \\ = \left[(\bar{A}\times)\cdot\dot{\alpha}_A(t)\right]\cdot\left\{\exp\left[(\bar{A}\times)\cdot\alpha_A(t)\right]\right\}\cdot\bar{B} \end{matrix}\right\}; \quad (43)$$

$$\begin{matrix} \dot{R}\left[\bar{A};\alpha_A(t)\right] = \left[(\bar{A}\times)\cdot\dot{\alpha}_A(t)\right]\cdot\left\{\exp\left[(\bar{A}\times)\cdot\alpha_A(t)\right]\right\} \\ \dot{R}\left[\bar{B};\beta_B(t)\right] = \left[(\bar{B}\times)\cdot\dot{\beta}_B(t)\right]\cdot\left\{\exp\left[(\bar{B}\times)\cdot\beta_B(t)\right]\right\} \end{matrix} \quad (44)$$

$$\left\{\begin{array}{l} {}^0\dot{\overline{C}} = \dfrac{d}{dt}\left\{R\left[\overline{A};\alpha_A(t)\right]\cdot R\left[\overline{B};\beta_B(t)\right]\cdot\overline{C}\right\} = \\ = \dfrac{d}{dt}\left\{\left\{\exp\left[(\overline{A}\times)\cdot\alpha_A(t)+(\overline{B}\times)\cdot\beta_B(t)\right]\right\}\cdot\overline{C}\right\} = \\ = \left[(\overline{A}\times)\cdot\dot{\alpha}_A(t)\right]\cdot R\left[\overline{A};\alpha_A(t)\right]\cdot R\left[\overline{B};\beta_B(t)\right]\cdot\overline{C} + \\ + R\left[\overline{A};\alpha_A(t)\right]\cdot\left[(\overline{B}\times)\cdot\dot{\beta}_B(t)\right]\cdot R\left[\overline{B};\beta_B(t)\right]\cdot\overline{C} \end{array}\right\} \quad (45)$$

The orientation angles and their derivatives, substituted into the generalized expressions conduct to (44) and (45), according to the meanings of symbols from [2] and [4]. The generalized transformations presented above, conduct to the set of twelve matrix expressions for the angular transfer matrix and the absolute rotational angular velocities.

3. ENERGIES OF HIGHER ORDER

The kinetic energy is a fundamental notion in systems dynamics being included in the Lagrange – Euler equations, based on which the dynamic control functions are achieved.

3.1 The Kinetic Energy in Explicit Form

According to König's theorem of the kinetic energy, this is a sum of two components: the first component characterizes the kinetic energy in translation motion and the second component defines the kinetic energy in the resultant rotation motion. The expression is:

$$E_K^i = \frac{1}{2}\cdot M_i\cdot {}^i\overline{v}_{C_i}^T\cdot {}^i\overline{v}_{C_i} + \frac{1}{2}\cdot {}^i\overline{\omega}_i^T\cdot {}^iI_i^*\cdot {}^i\overline{\omega}_i \qquad (46)$$

where, the velocity of the mass center is:

$$ {}^i\overline{v}_{C_i} = {}^i\overline{v}_i + {}^i\overline{\omega}_i\times {}^i\overline{r}_{C_i} \qquad (47)$$

while ${}^i\overline{\omega}_i$ and ${}^i\overline{v}_i$ are substituted by (18) – (23). Considering Fig. 2, the symbol ${}^iI_i^*$ represents the axial and centrifugal inertial tensor of the kinetic ensemble relative to the frame applied in the mass center C_i:

$$\left\{ {}^iI_i^* = \int\left\{{}^i\overline{r}_i^*\times\right\}\cdot\left\{{}^i\overline{r}_i^*\times\right\}^T dm\right\} = \\ = \begin{bmatrix} {}^iI_x^* & -{}^iI_{xy}^* & -{}^iI_{xz}^* \\ -{}^iI_{yx}^* & {}^iI_y^* & -{}^iI_{yz}^* \\ -{}^iI_{zx}^* & -{}^iI_{zy}^* & {}^iI_z^* \end{bmatrix} \qquad (48)$$

${}^i\overline{r}_{C_i}$ is the position vector of the mass center.

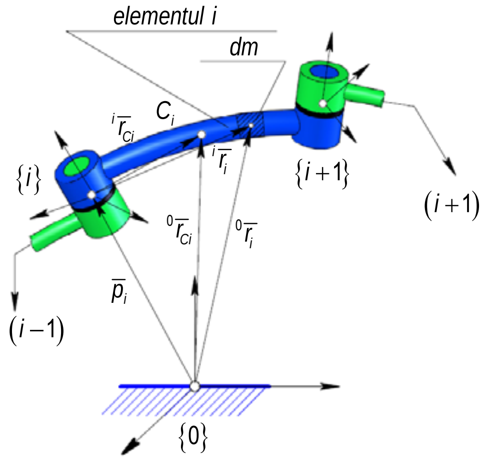

Fig. 2 A kinetic ensemble from MBS

In the following section, there are presented the results of an advanced study performed in order to emphasize the higher order energies that manifest themselves in certain mechanical systems and which correspond to higher order accelerations. Therefore, the expressions that define the acceleration energy of first order, also known as Appell's function, as well as the acceleration energy of second order are demonstrated under the explicit form.

