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MASS DISTRIBUTION IN ANALYTICAL DYNAMICS OF SYSTEMS

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Abstract: *In the case of the multibody systems (MBS), as example the mechanical robot structure, a few simplifying hypotheses, referring to mass properties, are implemented. According to these, the mass properties are continuously distributed between the fixed basis and the last kinetic ensemble from mechanical structure. As a result, in the dynamical study of MBS, the author of the paper has introduced the phrase "mass distribution" instead of mass geometry, typically too rigid solid. Mass distribution is based on the mass as fundamental notions in analytical dynamics of systems. At its turn, mass together with energy highlight the matter notion. But, mass is also highlighted by means of the two properties: gravitation and inertia. According to fundamental theorems from Newtonian dynamics, in the case of the translation motion the inertia property is highlighted by mass and position of the mass center. In the case of the resultant rotation motion the inertia property is characterized by mechanical moments of inertia and extension of these properties, known as inertial tensors and pseudoinertial tensors. Their matrix expressions are compulsory included in the dynamical notions like: kinetic energy, acceleration energy, angular momentum and their time derivatives according to differential equations of higher order, typically to analytical dynamics of systems.*

Key words: *analytical dynamics, mechanics, mass distribution, dynamics equations.*

1. INTRODUCTION

The multibody systems (MBS), and as result the mechanical robot structure, are characterized through the mass properties continuously distributed between the fixed basis and the last kinetic ensemble from mechanical structure [2] and [3]. Consequently, the author of the paper has introduced, in the dynamical study, the phrase "mass distribution" instead of mass geometry, typically too rigid body. Physically, mass distribution (MD) is based on the mass. At its turn, mass and energy highlight the matter notion. But, mass, as fundamental notion in analytical dynamics, is also highlighted by means of the two properties: gravitation and inertia. According to fundamental theorems from Newtonian dynamics, in the case of the translation motion the inertia is highlighted by mass and position of the mass center. In the case of the resultant rotation motion the inertia property is characterized by mechanical moments of inertia (Leonard Euler, 1758 year) and extension of these properties, known as inertial tensor and pseudoinertial tensor [2] and [8]. Their matrix expressions are compulsory included in the dynamical notions like: angular momentum, kinetic energy, acceleration energy of higher order, as well as their time derivatives, according to differential equations of higher order, typically to analytical dynamics [1] - [9].

In this paper, using the author researches, the properties of mass distribution (MD – type) will be developed [2] - [8]. In view of this, a few simplifying hypotheses are compulsory applied. So, every kinetic ensemble, belonging to MBS, is considered a rigid body, see Fig.1.

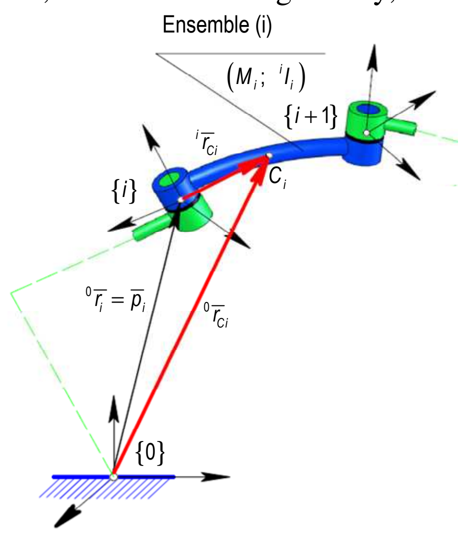

Fig.1 Kinetic Ensemble from MBS

In the same time, MD – type properties are based on mass and geometrical integrals. These can be applied, in exclusivity, in the case of the homogeneous bodies with a simple or regular geometrical shape. But, any homogeneous body is characterized by the constant density in the infinity of elementary mass (dm) continuous distributed inside of its geometrical shape:

$$dm = \{\rho_V \cdot dV;\ \rho_A \cdot dA;\ \rho_L \cdot dL\}. \tag{1}$$

According to set (1), mass (dm) and density are functions of geometry of the homogeneous body. Usually, every kinetic ensemble is a compound body with non-regular geometrical shape, for which mass integrals cannot be applied. As example, the kinetic ensemble $i = 1 \rightarrow n$ (Fig. 1) belonging to MBS is considered. Its geometrical shape shows as Fig. 2 and Fig.3.

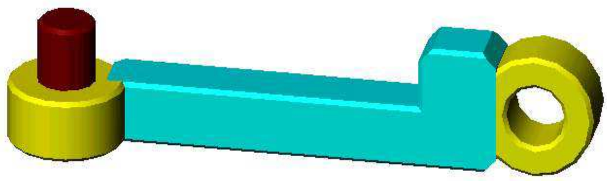

Fig.2 Kinetic Ensemble

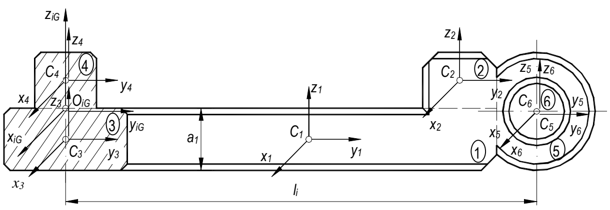

Fig.3 Kinetic Ensemble discomposed

As a result of the above aspects, every kinetic ensemble $i = 1 \rightarrow n$ is discomposed in a finite number of homogeneous bodies with a simple or regular geometrical shape, symbolized: $(j) \in (i)$, $j = 1 \rightarrow p_i$, and $p_i \in N\,(natural\ numbers)$.

On the basis of the above considerations and the formulations of the author, in this paper the MD – type properties will be develop: mass and position of the mass center, mechanical moments of inertia (axial, planar, polar and centrifugal moments of inertia), inertial tensor and its generalized variation law, pseudo inertial tensor, as well as the algorithm of the mass properties corresponding to MBS. Among of these, inertial tensor is a squared matrix of the mechanical moments of inertia, while the pseudoinertial tensor is a squared matrix of mass moments of zero, first and second order [2].

2. MASS. POSITION of MASS CENTER

Considering the geometrical and mass aspects from the first section, for define mass and position of the mass center corresponding to a kinetic ensemble $i = 1 \rightarrow n$, in the first step this is divided in $j = 1 \rightarrow p_i$ homogeneous body $(j) \in (i)$ with a simple geometrical shape [2]. So, for beginning mass and position of the mass center is determined for every homogeneous body $(j) \in (i)$, represented as example in Fig. 4.

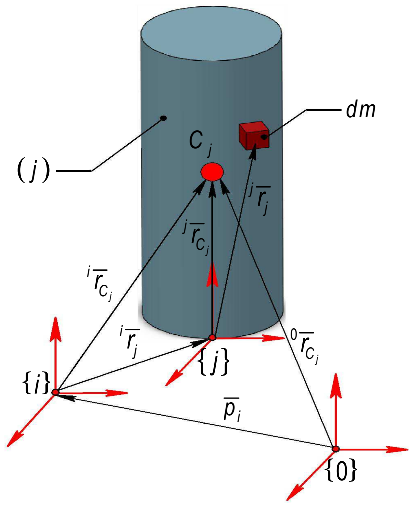

Fig. 4 Homogeneous Body $(j) \in (i)$

To this body is attached a Cartesian frame $\{j\}$, and then is discomposed in infinity of elementary mass (dm) continuous distributed inside of its geometrical shape. One out of these is positioned through ${}^{j}\bar{r}_j$. The total mass is:

$$m_j = \int dm; \tag{2}$$

where (dm) is substituted by (1) in function of geometry of the body. According to [2] and [4], the position of the mass center is defined with:

$${}^{j}\bar{r}_{C_j} = \frac{\int {}^{j}\bar{r}_j \cdot dm}{m_j}. \tag{3}$$

But, the position of the mass center C_j must be determined to frames $\{i\}$ or $\{0\}$ corresponding to kinematical structure of MBS. As a result, the expressions are written below as follows:

$${}^{(0)i}\bar{r}_{C_j} = {}^{(0)i}\bar{r}_j + {}^{(0)i}_{\ \ j}[R] \cdot {}^{j}\bar{r}_{C_j}; \tag{4}$$

$$\begin{bmatrix} {}^{(0)i}\bar{r}_{C_j} \\ 1 \end{bmatrix} = {}^{(0)i}_{\ \ j}[T] \cdot \begin{bmatrix} {}^{j}\bar{r}_{C_j} \\ 1 \end{bmatrix}; \tag{5}$$

where sometimes $\{j\}_{OR} \equiv \{i\}_{OR}$ is recommended. According to [2], the symbol: ${}^{(0)i}_{j}[R]$ and ${}^{(0)i}_{j}[T]$ expresses the rotation and respectively locating matrix between frames $\{j\}$ and $\{i\}$ or $\{0\}$ (Fig.4). When the above expressions (2), (3), (4) or (5) are applied for all bodies $j=1 \to p_i$, then mass and position of the mass center of the kinetic ensemble $i=1 \to n$ will be determined below as:

$$M_i = \sum_{j=1}^{p_i} \sigma_j \cdot m_j \,, \tag{6}$$

where $$\sigma_j = \{(1,\, j \in i);(0;\, j \notin i)\}; \tag{7}$$

$${}^{(0)i}\overline{r}_{C_i} = \frac{\sum_{j=1}^{p_i} \sigma_j \cdot {}^{(0)i}\overline{r}_{C_j} \cdot m_j}{M_i}\,, \tag{8}$$

$$\begin{bmatrix} {}^{0}\overline{r}_{C_i} \\ 1 \end{bmatrix} = {}^{0}_{i}[T] \cdot \begin{bmatrix} {}^{i}\overline{r}_{C_i} \\ 1 \end{bmatrix}. \tag{9}$$

Expression (6) is devoted to establishment the total mass of the kinetic ensemble, while (8) or (9) is corresponding to position of mass center.

