TECHNICAL UNIVERSITY OF CLUJ-NAPOCA

ACTA TECHNICA NAPOCENSIS

Series: Applied Mathematics, Mechanics, and Engineering
Vol. 60, Issue III, September, 2017

GENERALIZED FORCES IN ANALYTICAL DYNAMICS OF SYSTEMS

Iuliu NEGREAN

Abstract: *In the case of the multibody systems (MBS), as example the mechanical robot structure, and in accordance with differential principles typical to analytical dynamics of systems, the study of dynamical behavior is based on the generalized forces. They are developed in the direct connection with the generalized variables, also named independent parameters corresponding to holonomic mechanical systems. But, mechanically, the generalized forces are due to: driving sources of the mechanical motion, gravitational forces, manipulating loads, as well as complex frictions from physical links between the kinetic ensembles belonging to MBS. The expressions of definition of the generalized forces contain on the one hand kinematical parameters corresponding to absolute motions, on the other hand the mass properties. The last are highlighted by mass and position of the mass center, inertial tensors and pseudoinertial tensors. By means of the especially researches of the author, in this paper new formulations concerning the kinematical parameters, generalized forces and dynamics equations of the current and sudden motions will be presented. The dynamics study will be also contain acceleration energy and its time derivatives according to differential equations of higher order, typically to analytical dynamics of systems.*

Key words: *analytical dynamics, mechanics, generalized forces, dynamics equations, robotics.*

1. INTRODUCTION

In the case of the multibody systems (MBS), as example the mechanical robot structure, see Fig.1, and according with differential principles answerable to analytical dynamics of systems, the study of dynamical behavior is based on, among others, the generalized forces [3] – [15]. They are mechanically developed in the direct connection with the generalized variables, also named independent parameters (d.o.f.) which they univocally characterize the absolute motion for holonomic mechanical systems. But, mechanically, the generalized forces are due to: driving sources of the mechanical motion, gravitational forces, manipulating loads, as well as complex frictions from physical links between the kinetic ensembles belonging to MBS (for example see Fig.1). In the same time, the expressions of definition for the generalized dynamics forces contain the both kinematical parameters corresponding to absolute motions, geometrical features, and the mass properties. The last are highlighted by mass and position of the mass center, as well as inertial tensors and pseudoinertial tensors [4], [5], [7], [8] and [14].

On the basis of the especially researches of the author, in the four sections of this paper new formulations will be presented. So, the first section is devoted to the kinematical parameters typical to MBS (mechanical robot structures) with current and sudden motions. Second and third sections of the paper highlight, by means of transfer matrices, the generalized gravitational, manipulating and inertia forces, as well as the generalized friction forces. In the fourth section dynamics equations with acceleration energies and their time derivatives according to [9] - [14] are developed. They are typical to the analytical dynamics of systems with sudden motions.

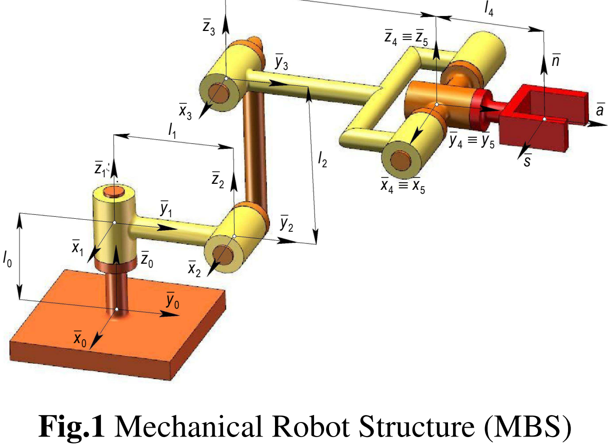


Fig.1 Mechanical Robot Structure (MBS)

2. KINEMATICS PARAMETERS OF MBS

The kinematical and dynamical study from this paper [4], [5], [7] is oriented on mechanical structure with opened kinematical chain, where the kinetic ensembles $i = 1 \rightarrow n$ are physically linked by driving joints of fifth order. (Example mechanical robot structure sees Fig.1).

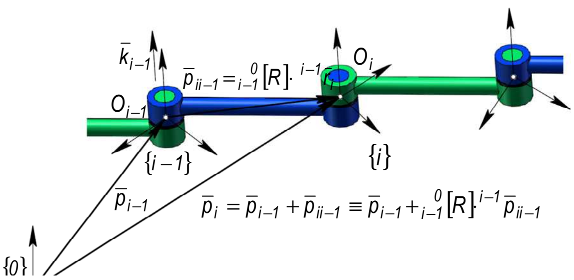


Fig.2 Sequence of Kinetic Ensembles

This is characterized by $(n\ d.o.f.)$, according to:

$$\bar{\theta} \neq \bar{\theta}^{(0)}; \quad \bar{\theta}(t) = \left[q_i(t); \quad i = 1 \rightarrow n\right]^T, \tag{1}$$

where $q_i(t)$ is the generalized coordinate from every driving axis. But, considering the current and sudden motions the generalized variables of higher order are developed as follows:

$$\left\{\begin{array}{l} \left\{\bar{\theta}(t); \dot{\bar{\theta}}(t); \ddot{\bar{\theta}}(t); \cdots; \overset{(m)}{\bar{\theta}}(t)\right\} = \\ = \left\{\begin{array}{c} q_i(t); \dot{q}_i(t); \ddot{q}_i(t); \cdots; \overset{(m)}{q_i}(t) \\ i = 1 \rightarrow n,\ m \geq 1 \end{array}\right\} \end{array}\right\}, \tag{2}$$

and (m) represents the time deriving order. The main objective of this section consists in the establishment of the absolute angular and linear velocities and accelerations for every kinetic ensemble from MBS. Unlike the classical approaches [3] – [5], [7] in the following a few formulations based on the time derivatives of the locating matrices will be developed.