3.2 The acceleration energy of first order

In keeping with [4] and [7], the acceleration energy of first order is defined with:

$$E_A^{(1)i} = \frac{1}{2}\cdot\int\dot{\overline{v}}_i^T\cdot\dot{\overline{v}}_i\cdot dm = \frac{1}{2}\cdot\int Trace\left(\ddot{\overline{r}}_i\cdot\ddot{\overline{r}}_i^T\right)\cdot dm\,; \quad (49)$$

where $\dot{\overline{v}}_i = \ddot{\overline{r}}_i$ represents the absolute acceleration of the elementary and infinitesimal mass dm, belonging to body, where $i = 1 \rightarrow n$, and the symbol $Trace$ corresponds to a squared matrix. Performing a few differentials and matrix transformations, presented at large in [4], it is obtained the expression for the acceleration energy of first order, corresponding to a rigid body, which is in fact, a component of a MBS, that is:

$$\begin{array}{l} E_A^{(1)i} = (-1)^{\Delta_m}\cdot\dfrac{1-\Delta_m}{1+3\cdot\Delta_m}\left\{\dfrac{1}{2}\cdot M_i\cdot {}^{(i)}\dot{\overline{v}}_{C_i}^T\cdot {}^{(i)}\dot{\overline{v}}_{C_i}\right\} + \\ +\Delta_m^2\cdot\left\{\dfrac{1}{2}\cdot {}^{(i)}\dot{\overline{\omega}}_i^T\cdot\left[{}^{(i)}I_i^*\cdot {}^{(i)}\dot{\overline{\omega}}_i + \left({}^{(i)}\overline{\omega}_i\times {}^{(i)}I_i^*\cdot {}^{(i)}\overline{\omega}_i\right)\right]\right\} + \\ +\Delta_m^2\cdot\left\{\dfrac{1}{2}\cdot {}^{(i)}\dot{\overline{\omega}}_i^T\cdot\left({}^{(i)}\overline{\omega}_i\times {}^{(i)}I_i^*\cdot {}^{(i)}\overline{\omega}_i\right)\right\} + \\ +\Delta_m^2\cdot\left\{\dfrac{1}{2}\cdot {}^{(i)}\overline{\omega}_i^T\cdot\left[{}^{(i)}\overline{\omega}_i^T\cdot Trace\left({}^{(i)}I_{pi}^*\right)\cdot {}^{(i)}\overline{\omega}_i - \right.\right. \\ \left.\left. - {}^{(i)}\overline{\omega}_i^T\cdot {}^{(i)}I_{pi}^*\cdot {}^{(i)}\overline{\omega}_i\right]\cdot {}^{(i)}\overline{\omega}_i\right\} \end{array} \qquad (50)$$

The symbol Δ_m from (50) has the meaning:

$$\Delta_M = \left\{ \{-1;\text{General motion}\}; \{0;\text{Translation}\}; \{1;\text{Rotation}\} \right\}$$

In the equation above are marked out the mass distribution properties, where M_i is the mass corresponding to each kinetic link of the robot, ${}^iI_i^*$ is the axial and centrifugal inertial tensor (48) and ${}^iI_{pi}^*$ represents the planar centrifugal inertial tensor corresponding to the entire kinetic assembly (i), relative to frame $\{i\}$, applied in the mass center of each link C_i:

$$\left\{ {}^iI_{pi}^* = \int {}^i\overline{r}_i^* \cdot {}^i\overline{r}_i^{*T} dm \right\} = \begin{bmatrix} {}^iI_{xx}^* & {}^iI_{xy}^* & {}^iI_{xz}^* \\ {}^iI_{yx}^* & {}^iI_{yy}^* & {}^iI_{yz}^* \\ {}^iI_{zx}^* & {}^iI_{zy}^* & {}^iI_{zz}^* \end{bmatrix} \quad (51)$$

$${}^i\dot{\overline{v}}_{C_i} = {}_0^i[R] \cdot {}^0\dot{\overline{v}}_0^* + {}^i\dot{\overline{v}}_i + {}^i\dot{\overline{\omega}}_i \times {}^i\overline{r}_{C_i} + {}^i\overline{\omega}_i \times {}^i\overline{\omega}_i \times {}^i\overline{r}_{C_i}$$

In the same expression ${}^i\dot{\overline{v}}_{C_i}$ defines the acceleration of the mass center, while ${}^i\overline{\omega}_i$, ${}^i\dot{\overline{\omega}}_i$ and ${}^i\dot{\overline{v}}_i$ are substituted by (18) – (23).