3. INERTIAL TENSOR for RIGID BODY

When the kinetic ensemble $i=1 \to n$, from MBS, is characterized by the resultant rotation motion, then the inertia property is highlighted by mechanical moments of inertia and extension of these properties, known as inertial tensor and pseudoinertial tensor [2] - [4]. Similarly with the previous section, for beginning the above inertia properties will be determined in the case to every homogeneous body $(j) \in (i)$, example Fig.5.

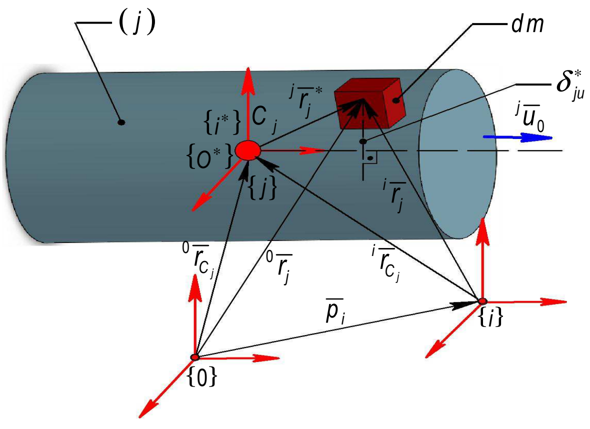

Fig. 5 Homogeneous Body $(j) \in (i)$

Every homogeneous body $(j) \in (i)$ was analyzed from view point of the mass (2) and position of the mass center (3), (4) or (5). As a result, mass m_j and mass center C_j are well defined. In keeping with Fig.5, mass center C_j is the origin for three concurrent frames, below presented:

$$C_j \in \left\{ \{j\};\, \{i^*\}_{OR} \equiv \{i\}_{OR};\, \{0^*\}_{OR} \equiv \{0\}_{OR} \right\}. \tag{10}$$

Considering the formulations of the author, a few symbols and notations are implemented:

$$u_j = \{x_j; y_j; z_j\};\; v_j = \{y_j; z_j; x_j\};\; w_j = \{z_j; x_j; y_j\} \tag{11}$$

$$\left\{ \begin{array}{l} {}^{(0)i}\overline{u}_j = \left\{ {}^{(0)i}\overline{x}_j; {}^{(0)i}\overline{y}_j; {}^{(0)i}\overline{z}_j \right\} \\ {}^{j}\overline{u}_{i(0)} = \left\{ {}^{j}\overline{x}_{i(0)}; {}^{j}\overline{y}_{i(0)}; {}^{j}\overline{z}_{i(0)} \right\} \end{array} \right\}, \tag{12}$$

$$\left\{ \begin{array}{l} {}^{(0)i}\overline{v}_j = \left\{ {}^{(0)i}\overline{y}_j; {}^{(0)i}\overline{z}_j; {}^{(0)i}\overline{x}_j \right\} \\ {}^{j}\overline{v}_{i(0)} = \left\{ {}^{j}\overline{y}_{i(0)}; {}^{j}\overline{z}_{i(0)}; {}^{j}\overline{x}_{i(0)} \right\} \end{array} \right\}, \tag{13}$$

$$\left\{ \begin{array}{l} {}^{(0)i}\overline{w}_j = \left\{ {}^{(0)i}\overline{z}_j; {}^{(0)i}\overline{x}_j; {}^{(0)i}\overline{y}_j \right\} \\ {}^{j}\overline{w}_{i(0)} = \left\{ {}^{j}\overline{z}_{i(0)}; {}^{j}\overline{x}_{i(0)}; {}^{j}\overline{y}_{i(0)} \right\} \end{array} \right\}; \tag{14}$$

where (11) are Cartesian coordinates or axes, while (12), (13) and (14) are the unit vectors between frames $\{j\}$ and $\{i\}$ or $\{0\}$, respectively between $\{i\}$ or $\{0\}$ and $\{j\}$. On the basis of the above notations, the rotation matrix is written:

$${}^{(0)i}_{j}[R] = \begin{bmatrix} {}^{(0)i}\overline{x}_j & {}^{(0)i}\overline{y}_j & {}^{(0)i}\overline{z}_j \end{bmatrix} = \begin{bmatrix} {}^{j}\overline{x}^T_{i(0)} \\ {}^{j}\overline{y}^T_{i(0)} \\ {}^{j}\overline{z}^T_{i(0)} \end{bmatrix}, \tag{15}$$

$${}^{(0)i}_{j}[R]^T = \begin{bmatrix} {}^{(0)i}\overline{x}^T_j \\ {}^{(0)i}\overline{y}^T_j \\ {}^{(0)i}\overline{z}^T_j \end{bmatrix} = \begin{bmatrix} {}^{j}\overline{x}_{i(0)} & {}^{j}\overline{y}_{i(0)} & {}^{j}\overline{z}_{i(0)} \end{bmatrix}. \tag{16}$$

The body $(j) \in (i)$ is discomposed in infinity of elementary mass (dm) continuous distributed inside of its geometrical shape. According to Fig. 5, one out of these is positioned through:

$${}^{j}\overline{r}^*_j = \begin{bmatrix} {}^{j}x^*_j & {}^{j}y^*_j & {}^{j}z^*_j \end{bmatrix}^T, \tag{17}$$

and obviously there are $$\int {}^{(0)(i)j}\overline{r}^*_j \cdot dm = 0\,, \tag{18}$$

as well as $${}^{(0)i}\overline{r}_{C_j} = \frac{\int {}^{(0)i}\overline{r}_j \cdot dm}{m_j}; \tag{19}$$

$${}^{(0)i}\overline{r}^*_j = {}^{(0)i}_{j}[R] \cdot {}^{j}\overline{r}^*_j;\; {}^{j}\overline{r}^*_j = {}^{(0)i}_{j}[R]^T \cdot {}^{(0)i}\overline{r}^*_j \tag{20}$$

where (17) and (20) are the position to frames:

$$\{\{j\};\{i^*\};\{0^*\}\}\in C_j.$$

Expression (20) is rewritten below as follows:

$$\begin{bmatrix} {}^{(0)i}x_j^* \\ {}^{(0)i}y_j^* \\ {}^{(0)i}z_j^* \end{bmatrix} = \begin{bmatrix} {}^j\overline{x}_{i(0)}^T \\ {}^j\overline{y}_{i(0)}^T \\ {}^j\overline{z}_{i(0)}^T \end{bmatrix} \cdot {}^j\overline{r}_j^* . \tag{21}$$

Another notation, compulsory applied in this study, is the skew symmetric matrix associated to position vector (17), as well its transpose:

$$\left[{}^j\overline{r}_j^*\times\right] = \begin{bmatrix} 0 & -{}^jz_j^* & {}^jy_j^* \\ {}^jz_j^* & 0 & -{}^jx_j^* \\ -{}^jy_j^* & {}^jx_j^* & 0 \end{bmatrix}, \tag{22}$$

$$\left[{}^j\overline{r}_j^*\times\right]^T = -\left[{}^j\overline{r}_j^*\times\right]. \tag{23}$$

Whereas the inertia properties must defined to $\{i\}$ or $\{0\}$, elementary mass is also positioned as:

$${}^{(0)i}\overline{r}_j = {}^{(0)i}\overline{r}_{C_j} + {}^{(0)i}_{\ \ j}[R]\cdot {}^j\overline{r}_j^* ; \tag{24}$$

$${}^{(0)i}_{\ \ j}[T] = \begin{bmatrix} {}^{(0)i}_{\ \ j}[R] & {}^{(0)i}\overline{r}_{C_j} \\ [0]_{(1\times 3)} & 1 \end{bmatrix}, \tag{25}$$

$${}^{(0)i}_{\ \ j}[T]^T = \begin{bmatrix} {}^{(0)i}_{\ \ j}[R]^T & [0]_{(3\times 1)} \\ {}^{(0)i}\overline{r}_{C_j}^T & 1 \end{bmatrix}, \tag{26}$$

$$\begin{bmatrix} {}^{(0)i}\overline{r}_j \\ 1 \end{bmatrix} = {}^{(0)i}_{\ \ j}[T]\cdot \begin{bmatrix} {}^j\overline{r}_j^* \\ 1 \end{bmatrix}; \tag{27}$$

where (27) with (25) shows the position of elementary mass in homogeneous coordinates.