So, in the Fig.2 a sequence of two kinetic ensembles belonging to MBS is subjected to kinematical study. According to [5] – [7], the locating matrices are the following:

$$\left\{\begin{array}{c} {}_{i}^{0}[T](t) = {}_{i}^{0}[T](t) \cdot {}_{i}^{i-1}[T](t) = \\ \begin{bmatrix} {}_{i}^{0}[R](t) & \bar{p}_i(t) \\ 0\ 0\ 0 & 1 \end{bmatrix} = \\ = \begin{bmatrix} {}_{i-1}^{0}[R](t) \cdot {}_{i}^{i-1}[R] & \bar{p}_{i-1} + \bar{p}_{ii-1} \\ 0\ \ 0\ \ 0 & 1 \end{bmatrix} \end{array}\right\}, \tag{3}$$

They define the locating (position – orientation) of the moving frame $\{i\}$ versus $\{0\}$. The components with the same significance are written below thus:

$${}_{i-1}^{0}[T](t) = \begin{bmatrix} {}_{i-1}^{0}[R](t) & \bar{p}_{i-1}(t) \\ 0\ 0\ 0 & 1 \end{bmatrix}, \tag{4}$$

$${}_{i}^{i-1}[T](t) = \begin{bmatrix} {}_{i}^{i-1}[R] & {}^{i-1}\bar{p}_{ii-1} \\ 0\ 0\ 0 & 1 \end{bmatrix}, \tag{5}$$

The matrix components from (3) – (5) are:

$${}_{i}^{i-1}[R] = R_{ii-1} \cdot R\left(\bar{k}_i; q_i(t) \cdot \Delta_i\right), \tag{6}$$

$${}^{i-1}\bar{p}_{ii-1} = {}^{i-1}\bar{p}_{ii-1}^{(0)} + \left(1 - \Delta_i\right) \cdot q_i(t) \cdot {}^{i-1}\bar{k}_i, \tag{7}$$

$${}_{i}^{0}[R](t) = {}_{i-1}^{0}[R](t) \cdot {}_{i}^{i-1}[R], \tag{8}$$

$$\bar{p}_i(t) = \bar{p}_{i-1}(t) + \bar{p}_{ii-1}(t) = \bar{p}_{i-1}(t) + {}_{i-1}^{0}[R] \cdot {}^{i-1}\bar{p}_{ii-1}, \tag{9}$$

$$\Delta_i = \left\{[1,\ i = R]; [0,\ i = T]\right\}; \tag{10}$$

The symbol (10) shows the driving joint type.

On matrix (3) is applied the first time derivative:

$$\left\{\begin{array}{c} {}_{i}^{0}\left[\dot{T}\right](t) = \begin{bmatrix} {}_{i}^{0}\left[\dot{R}\right](t) & \dot{\bar{p}}_i(t) \\ 0\ 0\ 0 & 0 \end{bmatrix} = \\ = {}_{i-1}^{0}\left[\dot{T}\right](t) \cdot {}_{i}^{i-1}[T](t) + {}_{i-1}^{0}[T](t) \cdot {}_{i}^{i-1}\left[\dot{T}\right](t) \end{array}\right\}. \tag{11}$$

The matrix components from right member are:

$$\left\{\begin{array}{c} {}_{i-1}^{0}\left[\dot{T}\right](t) \cdot {}_{i}^{i-1}[T](t) = \\ = \begin{bmatrix} {}_{i-1}^{0}\left[\dot{R}\right] \cdot {}_{i}^{i-1}[R] & \dot{\bar{p}}_{i-1} + {}_{i-1}^{0}\left[\dot{R}\right] \cdot {}^{i-1}\bar{p}_{ii-1} \\ 0\ 0\ 0 & 1 \end{bmatrix} \end{array}\right\}; \tag{12}$$

$$\left\{\begin{array}{c} {}_{i-1}^{0}[T](t) \cdot {}_{i}^{i-1}\left[\dot{T}\right](t) = \\ = \begin{bmatrix} {}_{i-1}^{0}[R](t) & \bar{p}_{i-1}(t) \\ 0\ 0\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} \Delta_i \cdot {}_{i}^{i-1}\left[\dot{R}\right] & \left(1 - \Delta_i\right) \cdot {}^{i-1}\dot{\bar{p}}_{ii-1} \\ 0\ 0\ 0 & 1 \end{bmatrix} \\ = {}_{i-1}^{0}[T](t) \cdot \begin{bmatrix} \dot{R}\left[\bar{k}_i; q_i(t) \cdot \Delta_i\right] & \left(1 - \Delta_i\right) \cdot \dot{q}_i(t) \cdot {}^{i-1}\bar{k}_i \\ 0\ 0\ 0 & 1 \end{bmatrix} \end{array}\right\}. \tag{13}$$

Considering (12) and (13), matrix (11) becomes:

$$\left\{\begin{array}{c} {}_{i}^{0}\left[\dot{T}\right]\left[q_j(t); \dot{q}_j(t); j = 1 \rightarrow i\right] = \begin{bmatrix} {}_{i}^{0}\left[\dot{R}\right](t) & \dot{\bar{p}}_i(t) \\ 0\ 0\ 0 & 0 \end{bmatrix} = \\ + \begin{bmatrix} {}_{i-1}^{0}\left[\dot{R}\right] \cdot {}_{i}^{i-1}[R] & \dot{\bar{p}}_{i-1} + {}_{i-1}^{0}\left[\dot{R}\right] \cdot {}^{i-1}\bar{p}_{ii-1} \\ 0\ 0\ 0 & 1 \end{bmatrix} + \\ \begin{bmatrix} \Delta_i \cdot {}_{i-1}^{0}[R] \cdot {}_{i}^{i-1}\left[\dot{R}\right] & \left(1 - \Delta_i\right) \cdot \dot{q}_i(t) \cdot {}_{i-1}^{0}[R] \cdot {}^{i-1}\bar{k}_i + \bar{p}_{i-1} \\ 0\ 0\ 0 & 1 \end{bmatrix} \end{array}\right\} \tag{14}$$