3.3 The acceleration energy of second order

Therefore, the dynamic analysis requires higher order differential, of at least third order. Corresponding to those equations, in the following the new formulation [4] for the acceleration energy of second order is presented:

$$E_A^{(2)} = \frac{1}{2} \cdot \int {}^i\ddot{\overline{v}}_i^T \cdot {}^i\ddot{\overline{v}}_i \cdot dm = \frac{1}{2} \cdot Trace\left(\dddot{\overline{r}} \cdot \dddot{\overline{r}}^T \right) \cdot dm \; ;(52)$$

where $\ddot{\overline{v}}_i = \dddot{\overline{r}}_i$ is the absolute acceleration of second order for the elementary mass dm, belonging to body (S_i), and $i = 1 \to n$:

$$\left\{ \begin{array}{c} E_A^{(2)}\left(\overline{\theta}; \dot{\overline{\theta}}; \ddot{\overline{\theta}}; \dddot{\overline{\theta}} \right) = \sum\limits_{i=1}^{n} \Big[\frac{1}{2} \cdot M_i \cdot {}^i\ddot{\overline{v}}_{C_i}^T \cdot {}^i\ddot{\overline{v}}_{C_i} + \\ + \frac{1}{2} \cdot {}^i\ddot{\overline{\omega}}_i^T \cdot {}^iI_i^* \cdot {}^i\ddot{\overline{\omega}}_i + 2 \cdot {}^i\overline{\omega}_i^T \cdot \left({}^i\ddot{\overline{\omega}}_i \times {}^iI_{pi}^* \cdot {}^i\dot{\overline{\omega}}_i \right) + \\ + {}^i\dot{\overline{\omega}}_i^T \cdot \left({}^i\ddot{\overline{\omega}}_i \times {}^iI_{pi}^* \cdot {}^i\overline{\omega}_i \right) - \\ - {}^i\overline{\omega}_i^T \cdot \left({}^i\ddot{\overline{\omega}}_i^T \cdot {}^iI_i^* \cdot {}^i\overline{\omega}_i \right) \cdot {}^i\overline{\omega}_i \Big] + \\ + \sum\limits_{i=1}^{n} \Big[2 \cdot {}^i\dot{\overline{\omega}}_i^T \cdot \left({}^i\overline{\omega}_i^T \cdot {}^iI_i^* \cdot {}^i\dot{\overline{\omega}}_i \right) \cdot {}^i\overline{\omega}_i + \\ + {}^i\overline{\omega}_i^T \cdot \left[{}^i\overline{\omega}_i^T \cdot \left({}^i\dot{\overline{\omega}}_i \times {}^iI_{pi}^* \cdot {}^i\overline{\omega}_i \right) \right] \cdot {}^i\overline{\omega}_i \Big] + \\ + \sum\limits_{i=1}^{n} \Big[\frac{1}{2} \cdot {}^i\overline{\omega}_i^T \cdot \left[{}^i\overline{\omega}_i^T \cdot \left({}^i\overline{\omega}_i^T \cdot {}^iI_i^* \cdot {}^i\overline{\omega}_i \right) \cdot {}^i\overline{\omega}_i \right] \cdot {}^i\overline{\omega}_i \Big] \end{array} \right\} (53)$$

In the defining equation of the acceleration energy of the second order, there are:

$$\begin{array}{c} {}^i\ddot{\overline{v}}_{C_i} = {}^i\ddot{\overline{v}}_i + {}^i\ddot{\overline{\omega}}_i \times {}^i\overline{r}_{C_i} + 2 \cdot {}^i\dot{\overline{\omega}}_i \times \left({}^i\overline{\omega}_i \times {}^i\overline{r}_{C_i} \right) + \\ + {}^i\overline{\omega}_i \times \left({}^i\dot{\overline{\omega}}_i \times {}^i\overline{r}_{C_i} \right) + {}^i\overline{\omega}_i \times \left[{}^i\overline{\omega}_i \times \left({}^i\overline{\omega}_i \times {}^i\overline{r}_{C_i} \right) \right] \end{array}$$

the acceleration of second order of the mass center for the kinetic ensemble (i), while ${}^i\overline{\omega}_i$, ${}^i\dot{\overline{\omega}}_i$, ${}^i\ddot{\overline{v}}_i$ and are substituted by (18) – (23).

4. DIFFERENTIAL PRINCIPLES

According to previously presented sections, the advanced study performed on multibody mechanical systems has proved the existence of some superior forms of motion energies, corresponding to higher order accelerations. Therefore, there were determined the expressions for the acceleration energy of first $E_A^{(1)}$ and second order $E_A^{(2)}$, defined with (50) and (53) in an explicit form that can be applied for multibody mechanical systems. Forward there are going to be presented new formulations regarding the generalized Gibbs - Appell's equations and differential equations of third order. They are based on D'Alembert - Lagrange's principle, corresponding to MBS. Considering the symbols from the Fig.2, this is written below as follows:

$$\left\{ \begin{array}{c} \sum\limits_{i=1}^{n} M_i \cdot \dot{\overline{v}}_{C_i}^T \cdot \frac{\partial \overline{r}_{C_i}}{\partial q_j} + \\ + \sum\limits_{i=1}^{n} \left(I_i^* \cdot \dot{\overline{\omega}}_i - \overline{\omega}_i \times I_i^* \cdot \overline{\omega}_i \right)^T \cdot \frac{\partial \overline{\Omega}_i}{\partial q_j} \cdot \Delta_j = \\ = \sum\limits_{i=1}^{n} \overline{F}_i^{*T} \cdot \frac{\partial \overline{r}_{C_i}}{\partial q_j} + \sum\limits_{i=1}^{n} \overline{N}_i^{*T} \cdot \frac{\partial \overline{\Omega}_i}{\partial q_j} \cdot \Delta_j \end{array} \right\} ; \quad (54)$$