Since every homogeneous body $(j)\in(i)$ is considered with simple geometrical shape, the mass and geometrical integrals are applied as:

$$\left\{ \begin{matrix} {}^jI_u^* = \int\left({}^jv_j^{*2} + {}^jw_j^{*2}\right)\cdot dm \\ {}^jI_{uv}^* = \int {}^ju_j^*\cdot {}^jv_j^*\cdot dm; \quad {}^jI_{uu}^* = \int {}^ju_j^{*2}\cdot dm \end{matrix} \right\}. \tag{29}$$

So, axial, centrifugal and planar mechanical moments of inertia with respect to $\{j\}\in C_j$ are known, and they are included in the next set:

$$\left\{{}^jI_u^*;\ {}^jI_{uv}^*;\ {}^jI_{uu}^*;\ where\ j=1\rightarrow p_i;\ (j)\in(i)\right\}. \tag{30}$$

Considering the author formulations, in the next step must be determined axial, centrifugal and planar mechanical moments of inertia with respect to $\{\{i^*\};\{0^*\}\}\in C_j$, according to set:

$$\left\{{}^{(0)i}I_{ju}^*;\ {}^{(0)i}I_{juv}^*;\ {}^{(0)i}I_{juu}^*;\ where\ j=1\rightarrow p_i;\ (j)\in(i)\right\} \tag{31}$$

The moment of inertia is a mass integral, [2] - [9]. It is applied on the product between mass (dm) and the squared distance to an axis or plane, thus:

$$\left\{ \begin{matrix} {}^{(0)i}I_{ju}^* = \int \delta_{ju}^{*2}\cdot dm = \int\left({}^{(0)i}v_j^{*2} + {}^{(0)i}w_j^{*2}\right)\cdot dm \\ {}^{(0)i}I_{juv}^* = \int {}^{(0)i}u_j^*\cdot {}^{(0)i}v_j^*\cdot dm;\ {}^{(0)i}I_{juu}^* = \int {}^{(0)i}u_j^{*2}\cdot dm \end{matrix} \right\} \tag{32}$$

The distance from mass (dm) to an axis having as unit vector ${}^j\overline{u}_{i(0)}$ (see Fig. 5) is determined as:

$$\delta_{ju}^* = \left|{}^j\overline{u}_{i(0)}\times {}^j\overline{r}_j^*\right|, \tag{33}$$

$$\left\{ \begin{matrix} \delta_{ju}^{*2} = \left({}^j\overline{u}_{i(0)}\times {}^j\overline{r}_j^*\right)^T\cdot\left({}^j\overline{u}_{i(0)}\times {}^j\overline{r}_j^*\right) = \\ = {}^j\overline{u}_{i(0)}^T\cdot\left[{}^j\overline{r}_j^*\times\right]\cdot\left[{}^j\overline{r}_j^*\times\right]^T\cdot {}^j\overline{u}_{i(0)} \end{matrix} \right\}. \tag{34}$$

As a result, considering (32) and (34), axial mechanical moment of inertia is defined thus:

$$\left\{ \begin{matrix} {}^{(0)i}I_{ju}^* = \int \delta_{ju}^{*2}\cdot dm = \\ = {}^j\overline{u}_{i(0)}^T\cdot\left\{\int\left[{}^j\overline{r}_j^*\times\right]\cdot\left[{}^j\overline{r}_j^*\times\right]^T\cdot dm\right\}\cdot {}^j\overline{u}_{i(0)} \end{matrix} \right\}. \tag{35}$$

But, the mechanical moments of inertia, from (32), can be obtained on basis of coordinates included in the transfer matrix expression (21):

$$\left\{ \begin{matrix} {}^{(0)i}u_j^* = {}^j\overline{u}_{i(0)}^T\cdot {}^j\overline{r}_j^* \\ {}^{(0)i}v_j^* = {}^j\overline{v}_{i(0)}^T\cdot {}^j\overline{r}_j^* \\ {}^{(0)i}w_j^* = {}^j\overline{w}_{i(0)}^T\cdot {}^j\overline{r}_j^* \end{matrix} \right\}; \tag{36}$$

$$\left\{ \begin{matrix} {}^{(0)i}u_j^{*2} = {}^j\overline{u}_{i(0)}^T\cdot {}^j\overline{r}_j^*\cdot {}^j\overline{r}_j^{*T}\cdot {}^j\overline{u}_{i(0)} \\ {}^{(0)i}v_j^{*2} = {}^j\overline{v}_{i(0)}^T\cdot {}^j\overline{r}_j^*\cdot {}^j\overline{r}_j^{*T}\cdot {}^j\overline{v}_{i(0)} \\ {}^{(0)i}w_j^{*2} = {}^j\overline{w}_{i(0)}^T\cdot {}^j\overline{r}_j^*\cdot {}^j\overline{r}_j^{*T}\cdot {}^j\overline{w}_{i(0)} \end{matrix} \right\}. \tag{37}$$

The squared distance from mass (dm) to the mass center C_j is characterized by expression:

$$\left\{ \begin{matrix} {}^{(0)i}u_j^{*2} + {}^{(0)i}v_j^{*2} + {}^{(0)i}w_j^{*2} = \\ = {}^{(0)i}\overline{r}_j^{*T}\cdot {}^{(0)i}\overline{r}_j^* \equiv {}^j\overline{r}_j^{*T}\cdot {}^j\overline{r}_j^* \end{matrix} \right\}. \tag{38}$$

The matrix (3×3), from (37), is identical with:

$${}^j\overline{r}_j^*\cdot {}^j\overline{r}_j^{*T} = {}^j\overline{r}_j^{*T}\cdot {}^j\overline{r}_j^*\cdot I_3 - \left[{}^j\overline{r}_j^*\times\right]\cdot\left[{}^j\overline{r}_j^*\times\right]^T. \tag{39}$$

Substituting (39) in (37), expressions become:

$$\left\{ \begin{matrix} {}^{(0)i}u_j^{*2} = {}^j\overline{r}_j^{*T}\cdot {}^j\overline{r}_j^* - \\ -{}^j\overline{u}_{i(0)}^T\cdot\left[{}^j\overline{r}_j^*\times\right]\cdot\left[{}^j\overline{r}_j^*\times\right]^T\cdot {}^j\overline{u}_{i(0)} \end{matrix} \right\}, \tag{40}$$

$$\left\{ \begin{matrix} {}^{(0)i}v_j^{*2} = {}^j\overline{r}_j^{*T}\cdot {}^j\overline{r}_j^* - \\ -{}^j\overline{v}_{i(0)}^T\cdot\left[{}^j\overline{r}_j^*\times\right]\cdot\left[{}^j\overline{r}_j^*\times\right]^T\cdot {}^j\overline{v}_{i(0)} \end{matrix} \right\}, \tag{41}$$

$$\left\{\begin{matrix} {}^{(0)i}w_j^{*2} = {}^j\bar{r}_j^{*T}\cdot{}^j\bar{r}_j^{*} - \\ -{}^j\bar{w}_{i(0)}^T\cdot\left[{}^j\bar{r}_j^{*}\times\right]\cdot\left[{}^j\bar{r}_j^{*}\times\right]^T\cdot{}^j\bar{w}_{i(0)} \end{matrix}\right\}; \quad (42)$$

$$\left\{\begin{matrix} {}^{(0)i}v_j^{*2} + {}^{(0)i}w_j^{*2} = {}^{(0)i}\bar{r}_j^{*T}\cdot{}^{(0)i}\bar{r}_j^{*} - {}^{(0)i}u_j^{*2} = \\ = -{}^j\bar{u}_{i(0)}^T\cdot\left[{}^j\bar{r}_j^{*}\times\right]\cdot\left[{}^j\bar{r}_j^{*}\times\right]^T\cdot{}^j\bar{u}_{i(0)} \equiv \delta_{ju}^{*2} \end{matrix}\right\}. \quad (43)$$

Therefore, using either (34) or (43), the axial mechanical moment of inertia (35) is rewritten:

$${}^{(0)i}I_{ju}^{*} = {}^j\bar{u}_{i(0)}^T\cdot\left\{\int\left[{}^j\bar{r}_j^{*}\times\right]\cdot\left[{}^j\bar{r}_j^{*}\times\right]^T dm\right\}\cdot{}^j\bar{u}_{i(0)}. \quad (44)$$