According to [7], matrix (14) is identical with:

$$\left\{\begin{array}{c} {}_{i}^{0}\left[\dot{T}\right]\left[q_j(t); \dot{q}_j(t); j = 1 \rightarrow i\right] = \\ = \begin{bmatrix} \left(\dot{\bar{\psi}}_i \times\right) & \dot{\bar{p}}_i - \dot{\bar{\psi}}_i \times \bar{p}_i \\ 0\ \ 0\ \ 0 & 0 \end{bmatrix} \cdot {}_{i}^{0}[T](t) \end{array}\right\}, \tag{15}$$

and $\bar{\psi}_i$ is orientation vector from $\{i\}$ versus $\{0\}$.

Considering the time derivative property (15), on the matrix (11) a few transformations are:

$$\left\{\begin{aligned}&{}^0_i\left[\dot T\right](t)\cdot{}^0_i[T]^{-1}(t)=\\&=\begin{bmatrix}{}^0_i\left[\dot R\right]\cdot{}^0_i[R]^T & \dot{\bar p}_i(t)-{}^0_i\left[\dot R\right]\cdot{}^0_i[R]^T\cdot\bar p_i(t)\\ 0\;0\;0 & 0\end{bmatrix}\\&=\begin{bmatrix}\{{}^0\bar\omega_i\times\} & \dot{\bar p}_i(t)-\{{}^0\bar\omega_i\times\}\cdot\bar p_i(t)\\ 0\;0\;0 & 0\end{bmatrix}\end{aligned}\right\}, \quad (16)$$

$$\{{}^0\bar\omega_i\times\}={}^0_i\left[\dot R\right]\cdot{}^0_i[R]^T,\ \{{}^i\bar\omega_i\times\}={}^0_i[R]^T\cdot{}^0_i\left[\dot R\right];\quad (17)$$

where properties (17) are according to [7] – [8]. Using (11), expression (16) is written again as:

$$\left\{\begin{aligned}&{}^0_i\left[\dot T\right](t)\cdot{}^0_i[T]^{-1}(t)=\\&={}^0_{i-1}\left[\dot T\right](t)\cdot{}^{i-1}_i[T](t)\cdot{}^0_i[T]^{-1}(t)+\\&+{}^0_{i-1}[T](t)\cdot{}^{i-1}_i\left[\dot T\right](t)\cdot{}^0_i[T]^{-1}(t)\end{aligned}\right\}. \quad (18)$$

The first matrix term from (18) becomes:

$$\left\{\begin{aligned}&{}^0_{i-1}\left[\dot T\right](t)\cdot{}^{i-1}_i[T](t)\cdot{}^0_i[T]^{-1}(t)=\\&={}^0_{i-1}\left[\dot T\right](t)\cdot{}^{i-1}_i[T](t)\cdot{}^{i-1}_i[T]^{-1}(t)\cdot{}^0_{i-1}[T]^{-1}(t)=\\&={}^0_{i-1}\left[\dot T\right](t)\cdot{}^0_{i-1}[T]^{-1}(t)=\\&=\begin{bmatrix}\{{}^0\bar\omega_{i-1}\times\} & \dot{\bar p}_{i-1}(t)-\{{}^0\bar\omega_{i-1}\times\}\cdot\bar p_{i-1}(t)\\ 0\;0\;0 & 0\end{bmatrix}\end{aligned}\right\}, \quad (19)$$

where $\{{}^0\bar\omega_{i-1}\times\}={}^0_{i-1}\left[\dot R\right]\cdot{}^0_{i-1}[R]^T$, (see (17)).

The second matrix term from (18) is shown as:

$$\left\{\begin{aligned}&{}^0_{i-1}[T](t)\cdot{}^{i-1}_i\left[\dot T\right](t)\cdot{}^0_i[T]^{-1}(t)=\\&={}^0_{i-1}[T](t)\cdot\left\{{}^{i-1}_i\left[\dot T\right](t)\cdot{}^{i-1}_i[T]^{-1}(t)\right\}\cdot{}^0_{i-1}[T]^{-1}(t)\\&=\begin{bmatrix}\left[dR\left(\Delta_i\cdot\dot q_i\right)\right] & d\bar p\left[\left(1-\Delta_i\right)\cdot\dot q_i\right]\\ 0\;0\;0 & 0\end{bmatrix}\end{aligned}\right\} \quad (20)$$

where $\left\{\Delta_i\cdot\dot q_i(t)\cdot{}^{i-1}\bar k_i\times\right\}=\Delta_i\cdot{}^{i-1}_i\left[\dot R\right]\cdot{}^{i-1}_i[R]^T$. (21)

The components from (20) are developed thus:

$$\left[dR\left(\Delta_i\cdot\dot q_i\right)\right]=\left\{\Delta_i\cdot\dot q_i(t)\cdot{}^0\bar k_i\times\right\}, \quad (22)$$

$$\left\{\begin{aligned}&d\bar p\left[\left(1-\Delta_i\right)\cdot\dot q_i\right]=\left(1-\Delta_i\right)\cdot\dot q_i(t)\cdot{}^0\bar k_i-\\&-\left\{\Delta_i\cdot\dot q_i(t)\cdot{}^0\bar k_i\times\right\}\cdot\left\{\bar p_{i-1}+{}^0_{i-1}[R]\cdot{}^{i-1}\bar p_{ii-1}\right\}\end{aligned}\right\}. \quad (23)$$