According to (26) and (27), the position vectors of the mass center and the orientation column vector form (54) have the following significances:

$$\overline{r}_{C_i}(t) = \overline{r}_{C_i}\left[q_j(t),\ j = 1 \to k \right]; \quad (55)$$

$$\overline{\Omega}_i(t) = \overline{\Omega}_i\left[q_j(t) \cdot \Delta_j,\ j = 1 \to k \right].$$

The right member from (54), symbolized as:

$$Q_j^* = \sum_{i=1}^{n} \overline{F}_i^{*T} \cdot \frac{\partial \overline{r}_{C_i}}{\partial q_j} + \sum_{i=1}^{n} \overline{N}_i^{*T} \cdot \frac{\partial \overline{\Omega}_i}{\partial q_j} \cdot \Delta_j \; ; \quad (56)$$

is known as the generalized force, see [2] – [8].

4.1 The Generalized Appell's Equations

According to references [5] and [6], the Gibbs — Appell's equations of motion have been established by taking into study a discrete system of material points, subjected to holonomous and nonholonomous linkages. In this section, there are presented a few formulations regarding the generalized Appell's equations, by extending the study in the area of holonomous multibody mechanical systems.

The expression for the acceleration energy of first order (50) is also obtained by applying D'Alembert – Lagrange's principle, in the case of the multibody systems [4]. So, after a few differential transformations in the left member from (54), the acceleration energy of first order is a function of the generalized accelerations:

$$\left\{\begin{aligned}&\sum_{i=1}^{n} M_i \cdot \dot{\bar{v}}_{C_i}^{T} \cdot \frac{\partial \bar{r}_{C_i}}{\partial q_j} + \sum_{i=1}^{n}\left(I_i^{*} \cdot \dot{\bar{\omega}}_i + \bar{\omega}_i \times I_i^{*} \cdot \bar{\omega}_i\right)^{T} \cdot \frac{\partial \bar{\Omega}_i}{\partial q_j} \cdot \Delta_j \\ &= \frac{\partial}{\partial \ddot{q}_j}\left[\frac{1}{2}\sum_{i=1}^{n} M_i \cdot \dot{\bar{v}}_{C_i}^{T} \cdot \dot{\bar{v}}_{C_i} + \frac{1}{2}\sum_{i=1}^{n} \dot{\bar{\omega}}_i^{T} \cdot I_i^{*} \cdot \dot{\bar{\omega}}_i + \right. \\ &\left. + \sum_{i=1}^{n} \dot{\bar{\omega}}_i^{T} \cdot \left(\bar{\omega}_i \times I_i^{*} \cdot \bar{\omega}_i\right)\right] = \frac{\partial E_A^{(1)}}{\partial \ddot{q}_j}\end{aligned}\right\} \tag{57}$$

Therefore, the differential equations of mechanical motion for the multibody system are in fact a generalization of the Gibbs –Appell equations:

$$\frac{\partial E_A^{(1)}}{\partial \ddot{q}_j} = \frac{d}{dt}\frac{\partial E_C}{\partial \dot{q}_j} - \frac{\partial E_C}{\partial q_j} = Q_j^{*},\ where\ \ j = 1 \rightarrow k \tag{58}$$

In keeping with [7], the differential equations (58) can be obtained, using the development:

$$\begin{aligned}&\frac{1}{m} \cdot \left[\frac{\partial \overset{(m)}{E_C}}{\partial \overset{(m+1)}{q_j}} - (m+1) \cdot \frac{\partial E_C}{\partial q_j}\right] = \\ &= \frac{d}{dt}\frac{\partial E_C}{\partial \dot{q}_j} - \frac{\partial E_C}{\partial q_j} = Q_j^{*},\quad where\ \ m = 1,2,\ldots\end{aligned} \tag{59}$$

In this paper, a new formulation is proposed:

$$\begin{aligned}&\frac{\partial \overset{(m-1)}{E_A^{(1)}}}{\partial \overset{(m+1)}{q_j}} + Q_g^j = \frac{d}{dt}\frac{\partial E_C}{\partial \dot{q}_j} - \frac{\partial E_C}{\partial q_j} + Q_g^j = Q_j^{*} = \\ &= Q_m^j\left[\bar{\theta}(t); \dot{\bar{\theta}}(t); \ddot{\bar{\theta}}(t)\right];\ \overset{(0)}{E_A^{(1)}} = E_A^{(1)};\ m = 1,2,\ldots\end{aligned} \tag{60}$$

In the above equations, the symbol (m) represents the order of the time derivative. But, unlike (59), the development (60) is based on the acceleration energy of first order. In these equations, the symbols Q_g^j and Q_m^j are the generalized gravitational and driving forces, according to [2] and [4].