Mass integral from (35) or (44) is symbolized:

$$\left\{\begin{matrix} {}^jI_j^{*} = \int\left[{}^j\bar{r}_j^{*}\times\right]\cdot\left[{}^j\bar{r}_j^{*}\times\right]^T dm = \\ = \begin{bmatrix} {}^jI_x^{*} & -{}^jI_{xy}^{*} & -{}^jI_{xz}^{*} \\ -{}^jI_{yx}^{*} & {}^jI_y^{*} & -{}^jI_{yz}^{*} \\ -{}^jI_{zx}^{*} & -{}^jI_{zy}^{*} & {}^jI_z^{*} \end{bmatrix} \end{matrix}\right\}, \quad (45)$$

$$\left\{\begin{matrix} {}^jI_x^{*} = \int\left({}^jy_j^{*2} + {}^jz_j^{*2}\right)\cdot dm \\ {}^jI_y^{*} = \int\left({}^jz_j^{*2} + {}^jx_j^{*2}\right)\cdot dm \\ {}^jI_z^{*} = \int\left({}^jx_j^{*2} + {}^jy_j^{*2}\right)\cdot dm \end{matrix}\right\}, \quad (46)$$

$$\left\{\begin{matrix} {}^jI_{xy}^{*} = \int {}^jx_j^{*}\cdot{}^jy_j^{*}\cdot dm \\ {}^jI_{yz}^{*} = \int {}^jy_j^{*}\cdot{}^jz_j^{*}\cdot dm \\ {}^jI_{zx}^{*} = \int {}^jz_j^{*}\cdot{}^jx_j^{*}\cdot dm \end{matrix}\right\}. \quad (47)$$

In the matrix (45), the main diagonal contains (46) named axial moments of inertia, while symmetrically and negative to main diagonal are (47) known as centrifugal moments of inertia.

So, according to [2], [3] and [4], the matrix of inertia moments (45) is known as inertial tensor axial and centrifugal of the body $(j)\in(i)$ with respect to $\{j\}$ applied in the mass center C_j.

Considering the inertial tensor, symbolized by (45), the axial mechanical moment of inertia with respect to $\left\{\{i^*\};\ \{0^*\}\right\}\in C_j$ is determined as:

$${}^{(0)i}I_{ju}^{*} = {}^j\bar{u}_{i(0)}^T\cdot{}^jI_j^{*}\cdot{}^j\bar{u}_{i(0)} \quad (48)$$

$$\left\{\underset{(3\times3)}{Diag}\left[{}^{(0)i}I_{ju}^{*},\ u=\{x;y;z\}\right] = \begin{bmatrix} {}^{(0)i}I_{jx}^{*} & & \\ & {}^{(0)i}I_{jy}^{*} & \\ & & {}^{(0)i}I_{jz}^{*} \end{bmatrix}\right\} =$$

$$= \left\{\underset{(3\times3)}{Diag}\left\{\begin{bmatrix} {}^j\bar{x}_{i(0)}^T \\ {}^j\bar{y}_{i(0)}^T \\ {}^j\bar{z}_{i(0)}^T \end{bmatrix}\cdot{}^jI_j^{*}\cdot\left[{}^j\bar{x}_{i(0)}\quad{}^j\bar{y}_{i(0)}\quad{}^j\bar{z}_{i(0)}\right]\right\}\right\}; \quad (49)$$

$$\left\{\begin{matrix} \underset{(3\times3)}{Diag}\left\{\begin{bmatrix} {}^j\bar{x}_{i(0)}^T \\ {}^j\bar{y}_{i(0)}^T \\ {}^j\bar{z}_{i(0)}^T \end{bmatrix}\cdot{}^jI_j^{*}\cdot\left[{}^j\bar{x}_{i(0)}\quad{}^j\bar{y}_{i(0)}\quad{}^j\bar{z}_{i(0)}\right]\right\} = \\ = \underset{(3\times3)}{Diag}\left\{{}_j^{(0)i}[R]\cdot{}^jI_j^{*}\cdot{}_j^{(0)i}[R]^T\right\} \end{matrix}\right\}. \quad (50)$$

Using (31) and (32), in the following the centrifugal moment of inertia is determined as:

$$\left\{\begin{matrix} {}^{(0)i}I_{juv}^{*} = \int {}^{(0)i}u_j^{*}\cdot{}^{(0)i}v_j^{*}\cdot dm = \\ = {}^j\bar{u}_{i(0)}^T\cdot\int {}^j\bar{r}_j^{*}\cdot{}^j\bar{r}_j^{*T}\cdot dm\cdot{}^j\bar{v}_{i(0)} \end{matrix}\right\}. \quad (51)$$

The matrix (3×3), defined by (39) is rewritten:

$${}^j\bar{r}_j^{*}\cdot{}^j\bar{r}_j^{*T} = {}^j\bar{r}_j^{*T}\cdot{}^j\bar{r}_j^{*}\cdot I_3 - \left[{}^j\bar{r}_j^{*}\times\right]\cdot\left[{}^j\bar{r}_j^{*}\times\right]^T. \quad (52)$$

This is substituted in (51), and it changes thus:

$$\left\{\begin{matrix} {}^{(0)i}I_{juv}^{*} = \int\left[{}^j\bar{r}_j^{*T}\cdot{}^j\bar{r}_j^{*}\cdot{}^j\bar{u}_{i(0)}^T\cdot{}^j\bar{v}_{i(0)} - \right. \\ \left. -{}^j\bar{u}_{i(0)}^T\cdot\left[{}^j\bar{r}_j^{*}\times\right]\cdot\left[{}^j\bar{r}_j^{*}\times\right]^T\cdot{}^j\bar{v}_{i(0)}\right]\cdot dm \end{matrix}\right\}, \quad (53)$$

$$\left\{\begin{matrix} -{}^j\bar{u}_{i(0)}^T\cdot\left\{\int\left[{}^j\bar{r}_j^{*}\times\right]\cdot\left[{}^j\bar{r}_j^{*}\times\right]^T\cdot dm\right\}\cdot\bar{v}_{i(0)} = \\ = -{}^j\bar{u}_{i(0)}^T\cdot{}^jI_j^{*}\cdot{}^j\bar{v}_{i(0)} = {}^{(0)i}I_{juv}^{*} \end{matrix}\right\}; \quad (54)$$

where (45) was substituted. Considering (15) and (16), centrifugal moments of inertia become:

$$\left\{\begin{matrix} \begin{bmatrix} & -{}^{(0)i}I_{jxy}^{*} & -{}^{(0)i}I_{jxz}^{*} \\ -{}^{(0)i}I_{jyx}^{*} & & -{}^{(0)i}I_{jyz}^{*} \\ -{}^{(0)i}I_{jzx}^{*} & -{}^{(0)i}I_{jzy}^{*} & \end{bmatrix} = \\ = \begin{bmatrix} {}^j\bar{x}_{i(0)}^T \\ {}^j\bar{y}_{i(0)}^T \\ {}^j\bar{z}_{i(0)}^T \end{bmatrix}\cdot{}^jI_j^{*}\cdot\left[{}^j\bar{x}_{i(0)}\quad{}^j\bar{y}_{i(0)}\quad{}^j\bar{z}_{i(0)}\right] - \\ -\underset{(3\times3)}{Diag}\left[{}^{(0)i}I_{ju}^{*},\ u=\{x;y;z\}\right] \end{matrix}\right\}, \quad (55)$$

$$\left\{\begin{matrix} \begin{bmatrix} & -{}^{(0)i}I_{jxy}^{*} & -{}^{(0)i}I_{jxz}^{*} \\ -{}^{(0)i}I_{jyx}^{*} & & -{}^{(0)i}I_{jyz}^{*} \\ -{}^{(0)i}I_{jzx}^{*} & -{}^{(0)i}I_{jzy}^{*} & \end{bmatrix} = \\ = {}_j^{(0)i}[R]\cdot{}^jI_j^{*}\cdot{}_j^{(0)i}[R]^T - \\ -\underset{(3\times3)}{Diag}\left\{{}_j^{(0)i}[R]\cdot{}^jI_j^{*}\cdot{}_j^{(0)i}[R]^T\right\} \end{matrix}\right\}. \quad (56)$$

So, (50) and (56) are included in the matrix of axial and centrifugal moments of inertia thus:

$$\left\{\begin{matrix} {}^{(0)i}I_j^{*} = \int\left[{}^{(0)i}\bar{r}_j^{*}\times\right]\cdot\left[{}^{(0)i}\bar{r}_j^{*}\times\right]^T dm = \\ = {}_j^{(0)i}[R]\cdot{}^jI_j^{*}\cdot{}_j^{(0)i}[R]^T \end{matrix}\right\}, \quad (57)$$

$$\left\{\begin{array}{c} {}^{(0)i}I_j^* = {}^{(0)i}_{\ j}[R]\cdot {}^{j}I_j^*\cdot {}^{(0)i}_{\ j}[R]^T = \\ = \begin{bmatrix} {}^{(0)i}I_{jx}^* & -{}^{(0)i}I_{jxy}^* & -{}^{(0)i}I_{jxz}^* \\ -{}^{(0)i}I_{jyx}^* & {}^{(0)i}I_{jy}^* & -{}^{(0)i}I_{jyz}^* \\ -{}^{(0)i}I_{jzx}^* & -{}^{(0)i}I_{jzy}^* & {}^{(0)i}I_{jz}^* \end{bmatrix} \end{array}\right\}. \tag{58}$$

The matrix (57) and (58) as developed form is named inertial tensor axial and centrifugal of the body $(j)\in(i)$ relative to $\{\{i^*\};\{0^*\}\}\in C_j$ applied in the mass center C_j. In the same time, the expressions (57) and (58) represent the variation law of the inertial tensor with respect to concurrent frames in the mass center C_j.