Taking into account on the one hand (16), and on the other hand (18) with the components (19), as well as (20) – (23) the following matrix and differential identity is obtained below:

$$\left\{\begin{aligned}&\begin{bmatrix}\{{}^0\bar\omega_i\times\} & \dot{\bar p}_i(t)-\{{}^0\bar\omega_i\times\}\cdot\bar p_i(t)\\ 0\;0\;0 & 0\end{bmatrix}=\\&=\begin{bmatrix}\{{}^0\bar\omega_{i-1}\times\} & \dot{\bar p}_{i-1}(t)-\{{}^0\bar\omega_{i-1}\times\}\cdot\bar p_{i-1}(t)\\ 0\;0\;0 & 0\end{bmatrix}+\\&+\begin{bmatrix}\left[dR\left(\Delta_i\cdot\dot q_i\right)\right] & d\bar p\left[\left(1-\Delta_i\right)\cdot\dot q_i\right]\\ 0\;0\;0 & 0\end{bmatrix}\end{aligned}\right\}. \quad (24)$$

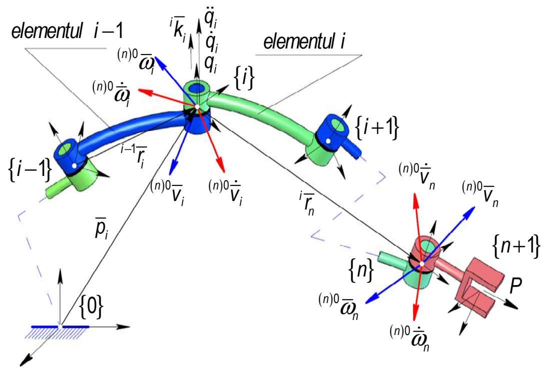


Fig. 3 Kinematical Parameters for MBS

Identifying the angular (rotation) components from the above matrix identity (24), it obtains:

$$\{{}^0\bar\omega_i\times\}=\{{}^0\bar\omega_{i-1}\times\}+\left\{\Delta_i\cdot\dot q_i(t)\cdot{}^0\bar k_i\times\right\}. \quad (25)$$

In above identity of skew-symmetric matrices, vector equation of angular velocity is selected:

$${}^0\bar\omega_i(t)={}^0\bar\omega_{i-1}(t)+\Delta_i\cdot\dot q_i(t)\cdot{}^0\bar k_i(t). \quad (26)$$

It represents the equation of definition of the angular rotation velocity vector, corresponding to absolute rotation of the kinetic ensemble from MBS with opened kinematical chain (see Fig.3). Identifying the position (linear) components (last column) from the matrix identity (24), it obtains:

$$\left\{\begin{aligned}&\dot{\bar p}_i(t)-\{{}^0\bar\omega_i\times\}\cdot\bar p_i(t)=\dot{\bar p}_{i-1}(t)-\\&-\{{}^0\bar\omega_{i-1}\times\}\cdot\bar p_{i-1}(t)+\\&+\left(1-\Delta_i\right)\cdot\dot q_i(t)\cdot{}^0\bar k_i-\\&-\left\{\Delta_i\cdot\dot q_i(t)\cdot{}^0\bar k_i\times\right\}\cdot\left\{\bar p_{i-1}+{}^0_{i-1}[R]\cdot{}^{i-1}\bar p_{ii-1}\right\}\end{aligned}\right\}. \quad (27)$$

After a few transformations (27) is changed as:

$$\left\{\begin{aligned}&\dot{\bar p}_i(t)=\dot{\bar p}_{i-1}(t)+\{{}^0\bar\omega_{i-1}\times\}\cdot{}^0_{i-1}[R]\cdot{}^{i-1}\bar p_{ii-1}+\\&+\left(1-\Delta_i\right)\cdot\dot q_i(t)\cdot{}^0\bar k_i\end{aligned}\right\}. \quad (28)$$

Using the definition of the linear velocity for the origin of frames: $\{i\}$ and $\{i-1\}$, (see [5] and [7]), the equation (28) is written below as follows:

$$\left\{\begin{aligned}&{}^0\bar v_i={}^0\bar v_{i-1}+\{{}^0\bar\omega_{i-1}\times\}\cdot{}^0_{i-1}[R]\cdot{}^{i-1}\bar p_{ii-1}+\\&+\left(1-\Delta_i\right)\cdot\dot q_i(t)\cdot{}^0\bar k_i\end{aligned}\right\}. \quad (29)$$

It represents the equation of definition of the linear velocity vector, corresponding to absolute motion of the origin $O_i \in \{i\}$ belonging to kinetic ensemble from MBS with opened chain (Fig.3).
Applying the absolute time derivatives of first order on (26) and (29), and performing a few differential transformations, the equations of definition for angular and linear accelerations vectors are obtained: ${}^{0}\dot{\overline{\omega}}_i$ and respectively ${}^{0}\dot{\overline{v}}_i$. But, especially in the dynamics equations the above kinematical parameters are required by the components with respect own frame $\{i\}$. The angular and linear velocities and accelerations, corresponding to every kinetic ensemble (Fig.3) are below presented by means of the definition equations with respect own frame $\{i\}$ thus:

$$ {}^{i}\overline{\omega}_i = {}_{i-1}^{\;\;i}[R]\cdot {}^{i-1}\overline{\omega}_{i-1} + \Delta_i \cdot \dot{q}_i \cdot {}^{i}\overline{k}_i \,; \quad (30) $$

$$ \left\{ \begin{array}{c} {}^{i}\overline{v}_i = {}_{i-1}^{\;\;i}[R]\cdot\left\{ {}^{i-1}\overline{v}_{i-1} + {}^{i-1}\overline{\omega}_{i-1} \times {}^{i-1}\overline{p}_{ii-1} \right\} + \\ + \left(1-\Delta_i\right)\cdot \dot{q}_i \cdot {}^{i}\overline{k}_i \end{array} \right\} ; \quad (31) $$