4.2 The differential equations of third order

In order to determine the differential equations of third order, a few considerations must be done. The starting equation is (54). Applying a few differential transformations, the D'Alembert – Lagrange's principle is changed:

$$\begin{aligned}&\sum_{i=1}^{n}\left(\dot{\bar{F}}_i^{*} - M_i \cdot \ddot{\bar{v}}_{C_i}\right) \cdot \frac{\partial \bar{r}_{C_i}}{\partial q_j} + \sum_{i=1}^{n}\left(\bar{F}_i^{*} - M_i \cdot \dot{\bar{v}}_{C_i}\right) \cdot \frac{d}{dt}\left(\frac{\partial \bar{r}_{C_i}}{\partial q_j}\right) \\ &+ \sum_{i=1}^{n}\left[\dot{\bar{N}}_i^{*} - \frac{d}{dt}\left(I_i^{*} \cdot \dot{\bar{\omega}}_i + \bar{\omega}_i \times I_i^{*} \cdot \bar{\omega}_i\right)\right] \cdot \frac{\partial \bar{\Omega}_i}{\partial q_j} \cdot \Delta_j + \\ &+ \sum_{i=1}^{n}\left[\bar{N}_i^{*} - I_i^{*} \cdot \dot{\bar{\omega}}_i - \bar{\omega}_i \times I_i^{*} \cdot \bar{\omega}_i\right] \cdot \frac{d}{dt}\left(\frac{\partial \bar{\Omega}_i}{\partial q_j} \cdot \Delta_j\right) = 0\end{aligned} \tag{61}$$

where, $\dot{\bar{F}}_i^{*}$ and $\dot{\bar{N}}_i^{*}$ represent the first time derivative of the components of the torsor of active forces. Applying some differential and matrix transformations, as in [4] in (61), it results the following expression:

$$\begin{aligned}&\sum_{i=1}^{n} M_i \cdot \ddot{\bar{v}}_{C_i} \cdot \frac{\partial \bar{r}_{C_i}}{\partial q_j} + \sum_{i=1}^{n}\frac{d}{dt}\left(I_i^{*} \cdot \dot{\bar{\omega}}_i + \bar{\omega}_i \times I_i^{*} \cdot \bar{\omega}_i\right) \cdot \frac{\partial \bar{\Omega}_i}{\partial q_j} \cdot \Delta_j + \\ &+ \sum_{i=1}^{n} M_i \cdot \dot{\bar{v}}_{C_i}^{T} \cdot \frac{d}{dt}\left(\frac{\partial \bar{r}_{C_i}}{\partial q_j}\right) + \\ &+ \sum_{i=1}^{n}\left(I_i^{*} \cdot \dot{\bar{\omega}}_i + \bar{\omega}_i \times I_i^{*} \cdot \bar{\omega}_i\right)^{T} \cdot \frac{d}{dt}\left(\frac{\partial \bar{\Omega}_i}{\partial q_j} \cdot \Delta_j\right) = \\ &= \frac{\partial}{\partial \dddot{q}_j}\left[\frac{1}{2} \cdot \sum_{i=1}^{n} M_i \cdot \ddot{\bar{v}}_{C_i}^{T} \cdot \ddot{\bar{v}}_{C_i} + \frac{1}{2} \cdot \sum_{i=1}^{n} \ddot{\bar{\omega}}_i^{T} \cdot I_i^{*} \cdot \ddot{\bar{\omega}}_i + \right. \\ &+ 2 \cdot \sum_{i=1}^{n} \bar{\omega}_i^{T} \cdot \left(\ddot{\bar{\omega}}_i \times I_{pi}^{*} \cdot \dot{\bar{\omega}}_i\right) \\ &\left. + \sum_{i=1}^{n} \dot{\bar{\omega}}_i^{T} \cdot \left(\ddot{\bar{\omega}}_i \times I_{pi}^{*} \cdot \bar{\omega}_i\right) - \sum_{i=1}^{n} \bar{\omega}_i^{T} \cdot \left(\ddot{\bar{\omega}}_i \cdot I_i^{*} \cdot \bar{\omega}_i\right) \cdot \bar{\omega}_i\right] + \\ &+ \frac{\partial}{\partial \dot{q}_j}\left(\frac{1}{2} \cdot \sum_{i=1}^{n} M_i \cdot \dot{\bar{v}}_{C_i}^{T} \cdot \dot{\bar{v}}_{C_i} + \frac{1}{2}\sum_{i=1}^{n} \dot{\bar{\omega}}_i^{T} \cdot I_i^{*} \cdot \dot{\bar{\omega}}_i \cdot \Delta_j \right. \\ &\left. + \sum_{i=1}^{n}\left(\bar{\omega}_i \times I_i^{*} \cdot \bar{\omega}_i\right)^{T} \cdot \dot{\bar{\omega}}_i \cdot \Delta_j\right)\end{aligned} \tag{62}$$

According to [4], it can be remarked that the right member from (62) contains the acceleration energy of second order (53), but in this case, it is defined as a function of the generalized accelerations of second order as follows:

$$E_A^{(2)i} = \frac{1}{2}\cdot M_i \cdot \ddot{\bar{v}}_{C_i}^T \cdot \ddot{\bar{v}}_{C_i} + \frac{1}{2}\cdot \ddot{\bar{\omega}}_i^T \cdot I_i^* \cdot \ddot{\bar{\omega}}_i + \\ +2\cdot \bar{\omega}_i^T \cdot \left(\ddot{\bar{\omega}}_i \times I_{pi}^* \cdot \dot{\bar{\omega}}_i\right) + \\ +\dot{\bar{\omega}}_i^T \cdot \left(\ddot{\bar{\omega}}_i \times I_{pi}^* \cdot \bar{\omega}_i\right) - \bar{\omega}_i^T \cdot \left(\ddot{\bar{\omega}}_i \cdot I_i^* \cdot \bar{\omega}_i\right)\cdot \bar{\omega}_i \tag{63}$$