On the basis of (31) and (32), in the following the planar moment of inertia is studied with:

$$\left\{\begin{array}{c} {}^{(0)i}I_{juu}^* = \int {}^{(0)i}u_j^{*2}\cdot dm = \\ = {}^{j}\overline{u}_{i(0)}^T\cdot\int {}^{j}\overline{r}_j^*\cdot {}^{j}\overline{r}_j^{*T}\cdot dm\cdot {}^{j}\overline{u}_{i(0)} \end{array}\right\}. \tag{59}$$

Substituting (52) in (59) and performing the mass integrals, the following matrix is obtained:

$$\left\{\begin{array}{c} {}^{j}I_{pj}^* = \int {}^{j}\overline{r}_j^*\cdot {}^{j}\overline{r}_j^{*T}\cdot dm = \\ = \begin{bmatrix} {}^{j}I_{xx}^* & {}^{j}I_{xy}^* & {}^{j}I_{xz}^* \\ {}^{j}I_{yx}^* & {}^{j}I_{yy}^* & {}^{j}I_{yz}^* \\ {}^{j}I_{zx}^* & {}^{j}I_{zy}^* & {}^{j}I_{zz}^* \end{bmatrix} \end{array}\right\}. \tag{60}$$

The matrix of inertia moments (60) is known as inertial tensor planar and centrifugal of the body $(j)\in(i)$ to $\{j\}$, applied in mass center C_j. Considering the inertial tensor, symbolized by (60), the planar mechanical moments of inertia with respect to $\{\{i^*\};\{0^*\}\}\in C_j$ are determined as:

$${}^{(0)i}I_{juu}^* = {}^{j}\overline{u}_{i(0)}^T\cdot {}^{j}I_{pj}^*\cdot {}^{j}\overline{u}_{i(0)}; \tag{61}$$

$$\left\{\begin{array}{c} \underset{(3\times3)}{Diag}\left[{}^{(0)i}I_{juu}^*,\ u=\{x;y;z\}\right] = \\ = \begin{bmatrix} {}^{(0)i}I_{jxx}^* & & \\ & {}^{(0)i}I_{jyy}^* & \\ & & {}^{(0)i}I_{jzz}^* \end{bmatrix} \end{array}\right\}, \tag{62}$$

$$\left\{\begin{array}{c} \underset{(3\times3)}{Diag}\left[{}^{(0)i}I_{juu}^*,\ u=\{x;y;z\}\right] = \\ = \underset{(3\times3)}{Diag}\left\{\begin{bmatrix} {}^{j}\overline{x}_{i(0)}^T \\ {}^{j}\overline{y}_{i(0)}^T \\ {}^{j}\overline{z}_{i(0)}^T \end{bmatrix}\cdot {}^{j}I_{pj}^*\cdot\begin{bmatrix} {}^{j}\overline{x}_{i(0)} & {}^{j}\overline{y}_{i(0)} & {}^{j}\overline{z}_{i(0)} \end{bmatrix}\right\} \end{array}\right\} \tag{63}$$

$$\left\{\begin{array}{c} \underset{(3\times3)}{Diag}\left\{\begin{bmatrix} {}^{j}\overline{x}_{i(0)}^T \\ {}^{j}\overline{y}_{i(0)}^T \\ {}^{j}\overline{z}_{i(0)}^T \end{bmatrix}\cdot {}^{j}I_{pj}^*\cdot\begin{bmatrix} {}^{j}\overline{x}_{i(0)} & {}^{j}\overline{y}_{i(0)} & {}^{j}\overline{z}_{i(0)} \end{bmatrix}\right\} = \\ = \underset{(3\times3)}{Diag}\left\{{}^{(0)i}_{\ j}[R]\cdot {}^{j}I_{pj}^*\cdot {}^{(0)i}_{\ j}[R]^T\right\} \end{array}\right\} \tag{64}$$

On the basis of (55) and (56), in which (60) is substituted, the centrifugal moments of inertia are determined below with the expressions:

$$\left\{\begin{array}{c} \begin{bmatrix} & {}^{(0)i}I_{jxy}^* & {}^{(0)i}I_{jxz}^* \\ {}^{(0)i}I_{jyx}^* & & {}^{(0)i}I_{jyz}^* \\ {}^{(0)i}I_{jzx}^* & {}^{(0)i}I_{jzy}^* & \end{bmatrix} = \\ = {}^{(0)i}_{\ j}[R]\cdot {}^{j}I_{pj}^*\cdot {}^{(0)i}_{\ j}[R]^T - \\ -\underset{(3\times3)}{Diag}\left\{{}^{(0)i}_{\ j}[R]\cdot {}^{j}I_{pj}^*\cdot {}^{(0)i}_{\ j}[R]^T\right\} \end{array}\right\}. \tag{65}$$

Thus, (64) and (65) are included in a matrix of planar and centrifugal moments of inertia as:

$$\left\{\begin{array}{c} {}^{(0)i}I_{pj}^* = \int {}^{(0)i}\overline{r}_j^*\cdot {}^{(0)i}\overline{r}_j^{*T}\cdot dm = \\ = {}^{(0)i}_{\ j}[R]\cdot {}^{j}I_{pj}^*\cdot {}^{(0)i}_{\ j}[R]^T \end{array}\right\}, \tag{66}$$

$$\left\{\begin{array}{c} {}^{(0)i}I_{pj}^* = {}^{(0)i}_{\ j}[R]\cdot {}^{j}I_{pj}^*\cdot {}^{(0)i}_{\ j}[R]^T = \\ = \begin{bmatrix} {}^{(0)i}I_{jxx}^* & {}^{(0)i}I_{jxy}^* & {}^{(0)i}I_{jxz}^* \\ {}^{(0)i}I_{jyx}^* & {}^{(0)i}I_{jyy}^* & {}^{(0)i}I_{jyz}^* \\ {}^{(0)i}I_{jzx}^* & {}^{(0)i}I_{jzy}^* & {}^{(0)i}I_{jzz}^* \end{bmatrix} \end{array}\right\}. \tag{67}$$

The matrix (66) and (67) as developed form is named inertial tensor planar and centrifugal of the body $(j)\in(i)$ relative to $\{\{i^*\};\{0^*\}\}\in C_j$ applied in the mass center C_j. In the same time, the expressions (57) and (58) represent the variation law of the inertial tensor with respect to concurrent frames in the mass center C_j.

Often the fundamental notions and theorems, from analytical dynamics, are applied under the matrix form. So, the position vectors from (60) are written by means of the homogeneous coordinates. It obtains a new matrix as follows:

$$\left\{\begin{array}{c} {}^{j}I_{psj}^* = \int\begin{pmatrix} {}^{j}\overline{r}_j^* \\ 1 \end{pmatrix}\cdot\begin{pmatrix} {}^{j}\overline{r}_j^{*T} & 1 \end{pmatrix}\cdot dm = \\ = \begin{bmatrix} \int {}^{j}\overline{r}_j^*\cdot {}^{j}\overline{r}_j^{*T}\cdot dm & \int {}^{j}\overline{r}_j^*\cdot dm \\ \int {}^{j}\overline{r}_j^{*T}\cdot dm & m_j \end{bmatrix} = \\ = \begin{bmatrix} {}^{j}I_{pj}^* & [0]_{(3\times1)} \\ [0]_{(1\times3)} & m_j \end{bmatrix} \end{array}\right\}. \tag{68}$$

where (18) is substituted. It contains inertial tensor planar and centrifugal (60), as well mass of the body. This matrix is pseudoinertial tensor relative to $\{j\}$, applied in the mass center C_j.