$$ \left\{ \begin{array}{c} {}^{i}\dot{\overline{\omega}}_i = {}_{i-1}^{\;\;i}[R]\cdot {}^{i-1}\dot{\overline{\omega}}_{i-1} + \\ +\Delta_i \cdot \left\{ {}_{i-1}^{\;\;i}[R]\cdot {}^{i-1}\overline{\omega}_{i-1} \times \dot{q}_i \cdot {}^{i}\overline{k}_i + \ddot{q}_i \cdot {}^{i}\overline{k}_i \right\} \end{array} \right\} ; \quad (32) $$

$$ \left\{ \begin{array}{c} {}^{i}\dot{\overline{v}}_i = {}_{i-1}^{\;\;i}[R]\cdot\left[{}^{i-1}\dot{\overline{v}}_{i-1} + {}^{i-1}\dot{\overline{\omega}}_{i-1} \times {}^{i-1}\overline{p}_{ii-1} + \right. \\ \left. + {}^{i-1}\overline{\omega}_{i-1} \times {}^{i-1}\overline{\omega}_{i-1} \times {}^{i-1}\overline{p}_{ii-1} \right] + \\ + \left(1-\Delta_i\right)\cdot\left(2\cdot {}^{i}\overline{\omega}_i \times \dot{q}_i \cdot {}^{i}\overline{k}_i + \ddot{q}_i \cdot {}^{i}\overline{k}_i\right) \end{array} \right\} . \quad (33) $$

They are function in exclusivity of parameters included in $[0;i]$ kinematical interval [7]. So, they are applied by outward iterations $i = 1 \rightarrow n$. When $(i=1)$, within of the equations (30) – (33) the kinematical parameters of the fixe basis from MBS are substituted, according to next:

$$ \left\{ {}^{0}\overline{\omega}_0 = \overline{0},\; {}^{0}\dot{\overline{\omega}}_0 = \overline{0},\; {}^{0}\overline{v}_0 = 0,\; {}^{0}\dot{\overline{v}}_0 = 0 \right\} . \quad (34) $$

When $(i=n)$, the kinematical parameters of the last kinetic ensemble from MBS are obtained. They are operational velocities and accelerations:

$$ \left\{ \begin{array}{c} {}^{(n)0}\underset{(6\times1)}{\dot{X}}\left[\overline{\theta}(t);\dot{\overline{\theta}}(t)\right] = \\ \left\{ {}^{(n)0}\overline{v}_n^T\left[\overline{\theta}(t);\dot{\overline{\theta}}(t)\right] \quad {}^{(n)0}\overline{\omega}_n^T\left[\overline{\theta}(t);\dot{\overline{\theta}}(t)\right] \right\}^T \end{array} \right\} ; \quad (35) $$

$$ \left\{ \begin{array}{c} {}^{(n)0}\underset{(6\times1)}{\ddot{X}}\left[\overline{\theta}(t);\dot{\overline{\theta}}(t);\ddot{\overline{\theta}}(t)\right] = \\ = \left\{ \begin{array}{c} {}^{(n)0}\dot{\overline{v}}_n\left[\overline{\theta}(t);\dot{\overline{\theta}}(t);\ddot{\overline{\theta}}(t)\right] \\ {}^{(n)0}\dot{\overline{\omega}}_n\left[\overline{\theta}(t);\dot{\overline{\theta}}(t);\ddot{\overline{\theta}}(t)\right] \end{array} \right\} \end{array} \right\} ; \quad (36) $$

$$ {}^{(n)0}\dot{X}\left[\overline{\theta}(t);\dot{\overline{\theta}}(t)\right] = {}^{(n)0}R \cdot {}^{(0)n}\dot{X}\left[\overline{\theta}(t);\dot{\overline{\theta}}(t)\right] ; \quad (37) $$

$$ \left\{ \begin{array}{c} {}^{(n)0}\ddot{X}\left[\overline{\theta}(t);\dot{\overline{\theta}}(t);\ddot{\overline{\theta}}(t)\right] = \\ = {}^{(n)0}R \cdot {}^{(0)n}\ddot{X}\left[\overline{\theta}(t);\dot{\overline{\theta}}(t);\ddot{\overline{\theta}}(t)\right] \end{array} \right\} ; \quad (38) $$

$$ \underset{(6\times6)}{{}^{(n)0}R} = \begin{bmatrix} {}_{n}^{0}[R]^{(T)} & [0] \\ [0] & {}_{n}^{0}[R]^{(T)} \end{bmatrix} ; \quad (39) $$

The above expressions (35) and (37), as well (36) and (38) represent the linear and angular velocities as well accelerations, corresponding to motion of the last kinetic ensemble of MBS with respect to absolute Cartesian frame, [5] – [7]. To these is added the locating matrix (3) for $(i=n)$.

The operational velocities and accelerations (35) and (36), according to [7] and [14], are also determined by means of the transfer matrices answerable to kinematical modeling of MBS. In the view of this, for beginning the orientation vector of the last kinetic ensemble of MBS is:

$$ \overline{\psi}(t) = \left[\alpha_A(t) \quad \beta_B(t) \quad \gamma_C(t)\right]^T ; \quad (40) $$

$$ \left\{ \begin{array}{c} {}^{0}J_{\psi}\left[\alpha_A(t) - \beta_B(t) - \gamma_C(t)\right] = \\ \left[{}^{0}\overline{A} \quad R\left(\overline{A};\alpha_A\right)\cdot\overline{B} \quad R\left(\overline{A};\alpha_A\right)\cdot R\left(\overline{B};\beta_B\right)\cdot\overline{C} \right] \end{array} \right\} ; \quad (41) $$

$$ \dot{\overline{\psi}}(t) = {}^{0}J_{\psi}\left[\alpha_A(t) - \beta_B(t) - \gamma_C(t)\right]\cdot \dot{\overline{\psi}}(t) ; \quad (42) $$