The differential equations of motion, based on acceleration energy of second order, can be defined in the following form:

$$\left\{ \begin{array}{l} \dfrac{\partial E_A^{(2)}}{\partial \ddot{q}_j} = \dfrac{d}{dt}\dfrac{\partial E_A^{(1)}}{\partial \ddot{q}_j} - \dfrac{1}{2}\cdot\dfrac{\partial E_A^{(1)}}{\partial \dot{q}_j} = \\ = \dfrac{d^2}{dt^2}\left(\dfrac{\partial E_C}{\partial \dot{q}_j}\right) - \dfrac{d}{dt}\left(\dfrac{\partial E_C}{\partial q_j}\right) - \dfrac{1}{2}\cdot\dfrac{\partial E_A^{(1)}}{\partial \dot{q}_j} \end{array} \right\} \tag{64}$$

$$\left\{ \begin{array}{l} \dfrac{\partial E_A^{(2)}}{\partial \ddot{q}_j} + \dfrac{1}{2}\cdot\dfrac{\partial E_A^{(1)}}{\partial \dot{q}_j} + \dot{Q}_g^j = \\ = \dot{Q}_m^j\left[\bar{\theta}(t);\dot{\bar{\theta}}(t);\ddot{\bar{\theta}}(t);\dddot{\bar{\theta}}(t)\right] \end{array} \right\} \tag{65}$$

Considering the mathematical connection between the kinetic energy and acceleration energy [4], finally, the differential equations of third order are defined under the form:

$$\left\{ \begin{array}{l} \dfrac{\partial E_A^{(2)}}{\partial \ddot{q}_j} + \left(\dfrac{1}{2}\cdot\displaystyle\sum_{i=1}^{n}\dfrac{\partial^3 E_C}{\partial \dot{q}_j \cdot \partial \dot{q}_i^2}\cdot \ddot{q}_i^2 + \right. \\ +\displaystyle\sum_{i=1}^{n-1}\sum_{k=i+1}^{n}\dfrac{\partial^3 E_C}{\partial \dot{q}_j \cdot \partial \dot{q}_i \cdot \partial \dot{q}_k}\cdot \ddot{q}_i \cdot \ddot{q}_k + \\ \left. +\displaystyle\sum_{i=1}^{n}\sum_{k=1}^{n}\dfrac{\partial^3 E_C}{\partial \dot{q}_j \cdot \partial \dot{q}_i \cdot \partial q_k}\cdot \ddot{q}_i \cdot \dot{q}_k - \sum_{i=1}^{n}\dfrac{\partial^2 E_C}{\partial \dot{q}_j \cdot \partial q_i}\cdot \ddot{q}_i \right) \\ = \dot{Q}_m^j\left[\bar{\theta}(t);\dot{\bar{\theta}}(t);\ddot{\bar{\theta}}(t);\dddot{\bar{\theta}}(t)\right] - \dot{Q}_g^j \end{array} \right\} \tag{66}$$

where $\dot{Q}_g^j$ and $\dot{Q}_m^j$ are the first time derivative of the generalized gravitational force and the first time derivative of the generalized driving force.

Considering the expressions (60), for the differential equations of third order, the next development is also proposed:

$$\frac{\partial \overset{(m-3)}{E_A^{(2)}}}{\partial \overset{(m)}{q_j}} + \frac{1}{2}\cdot\frac{\partial E_A^{(1)}}{\partial \dot{q}_j} + \dot{Q}_g^j = \frac{d}{dt}\left(\frac{\partial \overset{(m-4)}{E_A^{(1)}}}{\partial \overset{(m-2)}{q_j}}\right) = \\ = \dot{Q}_m^j\left[\bar{\theta}(t);\dot{\bar{\theta}}(t);\ddot{\bar{\theta}}(t);\dddot{\bar{\theta}}(t)\right], \quad m = 4,5,\ldots \tag{67}$$

where the symbol (m) represents the order of the time derivative, applied on the acceleration energy of first and second order.

5. APPLICATIONS OF FORMULATIONS ON THE DIFFERENTIAL PRINCIPLES

It is considered the mechanical structure of a 2TR serial robot, whose kinematical structure is represented within of the Fig. 3. The robot performs two translations: along $\bar{x}_0$ and $\bar{z}_0$ axes and a rotation around $\bar{z}_0$. It is required to determine the moving differential equations based on differential principles.