Similarly with (58) and (67), for pseudoinertial tensor relative to $\{\{i^*\};\{0^*\}\}\in C_j$ it obtains next:

$$\left\{\begin{array}{c} {}^{(0)i}I^*_{psj}=\int\begin{pmatrix}{}^{(0)i}\overline{r}^*_j\\ 1\end{pmatrix}\cdot\begin{pmatrix}{}^{(0)i}\overline{r}^{*T}_j & 1\end{pmatrix}\cdot dm= \\ =\begin{bmatrix}\int {}^{(0)i}\overline{r}^*_j\cdot{}^{(0)i}\overline{r}^{*T}_j\cdot dm & \int {}^{(0)i}\overline{r}^*_j\cdot dm\\ \int {}^{(0)i}\overline{r}^{*T}_j\cdot dm & m_j\end{bmatrix}= \\ =\begin{bmatrix}{}^{(0)i}I^*_{pj} & [0]_{(3\times1)}\\ [0]_{(1\times3)} & m_j\end{bmatrix}\end{array}\right\}. \quad (69)$$

where the conditions (18) are substituted again. But, (69) can be written in another matrix form:

$${}^{(0)i}_{\;\;j}[T]^*=\begin{bmatrix}{}^{(0)i}_{\;\;j}[R] & [0]_{(3\times1)}\\ [0]_{(1\times3)} & 1\end{bmatrix}, \quad (70)$$

$${}^{(0)i}I^*_{psj}={}^{(0)i}_{\;\;j}[T]^*\cdot{}^{j}I^*_{pj}\cdot{}^{(0)i}_{\;\;j}[T]^{*T}. \quad (71)$$

So, the squared matrix (4×4) symmetrical and positive defined (69) or (71) is considered the variation law of the pseudoinertial tensor with respect to concurrent frames in mass center C_j.

4. VARIATION of INERTIAL TENSOR

In the previous section the inertial tensor axial and centrifugal as respectively planar and centrifugal together with its variation law relative to concurrent frames in mass center C_j, as well as pseudoinertial tensor have been determined by means of definition expressions. Consequently, for every homogeneous body $(j)\in(i)$ with simple geometrical shape [2], the mass properties are known by the input data:

$$\left\{m_j;\ {}^{j}\overline{r}_{C_j}\ {}^{j}I^*_j;\ {}^{j}I^*_j;\ {}^{j}I^*_{pj};\ {}^{j}I^*_{psj};\quad j=1\rightarrow p_i;(j)\in(i)\right\}.$$

On the basis of the expressions, included in the third section of the paper, the mass properties are calculated relative to $\{\{i^*\};\ \{0^*\}\}\in C_j$, that is:

$$\left\{m_j;\ {}^{(0)i}\overline{r}^*_{C_j}\ {}^{(0)i}I^*_j;\ {}^{(0)i}I^*_j;\ {}^{(0)i}I^*_{pj};\ {}^{(0)i}I^*_{psj};\ j=1\rightarrow p_i\right\}.$$

In the following steps, the axial, centrifugal and planar mechanical moments of inertia for every homogeneous body $(j)\in(i)$ must be calculated with respect to $\{i\}$ and $\{0\}$ corresponding to kinematical structure of MBS (see Fig.1):

$$\left\{{}^{(0)i}I_u;\ {}^{(0)i}I_{uv};\ {}^{(0)i}I_{uu};\ where\ j=1\rightarrow p_i;\ (j)\in(i)\right\} \quad (72)$$

They are included in the generalized variation law of the inertial and pseudoinertial tensors:

$$\left\{{}^{(0)i}I_j;\ {}^{(0)i}I_{pj};\ {}^{(0)i}I_{psj};\ where\ j=1\rightarrow p_i;\ (j)\in(i)\right\} \quad (73)$$

For beginning, the generalized variation law of inertial tensor axial and centrifugal of the body $(j)\in(i)$ relative to $\{\{i\};\ \{0\}\}$ is determined thus:

$${}^{(0)i}I_j=\int\left[{}^{(0)i}\overline{r}_j\times\right]\cdot\left[{}^{(0)i}\overline{r}_j\times\right]^T\cdot dm, \quad (74)$$

$$\left\{\begin{array}{c}\left[{}^{(0)i}\overline{r}_j\times\right]=\left[{}^{(0)i}\overline{r}_{C_j}\times\right]+{}^{(0)i}_{\;\;j}[R]\cdot\left[{}^{j}\overline{r}^*_j\times\right]\\ \left[{}^{(0)i}\overline{r}_j\times\right]^T=\left[{}^{(0)i}\overline{r}_{C_j}\times\right]^T+\left[{}^{j}\overline{r}^*_j\times\right]^T\cdot{}^{(0)i}_{\;\;j}[R]^T\end{array}\right\}. \quad (75)$$

Substituting (75) in (74), this is changed below:

$$\left\{\begin{array}{c}{}^{(0)i}I_j=\left[{}^{(0)i}\overline{r}_{C_j}\times\right]\cdot\left[{}^{(0)i}\overline{r}_{C_j}\times\right]^T\cdot\int dm+\\ +{}^{(0)i}_{\;\;j}[R]\cdot\left\{\int\left[{}^{j}\overline{r}^*_j\times\right]\cdot\left[{}^{j}\overline{r}^*_j\times\right]^T\cdot dm\right\}\cdot{}^{(0)i}_{\;\;j}[R]^T\end{array}\right\}. \quad (76)$$

First matrix from the right member shows as:

$$\left\{\begin{array}{c}{}^{(0)i}I_{C_j}=\left[{}^{(0)i}\overline{r}_{C_j}\times\right]\cdot\left[{}^{(0)i}\overline{r}_{C_j}\times\right]^T\cdot\int dm=\\ =m_j\cdot\left[{}^{(0)i}\overline{r}_{C_j}\times\right]\cdot\left[{}^{(0)i}\overline{r}_{C_j}\times\right]^T\end{array}\right\}, \quad (77)$$

$$\left\{\begin{array}{c}{}^{(0)i}I_{C_j}=m_j\cdot\left[{}^{(0)i}\overline{r}_{C_j}\times\right]\cdot\left[{}^{(0)i}\overline{r}_{C_j}\times\right]^T=\\ =\begin{bmatrix}{}^{(0)i}I_{C_jx} & -{}^{(0)i}I_{C_jxy} & -{}^{(0)i}I_{C_jxz}\\ -{}^{(0)i}I_{C_jyx} & {}^{(0)i}I_{C_jy} & -{}^{(0)i}I_{C_jyz}\\ -{}^{(0)i}I_{C_jzx} & -{}^{(0)i}I_{C_jzy} & {}^{(0)i}I_{C_jz}\end{bmatrix}\end{array}\right\}. \quad (78)$$

The components from (78) are determined with:

$$\left\{\begin{array}{l}{}^{(0)i}I_{C_jx}={}^{(0)i}y^2_{C_j}+{}^{(0)i}z^2_{C_j}\\ {}^{(0)i}I_{C_jy}={}^{(0)i}z^2_{C_j}+{}^{(0)i}x^2_{C_j}\\ {}^{(0)i}I_{C_jz}={}^{(0)i}x^2_{C_j}+{}^{(0)i}y^2_{C_j}\end{array}\right\}, \quad (79)$$

$$\left\{\begin{array}{l}{}^{(0)i}I_{C_jxy}={}^{(0)i}x_{C_j}\cdot{}^{(0)i}y_{C_j}\\ {}^{(0)i}I_{C_jyz}={}^{(0)i}y_{C_j}\cdot{}^{(0)i}z_{C_j}\\ {}^{(0)i}I_{C_jzx}={}^{(0)i}z_{C_j}\cdot{}^{(0)i}x_{C_j}\end{array}\right\}. \quad (80)$$

Considering (80), the expression (78) is named the inertia matrix axial and centrifugal of the mass center C_j relative to frames $\{\{i\};\ \{0\}\}$.

Therefore, the starting expression (74) takes the final form, written below as follows:

$$\left\{\begin{aligned} {}^{(0)i}I_j &= \int \left[{}^{(0)i}\overline{r}_j \times\right]\cdot\left[{}^{(0)i}\overline{r}_j \times\right]^T \cdot dm = \\ &= {}^{(0)i}I_{C_j} + {}^{(0)i}_{\ \ j}[R]\cdot {}^jI^*_j \cdot {}^{(0)i}_{\ \ j}[R]^T = {}^{(0)i}I_{C_j} + {}^{(0)i}I^*_j \end{aligned}\right\}. \quad (81)$$

This is named the generalized variation law of the inertial tensor axial and centrifugal of the body $(j)\in(i)$, with respect to frames $\{\{i\};\ \{0\}\}$.