where (41) represents the angular transfer matrix defined as function of set of orientation angles.
In the following, according to [7] – [14], column vectors corresponding to angular and linear transfer matrices for velocities and accelerations, as well as their time derivatives are presented for $(i = 1 \rightarrow n)$. So, the equations of definition are:

$$ \left\{ \begin{array}{c} \overline{\Omega}_i = {}^{0}\overline{k}_i \cdot \Delta_i = {}_{i}^{0}[R]\cdot {}^{i}\overline{k}_i \cdot \Delta_i = \\ = vect\left\{ \frac{\partial}{\partial q_i}\left\{ {}_{n}^{0}[R] \right\} \cdot {}_{n}^{0}[R]^T \right\} \cdot \Delta_i = \\ = {}^{0}J_{\psi}(t) \cdot \frac{\partial}{\partial q_i}\left[\overline{\psi}(t)\right]\cdot \Delta_i \end{array} \right\} ; \quad (43) $$

$$ \dot{\overline{\Omega}}_i = {}^{0}\dot{\overline{k}}_i \cdot \Delta_i = {}_{i}^{0}\left[\dot{R}\right]\cdot {}^{i}\overline{k}_i \cdot \Delta_i = {}^{0}\overline{\omega}_i \times {}^{0}\overline{k}_i \cdot \Delta_i ; \quad (44) $$

$$ \overline{V}_i = \frac{\partial \overline{p}_n}{\partial q_i} = {}^{0}\overline{k}_i \times \left(\overline{p}_n - \overline{p}_i\right)\cdot \Delta_i + \left(1-\Delta_i\right)\cdot {}^{0}\overline{k}_i ; \quad (45) $$

$$ \left\{ \begin{array}{c} \dot{\overline{V}}_i = \sum_{j=1}^{n} \frac{\partial^2 \overline{p}_n}{\partial q_i \cdot \partial q_j} \cdot \dot{q}_j = \\ = \frac{d}{dt}\left[{}^{0}\overline{k}_i \times \left(\overline{p}_n - \overline{p}_i\right)\cdot \Delta_i + \left(1-\Delta_i\right)\cdot {}^{0}\overline{k}_i \right] \end{array} \right\} ; \quad (46) $$

where (44) and (46) are based on property (17).

The components (43) and (44), respectively (45) and (46) are corresponding to transfer matrices for angular and respectively linear velocities and accelerations. Finally, the following are obtained:

$$\left\{ \begin{array}{l} {}^0J\left[\bar\theta(t)\right] = \begin{bmatrix} V\left[\bar\theta(t)\right] \\ \dots\dots\dots\dots \\ \Omega\left[\bar\theta(t)\right] \end{bmatrix} = \\ = \left[{}^0J_i(t) = \begin{pmatrix} \bar V_i(t) \\ \dots\dots \\ \bar\Omega_i(t) \end{pmatrix}; \; i = 1 \to n \right] \end{array} \right\}; \qquad (47)$$

$$\left\{ \begin{array}{l} {}^0\dot J\left[\bar\theta(t)\right] = \begin{bmatrix} \dot V\left[\bar\theta(t)\right] \\ \dots\dots\dots\dots \\ \dot\Omega\left[\bar\theta(t)\right] \end{bmatrix} = \\ = \left[{}^0\dot J_i(t) = \begin{pmatrix} \dot{\bar V}_i(t) \\ \dots\dots \\ \dot{\bar\Omega}_i(t) \end{pmatrix}; \; i = 1 \to n \right] \end{array} \right\}; \qquad (48)$$

$${}^{(n)0}J\left[\bar\theta(t)\right] = {}^{(n)0}R \cdot {}^{(0)n}J\left[\bar\theta(t)\right]; \qquad (49)$$

$${}^{(n)0}\dot J\left[\bar\theta(t)\right] = {}^{(n)0}R \cdot {}^{(0)n}\dot J\left[\bar\theta(t)\right]. \qquad (50)$$

According to [5] and [7], the matrix expression (47) represents the Jacobian matrix, also named the velocity transfer matrix, sometimes matrix of partial derivatives of the locating equations. The expression (48) is the time derivative of the Jacobian matrix, while (49) and (50) are the transfer relationships from one to another frame. The inverse of Jacobian matrix is determined as:

$${}^0J(\bar\theta)^{-1} = \left[{}^0J(\bar\theta)^T \cdot {}^0J(\bar\theta)\right]^{-1} \cdot {}^0J(\bar\theta)^T; \qquad (51)$$

$${}^0J(\bar\theta)^{-1} = {}^0J(\bar\theta)^T \cdot \left[{}^0J(\bar\theta) \cdot {}^0J(\bar\theta)^T\right]^{-1}. \qquad (52)$$

When the mechanical robot structure (Fig. 1) is dominated by sudden motions, the generalized and operational accelerations of higher order are developed. Considering the researches from [7] – [14], in the paper the expressions become:

$$\left\{ \begin{array}{l} {}^0\overset{(m)}{X}(t) = {}^0J\left[\bar\theta(t)\right] \cdot \overset{(m)}{\theta}(t) + \\ + \sum_{k=1}^{m-1} \frac{(m-1)!}{k!(m-k-1)!} \cdot {}^0\overset{(k)}{J}\left[\bar\theta(t)\right] \cdot \overset{[m-k]}{\theta}(t) = \\ = \sum_{k=1}^{m} \frac{(m-1)!}{(k-1)!(m-k)!} \cdot {}^0\overset{(k-1)}{J}\left[\bar\theta(t)\right] \cdot \overset{[m-(k-1)]}{\theta}(t) \end{array} \right\}; \qquad (53)$$

where (m) is the order of the time derivatives, the symbol ${}^0\overset{(m)}{X}(t)$ represents the column matrix of the operational accelerations of higher order, and $\overset{(m)}{\theta}(t)$ is the column matrix of generalized accelerations of higher order, according with:

$$\left\{ \begin{array}{l} \overset{(m)}{\theta}(t) = {}^0J\left[\bar\theta(t)\right]^{-1} \cdot {}^0\overset{(m)}{X}(t) - \\ - {}^0J\left[\bar\theta(t)\right]^{-1} \cdot \sum_{k=1}^{m-1} \frac{(m-1)!}{k!(m-k-1)!} \cdot {}^0J\left[\overset{(k)}{\bar\theta}(t)\right] \cdot \overset{(m-k)}{\theta}(t) \end{array} \right\} \qquad (54)$$