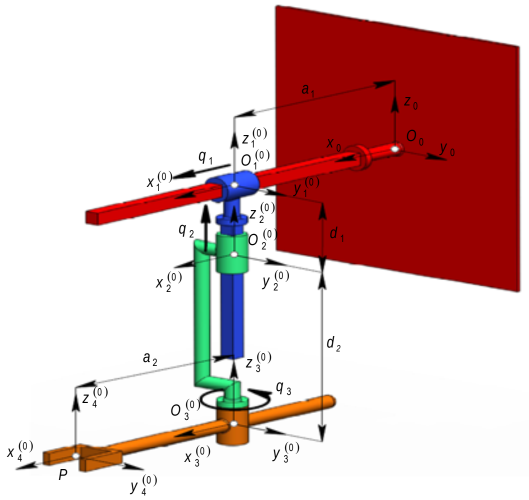

Fig. 3 The Kinematical Structure for 2TR Robot

The kinematical results are included into [3]. As a result in this paper, the kinetic energy under the explicit form is shown:

$$\left\{ \begin{array}{l} E_C = \dfrac{1}{2}\cdot\left[\left(M_1+M_2+M_3\right)\cdot\dot{q}_1^2 + \left(M_2+M_3\right)\cdot\dot{q}_2^2 + \right. \\ \qquad +M_3\cdot a_2^2\cdot\dot{q}_3^2 - \\ \qquad \left. -2\cdot\left(M_3\cdot a_2\cdot sq_3\right)\cdot\dot{q}_1\cdot\dot{q}_3 + I_z\cdot\dot{q}_3^2\right] \end{array} \right\} \tag{68}$$

The acceleration energy of first order in the same explicit form shows as:

$$\left\{ \begin{array}{l} E_A^{(1)} = \dfrac{1}{2}\cdot\left[\left(M_1+M_2+M_3\right)\cdot\ddot{q}_1^2 + \right. \\ \qquad \left. +\left(M_2+M_3\right)\cdot\ddot{q}_2^2 + M_3\cdot a_2^2\cdot\ddot{q}_3^2\right] - \\ -\left(M_3\cdot a_2\cdot\dot{q}_3^2\cdot cq_3\right)\cdot\ddot{q}_1 - \left(M_3\cdot a_2\cdot sq_3\right)\cdot\ddot{q}_1\cdot\ddot{q}_3 + \\ \dfrac{1}{2}\cdot\left[M_3\cdot a_2^2\cdot\dot{q}_3^4 + I_z\cdot\ddot{q}_3^2 + \left({}^3I_{xx} + {}^3I_{yy}\right)\cdot\dot{q}_3^4\right] \end{array} \right\} \tag{69}$$

In the above expressions there are included the mass properties, where: M_1, M_2 and M_3 represent the masses of the kinetic ensembles, I_z is the mechanical axial moment of inertia on z axis and ${}^3I_{xx}, {}^3I_{yy}$ are the mechanical planar moments of inertia of the third kinetic ensemble of the 2TR serial robot. Considering the variational principles from analytical mechanics, the dynamics equations can be also obtained by means of the differential expression for the kinetic energy. This is shown, for 2TR serial robot, as follows:

$$\left\{\begin{array}{c} \delta E_C = \left(M_1 + M_2 + M_3\right)\cdot \dot{q}_1 \cdot \delta \dot{q}_1 + \\ +\left(M_2 + M_3\right)\cdot \dot{q}_2 \cdot \delta \dot{q}_2 + \\ +M_3 \cdot a_2^2 \cdot \dot{q}_3 \cdot \delta \dot{q}_3 - \\ -\left(M_3 \cdot a_2 \cdot sq_3\right)\cdot\left(\dot{q}_3 \cdot \delta \dot{q}_1 + \dot{q}_1 \cdot \delta \dot{q}_3\right) - \\ -\left(M_3 \cdot a_2 \cdot cq_3\right)\cdot \dot{q}_1 \cdot \dot{q}_3 \cdot \delta q_3 + I_z \cdot \dot{q}_3 \cdot \delta \dot{q}_3 \end{array}\right\} \tag{70}$$

Taking into account the above considerations, the differential expression for the acceleration energy of first order for the 2TR serial robot is:

$$\left\{\begin{array}{c} \delta E_A = \left(M_1 + M_2 + M_3\right)\cdot \ddot{q}_1 \cdot \delta \ddot{q}_1 + \\ +\left(M_2 + M_3\right)\cdot \ddot{q}_2 \cdot \delta \ddot{q}_2 + \\ +M_3 \cdot a_2^2 \cdot \ddot{q}_3 \cdot \delta \ddot{q}_3 - \\ -\left(M_3 \cdot a_2 \cdot \dot{q}_3^2 \cdot cq_3\right)\cdot \ddot{q}_1 \cdot \delta \ddot{q}_1 - \\ -\left(M_3 \cdot a_2 \cdot sq_3\right)\cdot\left(\ddot{q}_3 \cdot \delta \ddot{q}_1 + \ddot{q}_1 \cdot \delta \ddot{q}_3\right) + \\ +I_z \cdot \ddot{q}_3 \cdot \delta \ddot{q}_3 \end{array}\right\} \tag{71}$$

The symbol δ from (70) and (71) has the significance of virtual differentiation operator. Considering (58) and (60), and then performing the calculus, the differential motion equations of the second order for the 2TR serial robot are:

$$\left[\begin{array}{c} Q_m^1 = \left(M_1 + M_2 + M_3\right)\cdot \ddot{q}_1 - M_3 \cdot a_2 \cdot sq_3 \cdot \ddot{q}_3 - \\ -M_3 \cdot a_2 \cdot \dot{q}_3^2 \cdot cq_3 \end{array}\right] \tag{72}$$

$$Q_m^2 = \left(M_2 + M_3\right)\cdot\left(\ddot{q}_2 + g\right); \tag{73}$$

$$Q_m^3 = M_3 \cdot a_2^2 \cdot \ddot{q}_3 - M_3 \cdot a_2 \cdot sq_3 \cdot \ddot{q}_1 + {}^3I_z \cdot \ddot{q}_3 \tag{74}$$

Expressions (72)-(74) highlight the generalized driving forces, characterizing the dynamic behavior of the *2TR* serial robot structure.