According to (73), the generalized variation law of inertial tensor planar and centrifugal of the body $(j)\in(i)$ relative to $\{\{i\};\ \{0\}\}$ is established in the following. The starting expression is:

$${}^{(0)i}I_{pj} = \int {}^{(0)i}\overline{r}_j \cdot {}^{(0)i}\overline{r}_j^T \cdot dm\,. \quad (82)$$

Substituting (24) in (82), this is changed thus:

$$\left\{\begin{aligned} &{}^{(0)i}I_{pj} = {}^{(0)i}\overline{r}_{C_j}\cdot {}^{(0)i}\overline{r}_{C_j}^{\ T}\cdot \int dm + \\ &+ {}^{(0)i}_{\ \ j}[R]\cdot\left\{\int {}^j\overline{r}_j^{*}\cdot {}^j\overline{r}_j^{*T}\cdot dm\right\}\cdot {}^{(0)i}_{\ \ j}[R]^T \end{aligned}\right\}. \quad (83)$$

First matrix from the right member shows as:

$${}^{(0)i}I_{pC_j} = {}^{(0)i}\overline{r}_{C_j}\cdot {}^{(0)i}\overline{r}_{C_j}^{\ T}\cdot\int dm = m_j\cdot {}^{(0)i}\overline{r}_{C_j}\cdot {}^{(0)i}\overline{r}_{C_j}^{\ T},$$

$$\left\{\begin{aligned} &{}^{(0)i}I_{pC_j} = m_j\cdot {}^{(0)i}\overline{r}_{C_j}\cdot {}^{(0)i}\overline{r}_{C_j}^{\ T} = \\ &= \begin{bmatrix} {}^{(0)i}I_{C_jxx} & {}^{(0)i}I_{C_jxy} & {}^{(0)i}I_{C_jxz} \\ {}^{(0)i}I_{C_jyx} & {}^{(0)i}I_{C_jyy} & {}^{(0)i}I_{C_jyz} \\ {}^{(0)i}I_{C_jzx} & {}^{(0)i}I_{C_jzy} & {}^{(0)i}I_{C_jzz} \end{bmatrix} \end{aligned}\right\}. \quad (84)$$

The components from (85) are determined with:

$$\left\{\begin{aligned} {}^{(0)i}I_{C_jxx} &= {}^{(0)i}x^2_{C_j} \\ {}^{(0)i}I_{C_jy} &= {}^{(0)i}y^2_{C_j} \\ {}^{(0)i}I_{C_jz} &= {}^{(0)i}z^2_{C_j} \end{aligned}\right.\ ;\ \left\{\begin{aligned} {}^{(0)i}I_{C_jxy} &= {}^{(0)i}x_{C_j}\cdot {}^{(0)i}y_{C_j} \\ {}^{(0)i}I_{C_jyz} &= {}^{(0)i}y_{C_j}\cdot {}^{(0)i}z_{C_j} \\ {}^{(0)i}I_{C_jzx} &= {}^{(0)i}z_{C_j}\cdot {}^{(0)i}x_{C_j} \end{aligned}\right. \quad (85)$$

Considering (85), the expression (84) is named the inertia matrix planar and centrifugal of the mass center C_j with respect to frames $\{\{i\};\ \{0\}\}$.

Therefore, the starting expression (82) takes the final form, written below as follows:

$$\left\{\begin{aligned} {}^{(0)i}I_{pj} &= \int {}^{(0)i}\overline{r}_j\cdot {}^{(0)i}\overline{r}_j^T\cdot dm = \\ &= {}^{(0)i}I_{pC_j} + {}^{(0)i}_{\ \ j}[R]\cdot {}^jI^*_{pj}\cdot {}^{(0)i}_{\ \ j}[R]^T = {}^{(0)i}I_{pC_j} + {}^{(0)i}I^*_{pj} \end{aligned}\right\} \quad (86)$$

This is named the generalized variation law of the inertial tensor planar and centrifugal of the body $(j)\in(i)$, with respect to frames $\{\{i\};\ \{0\}\}$.

On the basis of (68) – (71), in the following the variation law of the pseudoinertial tensor of the body $(j)\in(i)$ with respect to $\{\{i\};\ \{0\}\}$ is established. The starting expression is (82) where the position vectors are substituted by their homogeneous coordinates [2], that is:

$$\left\{\begin{aligned} &{}^{(0)i}I_{psj} = \int\begin{bmatrix} {}^{(0)i}\overline{r}_j \\ 1 \end{bmatrix}\cdot\begin{bmatrix} {}^{(0)i}\overline{r}_j^T & 1 \end{bmatrix}\cdot dm = \\ &= \begin{bmatrix} \int {}^{(0)i}\overline{r}_j\cdot {}^{(0)i}\overline{r}_j^T\cdot dm & \int {}^{(0)i}\overline{r}_j\cdot dm \\ \int {}^{(0)i}\overline{r}_j^T\cdot dm & \int dm \end{bmatrix} \end{aligned}\right\}. \quad (87)$$

Substituting (2), (19), as well as (82) in (87), this is changed in the following expression:

$${}^{(0)i}I_{psj} = \begin{bmatrix} {}^{(0)i}I_{pj} & {}^{(0)i}\overline{r}_{C_j}\cdot m_j \\ {}^{(0)i}\overline{r}_{C_j}^T\cdot m_j & m_j \end{bmatrix}. \quad (88)$$

Substituting (27) and (82) in (87), it obtains:

$$\left\{\begin{aligned} &{}^{(0)i}_{\ \ j}[T]\cdot\int\begin{bmatrix} {}^j\overline{r}_j^{*} \\ 1 \end{bmatrix}\cdot\begin{bmatrix} {}^j\overline{r}_j^{*T} & 1 \end{bmatrix}\cdot dm\cdot {}^{(0)i}_{\ \ j}[T]^T = \\ &= {}^{(0)i}_{\ \ j}[T]\cdot {}^jI^*_{psj}\cdot {}^{(0)i}_{\ \ j}[T]^T = {}^{(0)i}I_{psj} \end{aligned}\right\}. \quad (89)$$

Either (88) or (89) characterize the variation law of the pseudoinertial tensor of the body $(j)\in(i)$ relative to $\{\{i\};\ \{0\}\}$. In the last (89), the expression (71) and locating matrices below presented are substituted. As a result, it obtains:

$${}^{(0)i}_{(0^*)i^*}[T] = \begin{bmatrix} I_{3\times3} & {}^{(0)i}\overline{r}_{C_j} \\ [0]_{(1\times3)} & 1 \end{bmatrix}, \quad (90)$$

$${}^{(0)i}_{(0^*)i^*}[T]^T = \begin{bmatrix} I_{3\times3} & [0]_{(3\times1)} \\ {}^{(0)i}\overline{r}_{C_j}^T & 1 \end{bmatrix}; \quad (91)$$

$$\left\{\begin{aligned} &{}^{(0)i}I_{psj} = {}^{(0)i}_{(0^*)i^*}[T]\cdot {}^{(0)i}I^*_{psj}\cdot {}^{(0)i}_{(0^*)i^*}[T]^T = \\ &= \begin{bmatrix} {}^{(0)i}I_{pj} & {}^{(0)i}\overline{r}_{C_j}\cdot m_j \\ {}^{(0)i}\overline{r}_{C_j}^T\cdot m_j & m_j \end{bmatrix} \end{aligned}\right\}. \quad (92)$$

The expression (92) shows the variation law of the pseudoinertial tensor of the body $(j)\in(i)$ between $\{\{i^*\};\{0^*\}\}\in C_j$ parallel with $\{\{i\};\ \{0\}\}$.

Any expression (88), (89) or (92) characterizes the variation law of the pseudoinertial tensor of the body $(j)\in(i)$ relative to $\{\{i\};\ \{0\}\}$.

5. INERTIAL TENSOR for MBS

In the third and fourth section of this paper, the generalized variation laws of the inertial and pseudoinertial tensors (73) have been defined for every homogeneous body $(j)\in(i)$ with respect to $\{\{i\};\ \{0\}\}$. In this section, as function of above expressions, inertia properties for every kinetic ensemble $(i = 1 \to n)$, that is: axial, centrifugal and planar mechanical moments of inertia will be determined, in accordance with [2] and [3]:

$$\left\{ {}^{(0)i}I_u;\ {}^{(0)i}I_{uv};\ {}^{(0)i}I_{uu};\ \text{where } i = 1 \to n \right\}. \quad (93)$$

Considering the previous sections, these are included in inertial and pseudoinertial tensors:

$$\left\{ {}^{(0)i}I_i;\ {}^{(0)i}I_{pi};\ {}^{(0)i}I_{psi};\ \text{where } i = 1 \to n \right\}. \quad (94)$$