Considering the mathematical models from [7], the Jacobian matrix (47) can be also determined with matrix exponentials. But, the components of (53) and (54) are based on rotation matrices and position vectors, according (43) – (48). So, their time derivatives of higher order show as:

$$\underset{(6\times n)}{{}^0J}\left[\overset{(k)}{\bar\theta}(t)\right] \equiv \left[\underset{(6\times 1)}{{}^0J_i}\left[\overset{(k)}{\bar\theta}_i(t)\right] \text{ where } i = 1 \to n \right]; \qquad (55)$$

$$\underset{(6\times 1)}{{}^0J_i}\left[\overset{(k)}{\bar\theta}_i(t)\right] = \begin{pmatrix} \overset{(k)}{V}_i(t) \\ \dots\dots \\ \overset{(k)}{\Omega}_i(t) \end{pmatrix}; \qquad (56)$$

$$\left\{ \begin{array}{l} \frac{d^k}{dt^k}\left\{{}^0_i[R](t)\right\} = {}^0_i\overset{(k)}{[R]} = \sum_{j=1}^{i} \left\{ \frac{\partial}{\partial \overset{(m)}{q}_j} \left\{ {}^0_s\overset{(m)}{[R]} \right\} \cdot \Delta_j \cdot \overset{(k)}{q}_j \right\} + \\ \sum_{j=1}^{i} \sum_{r=1}^{k-1} \left\{ \frac{\prod_{p=1}^{r}(k-p)}{p!} \cdot \left\{ \frac{d^p}{dt^p} \left\{ \frac{\partial}{\partial \overset{(m)}{q}_j} \left\{ {}^0_s\overset{(m)}{[R]} \right\} \right\} \cdot \Delta_j \cdot \overset{(k-p)}{q}_j \right\} \right\} = \\ = \sum_{j=1}^{i} \left\{ \frac{\partial}{\partial \overset{(m)}{q}_j} \left\{ {}^0_s\overset{(m)}{[R]} \right\} \cdot \Delta_j \cdot \overset{(k)}{q}_j \right\} + \\ \sum_{j=1}^{i} \sum_{r=1}^{k-1} \left\{ \frac{\prod_{p=1}^{r}(k-p)}{p!} \cdot \left\{ \frac{p! \cdot m!}{(m+p)!} \cdot \frac{\partial}{\partial \overset{(m)}{q}_j} \left\{ {}^0_s\overset{(m+p)}{[R]} \right\} \cdot \Delta_j \cdot \overset{(k-p)}{q}_j \right\} \right\} \end{array} \right\}$$

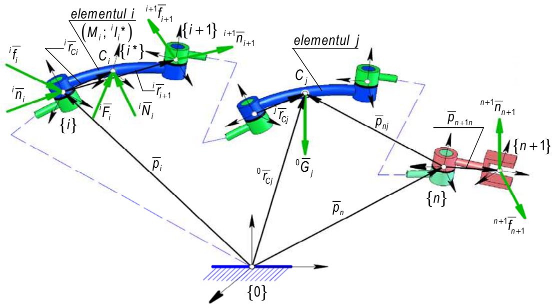
$$\left\{ \begin{array}{l} \frac{d^k \bar p_i(t)}{dt^k} = \overset{(k)}{\bar p}_i = \sum_{j=1}^{i} \left[\frac{\partial \overset{(m)}{\bar p}_i}{\partial \overset{(m)}{q}_j} \cdot \overset{(k)}{q}_j \right] + \\ \sum_{j=1}^{i} \sum_{r=1}^{k-1} \left\{ \frac{\prod_{p=1}^{r}(k-p)}{p!} \cdot \left[\frac{p! \cdot m!}{(m+p)!} \cdot \frac{\partial \overset{(m+p)}{\bar p}_i}{\partial \overset{(m)}{q}_j} \cdot \overset{(k-p)}{q}_j \right] \right\} \end{array} \right\} \qquad (57)$$

and
$$\left\{ \begin{array}{l} k \ge 1; \; k = \{1;2;3;4;5;\dots\} \\ m \ge (k+1); \; m = \{2;3;4;5;\dots\} \end{array} \right\}; \qquad (58)$$

where the symbols: (k) and (m) are the orders of time derivatives concerning (55) – (58).

3. GENERALIZED ACTIVE FORCES

In accordance with [3], [4] and [7], on every kinetic ensemble $(i = 1 \to n)$, belonging to the mechanical robot structure, as integrated part from MBS, are especially applied a system of external and active forces, manipulating loads,



as well as complex friction forces, see Fig.4.

Fig. 4 Distribution of the forces on MBS

In function of (static or dynamic) behavior in every physical link (driving joint of fifth order) generalized static or driving force is developed.