The acceleration energy of second order for the same *2TR* serial robot structure is:

$$\left\{\begin{array}{c} E_A^{(2)} = \frac{1}{2}\cdot\left[\left(M_1 + M_2 + M_3\right)\cdot \dddot{q}_1^2 + \left(M_2 + M_3\right)\cdot \dddot{q}_2^2 + \right. \\ +M_3 \cdot a_2^2 \cdot \dddot{q}_3^2 - \\ -I_z \cdot \dot{q}_3^3 \cdot \dddot{q}_3 - M_3 \cdot a_2^2 \cdot \dot{q}_3^3 \cdot \dddot{q}_3 - \\ -\left(M_3 \cdot a_2 \cdot sq_3\right)\cdot \dddot{q}_1 \cdot \dddot{q}_3 + \left(M_3 \cdot a_2 \cdot sq_3\right)\cdot \dddot{q}_1 \cdot \dot{q}_3^3 - \\ -3\cdot\left(M_3 \cdot a_2 \cdot \dot{q}_3 \cdot cq_3\right)\cdot \dddot{q}_1 \cdot \ddot{q}_3 + \\ \left. +\frac{1}{2}\cdot\left[M_3 \cdot a_2^2 \cdot \dot{q}_3^6 + 9\cdot M_3 \cdot a_2^2 \cdot \dot{q}_3^2 \cdot \ddot{q}_3^2 + I_z \cdot \dddot{q}_3^2\right]\right. \end{array}\right\} \tag{75}$$

This expression is obtained by applying (53). In the following, using (63), (65) and (67) the equations for the first time derivatives of the generalized driving forces are determined. They are written in the next mathematical form:

$$\left[\begin{array}{c} \dot{Q}_m^1 = \left(M_1 + M_2 + M_3\right)\cdot \dddot{q}_1 - M_3 \cdot a_2 \cdot sq_3 \cdot \dddot{q}_3 - \\ -3\cdot M_3 \cdot a_2 \cdot \ddot{q}_3 \cdot cq_3 \cdot \dot{q}_3 + M_3 \cdot a_2 \cdot sq_3 \cdot \dot{q}_3^3 \end{array}\right] \tag{76}$$

$$\dot{Q}_m^2 = \left(M_2 + M_3\right)\cdot\left(\dddot{q}_2 + g\right) \tag{77}$$

$$\left[\begin{array}{c} \dot{Q}_m^3 = M_3 \cdot a_2^2 \cdot \dddot{q}_3 - M_3 \cdot a_2 \cdot sq_3 \cdot \dddot{q}_1 - \\ -M_3 \cdot a_2 \cdot cq_3 \cdot \dot{q}_3 \cdot \ddot{q}_1 + {}^3I_z \cdot \dddot{q}_3 \end{array}\right]. \tag{78}$$

The equations (76)-(78) are also differential equations of third order that define the mechanical motion of the analyzed serial robot.

6. CONCLUSIONS

Within this paper, a few new formulations regarding advanced dynamics of multibody systems have been presented. In order to achieve this goal, in the first part of the paper, the forward kinematics equations of multibody systems have been presented as an algorithm. These equations have been developed using matrix exponentials that have undeniable advantages in the matrix study of any complex mechanical system. The study continued in this paper with the use of exponentials to find the resultant rotation matrix in a generalized form. The kinematic parameters expressions from the first part of the paper were used to express the energies of higher order. Therefore, new formulations for acceleration energy of first and second order were presented in this paper in an explicit form. The same expressions of energies of higher order mentioned above were then established using the D'Alembert-Lagrange's principle for multibody systems. Therefore, the paper presented a generalization of Gibbs-Appell's equations identical as expression with Lagrange's equations of second kind. Using the differential principle of D'Alembert-Lagrange, the third order differential equations of motion were established. In order to illustrate the essentials of higher order energies expressions, in the third part of the paper, an application regarding dynamics equations in the case of a serial structure for a robot of 2TR-type was presented. As a result of this application, the expressions of generalized driving forces of second order that have the generalized accelerations of second order as components, have been established in the symbolical form.

7. REFERENCES

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Formulări în dinamica avansată a sistemelor

Rezumat: În cadrul acestei lucrări sunt analizate, pe baza unor noi formulări, niște noțiuni importante de dinamică, cum ar fi energia accelerațiilor de ordinul întâi și de ordinul al doilea, precum și principii diferențiale din mecanica analitică. Acestea au fost aplicate pentru sisteme mecanice multicorp, cum ar fi roboții. La începutul lucrării, a fost prezentată pe scurt aplicarea funcțiilor exponențiale de matrice. Având la bază principiul lui D'Alembert-Lagrange în formă generalizată, au fost descrise câteva formulari ale ecuațiilor diferențiale de mișcare de ordinul al treilea. În partea finală a acestei lucrări, formulările au fost aplicate asupra structurii robotice de tipul 2TR.

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