For beginning, considering expression (81), the generalized variation law of the inertial tensor axial and centrifugal of the kinetic ensemble (i) relative to $\{\{i\};\ \{0\}\}$ is below determined:

$$\left\{ \begin{array}{c} {}^{(0)i}I_i = \sum_{j=1}^{p_i} \sigma_j \cdot {}^{(0)i}I_j = \\ = \sum_{j=1}^{p_i} \sigma_j \cdot {}^{(0)i}I_{C_j} + \sum_{j=1}^{p_i} \sigma_j \cdot {}^{(0)i}I_j^* = {}^{(0)i}I_{C_i} + {}^{(0)i}I_i^* \end{array} \right\}, \quad (95)$$

$${}^{(0)i}I_{C_i} = \sum_{j=1}^{p_i} \sigma_j \cdot {}^{(0)i}I_{C_j} = M_i \cdot \left[{}^{(0)i}\overline{r}_{C_i} \times \right] \cdot \left[{}^{(0)i}\overline{r}_{C_i} \times \right]^T \quad (96)$$

$${}^{(0)i}I_i^* = \sum_{j=1}^{p_i} \sigma_j \cdot {}^{(0)i}I_j^* = \begin{bmatrix} {}^{(0)i}I_x^* & -{}^{(0)i}I_{xy}^* & -{}^{(0)i}I_{xz}^* \\ -{}^{(0)i}I_{yx}^* & {}^{(0)i}I_y^* & -{}^{(0)i}I_{yz}^* \\ -{}^{(0)i}I_{zx}^* & -{}^{(0)i}I_{zy}^* & {}^{(0)i}I_z^* \end{bmatrix} \quad (97)$$

where (96) is the inertia matrix axial and centrifugal of mass center C_i, and (97) is the inertial tensor axial and centrifugal of the kinetic ensemble (i) relative to $\{\{i^*\};\ \{0^*\}\} \in C_i$. The inertial tensor (97) is included, see [2] – [8], in the explicit expressions for: kinetic energy, the time derivatives of the angular momentum, acceleration energies of higher order and so on. Considering (86), the generalized variation law of the inertial tensor planar and centrifugal of the kinetic ensemble (i), relative to $\{\{i\};\ \{0\}\}$, is established the following with expressions:

$$\left\{ \begin{array}{c} {}^{(0)i}I_{pi} = \sum_{j=1}^{p_i} \sigma_j \cdot {}^{(0)i}I_{pj} = {}^{(0)i}I_{pC_i} + {}^{(0)i}I_{pi}^* = \\ = \sum_{j=1}^{p_i} \sigma_j \cdot {}^{(0)i}I_{pC_j} + \sum_{j=1}^{p_i} \sigma_j \cdot {}^{(0)i}I_{pj}^* \end{array} \right\}, \quad (98)$$

$${}^{(0)i}I_{pi} = \sum_{j=1}^{p_i} \sigma_j \cdot {}^{(0)i}I_{pj} = \begin{bmatrix} {}^{(0)i}I_{xx} & {}^{(0)i}I_{xy} & {}^{(0)i}I_{xz} \\ {}^{(0)i}I_{yx} & {}^{(0)i}I_{yy} & {}^{(0)i}I_{yz} \\ {}^{(0)i}I_{zx} & {}^{(0)i}I_{zy} & {}^{(0)i}I_{zz} \end{bmatrix}, \quad (99)$$

$${}^{(0)i}I_{pC_i} = \sum_{j=1}^{p_i} \sigma_j \cdot {}^{(0)i}I_{pC_j} = M_i \cdot {}^{(0)i}\overline{r}_{C_i} \cdot {}^{(0)i}\overline{r}_{C_i}^{\,T}, \quad (100)$$

$${}^{(0)i}I_{pi}^* = \sum_{j=1}^{p_i} \sigma_j \cdot {}^{(0)i}I_{pj}^* = \begin{bmatrix} {}^{(0)i}I_{xx}^* & {}^{(0)i}I_{xy}^* & {}^{(0)i}I_{xz}^* \\ {}^{(0)i}I_{yx}^* & {}^{(0)i}I_{yy}^* & {}^{(0)i}I_{yz}^* \\ {}^{(0)i}I_{zx}^* & {}^{(0)i}I_{zy}^* & {}^{(0)i}I_{zz}^* \end{bmatrix}, \quad (101)$$

where (100) is the inertia matrix planar and centrifugal of mass center C_i, and (101) is the inertial tensor planar and centrifugal of the kinetic ensemble. (i) relative to $\{\{i^*\};\ \{0^*\}\} \in C_i$. In accordance with (94), the pseudoinertial tensor of the kinetic ensemble (i) relative to $\{\{i\};\ \{0\}\}$ is determined, using (87) – (92), thus:

$$\left\{ \begin{array}{c} {}^{(0)i}I_{psi} = \sum_{j=1}^{p_i} \sigma_j \cdot {}^{(0)i}I_{psj} = \\ = \begin{bmatrix} \sum_{j=1}^{p_i} \sigma_j \cdot {}^{(0)i}I_{pj} & \sum_{j=1}^{p_i} \sigma_j \cdot {}^{(0)i}\overline{r}_{C_j} \cdot m_j \\ \sum_{j=1}^{p_i} \sigma_j \cdot {}^{(0)i}\overline{r}_{C_j}^{\,T} \cdot m_j & \sum_{j=1}^{p_i} \sigma_j \cdot m_j \end{bmatrix} \end{array} \right\}, \quad (102)$$

$${}^{(0)i}I_{psi} = \begin{bmatrix} {}^{(0)i}I_{pi} & M_i \cdot {}^{(0)i}\overline{r}_{C_i} \\ M_i \cdot {}^{(0)i}\overline{r}_{C_i}^{\,T} & M_i \end{bmatrix}. \quad (103)$$

From expression (103), it is again noticed that pseudoinertial tensor is a squared matrix (4×4) symmetrical and positive defined. It contains inertial tensor planar and centrifugal (99), as well as statics moments (8) and total mass (6) of the kinetic ensemble (i). The pseudoinertial tensor, see [5] – [8], is also included in the matrix expressions of the fundamental notions and theorems, belonging to analytical dynamics.

6. CONCLUSIONS

The currently paper was devoted, especially, to presentation a few essential new formulations about the mass properties, compulsory included in analytical dynamics of the multibody systems (MBS). So, in the case of (MBS), for example, mechanical robot structure, a few simplifying hypotheses, on the mass properties, have been implemented. Consequently, the mass properties are continuously distributed between the fixed basis and the last kinetic ensemble from mechanical structure. That is why, in the dynamical study of MBS, the author of this paper has introduced the phrase “mass distribution” instead of mass geometry. But, mass distribution (MD) is based on the mass a fundamental notion in analytical dynamics of systems. At its turn, mass together with energy highlight the matter notion. In the same time, mass notion is also highlighted by means of the two properties: gravitation and inertia. Taking into account the fundamental notions and theorems of mechanics, within of fifth sections of this paper, the author has specified that in the case of the translation motion the inertia property is highlighted by mass and position of the mass center. In the case of the resultant rotation motion the inertia property is characterized by mechanical moments of inertia and extension of these properties, known as inertial tensor and pseudoinertial tensor.

Applying new formulations, in the first four sections of the paper was determined the definition expressions for mass properties: mass, position of the mass center, inertial tensor and its generalized variation law, as well pseudoinertial tensor. They are corresponding for every homogeneous body having simple geometrical shape. These mass properties are necessary in the fifth section of the paper, whose objective consists in the establishment the definition expressions for inertial and pseudoinertial tensor corresponding for every compound ensemble physically integrated in the multibody system.

All above mass properties are compulsory found in the dynamical notions like: angular momentum, kinetic energy, acceleration energies of higher order, and their time derivatives according to differential equations of higher order, from analytical dynamics of systems.

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Distribuția Maselor în Dinamica Analitică a Sistemelor

În cazul sistemelor mecanice multicorp (MBS), spre exemplu structura mecanică a robotului, sunt implementate câteva ipoteze simplificatoare cu privire la proprietățile maselor. Conform acestora, proprietățile maselor sunt continuu distribuite între baza fixă și ultimul ansamblu cinetic al structurii mecanice. Prin urmare, în studiul dinamic al MBS autorul lucrării a introdus sintagma “distribuția maselor” în loc de geometria maselor, specifică corpului rigid. Distribuția maselor este bazată pe noțiunea fundamentală de masă în dinamica analitică a sistemelor. La rândul ei, masa împreună cu energia evidențiază noțiunea de materie. Dar, masa este, de asemenea, evidențiată prin două proprietăți: gravitația și inerția. În conformitate cu teoremele fundamentale din dinamica newtoniană, în cazul mișcării de translație proprietatea de inerție este evidențiată prin masă și poziția centrului maselor. În cazul mișcării de rotație rezultantă proprietatea de inerție este caracterizată prin momentele de inerție mecanice și extensia acestor proprietăți, cunoscute ca tensori inerțiali și tensori pseudoinerțiali. Expresiile matriceale ale acestora sunt incluse, obligatoriu, în noțiunile dinamice cum sunt: energia cinetică, energia de accelerații, momentul cinetic și derivatele acestora în raport cu timpul, conform cu ecuațiile diferențiale de ordin superior, specifice dinamicii analitice a sistemelor.

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