For beginning, using the *classical algorithm* [4] and [7], generalized forces corresponding to static equilibrium are established. Considering theorems from statics of mechanical systems and a few transformations, the next expressions are obtained by inward iterations $(i = n \to 1)$, thus:

$$ {}^i\bar{f}_i = M_i \cdot {}^0_i[R]^T \cdot \bar{g} + {}^i_{i+1}[R] \cdot {}^{i+1}\bar{f}_{i+1}; \qquad (59)$$

$$\left\{ \begin{array}{l} {}^i\bar{n}_i = {}^i\bar{r}_{C_i} \times {}^0_i[R]^T \cdot \bar{g} \cdot M_i + \\ + {}^i\bar{r}_{i+1} \times {}^i_{i+1}[R] \cdot {}^{i+1}\bar{f}_{i+1} + {}^i_{i+1}[R] \cdot {}^{i+1}\bar{n}_{i+1} \end{array} \right\}; \qquad (60)$$

$$\bar{g} = \tau \cdot g \cdot \bar{k}_0,\ and\ \bar{k}_0 = \{\bar{x}_0; \bar{y}_0; \bar{z}_0\} \in \{0\}\ frame; \qquad (61)$$

$$\tau = -\bar{k}_g^T \cdot \bar{k}_0 = \left\{ \begin{array}{l} -1;\ \bar{k}_g^T \cdot \bar{k}_0 = 1 \\ 1;\ \bar{k}_g^T \cdot \bar{k}_0 = -1 \end{array} \right\},\ \bar{k}_g = {}^0\bar{g} / \left| {}^0\bar{g} \right|; \qquad (62)$$

$$Q_S^i = \left[{}^i\bar{f}_i^{\,T} \cdot (1 - \Delta_i) + {}^i\bar{n}_i^T \cdot \Delta_i \right] \cdot {}^i\bar{k}_i. \qquad (63)$$

The symbols from above expressions have the following significances: ${}^i\bar{f}_i$ and ${}^i\bar{n}_i$ are the action (force and moment of force) from $(i-1)$ on (i) ensemble of MBS (Fig.4); M_i and ${}^i\bar{r}_{C_i}$ are mass and position of the mass center; g module of gravitational acceleration (61); Q_S^i is generalized static force (63). It observes that all vectors are projected on driving axis whose unit vector is ${}^i\bar{k}_i$.

To highlight the influence of gravitational and manipulating loads, (59) and (60) are written as:

$$\left\{ \begin{array}{l} {}^i\bar{f}_i = \Delta_m^2 \cdot \sum_{j=i}^{n} M_j \cdot {}^0_i[R]^T \cdot \bar{g} + \\ + (-1)^{\Delta_M} \cdot \dfrac{1-\Delta_m}{1+\Delta_m} \cdot {}^i_{n+1}[R] \cdot {}^{n+1}\bar{f}_{n+1} \end{array} \right\}; \qquad (64)$$

$$\left\{ \begin{array}{l} {}^i\bar{n}_i = \Delta_m^2 \cdot \sum_{j=i}^{n} M_j \cdot {}^0_i[R]^T \cdot \left[\left({}^0\bar{r}_{C_j} - \bar{p}_i \right) \times \bar{g} \right] + \\ (-1)^{\Delta_m} \cdot \dfrac{1-\Delta_m}{1+\Delta_m} \cdot \left\{ {}^0_i[R]^T \cdot (\bar{p} - \bar{p}_i) \times {}^i_{n+1}[R] \cdot {}^{n+1}\bar{f}_{n+1} \right. \\ \left. + {}^i_{n+1}[R] \cdot {}^{n+1}\bar{n}_{n+1} \right\} \end{array} \right\} \qquad (65)$$

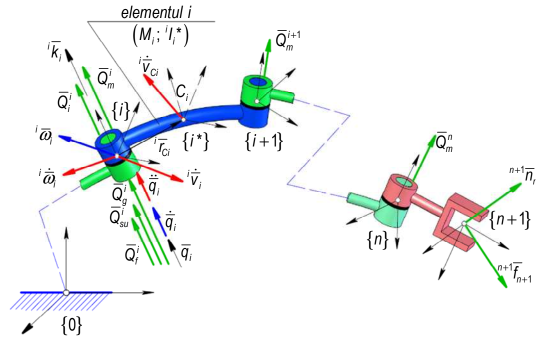


Fig. 5 Generalized Forces on MBS

$$\Delta_m = \{\{-1; (SU;\ M_i)\}; \{0; SU\}; \{1;\ M_i\}\}; \qquad (66)$$

The operator (66) highlights: gravitational loads by (M_i) and manipulating loads by symbol (SU). Unlike (59) and (60), the equations (64) and (65) are established by means of outward iterations $(i = 1 \to n)$, in accordance with [5] and [7]. So, they are substituted in (63), and this is changed:

$$Q_S^i = \left\{ {}^i\bar{f}_i^{\,T} \cdot (1 - \Delta_i) + {}^i\bar{n}_i^T \cdot \Delta_i \right\} \cdot {}^i\bar{k}_i = Q_g^i + Q_{SU}^i; \qquad (67)$$

where Q_g^i and Q_{SU}^i are generalized gravitational and respectively manipulating forces, according to [7] and [14]. They are also named generalized active forces. Unlike the classical approach based on the virtual work principle, these are determined, in accordance with [5], [7] and [14], by means of transfer matrices (previous section). Considering (64), (65) and (67), the starting equation for generalized gravitational force is:

$$\left\{ \begin{array}{l} Q_g^i = \sum_{j=i}^{n} M_j \cdot \left\{ \bar{g}^T \cdot {}^0_i[R] \cdot (1 - \Delta_i) + \right. \\ \left. + \bar{g}^T \cdot \left[\left({}^0\bar{r}_{C_j} - \bar{p}_i \right) \times \right]^T \cdot {}^0_i[R] \cdot \Delta_i \right\} \cdot {}^i\bar{k}_i \end{array} \right\}; \qquad (68)$$

$${}^0_i[R] \cdot {}^i\bar{k}_i = {}^0\bar{k}_i;\ \left\{ \begin{array}{l} Q_g^i = \sum_{j=i}^{n} M_j \cdot \bar{g}^T \cdot \left\{ {}^0\bar{k}_i \cdot (1 - \Delta_i) + \right. \\ \left. + {}^0\bar{k}_i \times \left({}^0\bar{r}_{C_j} - \bar{p}_i \right) \cdot \Delta_i \right\} \end{array} \right\} \qquad (69)$$

Taking into account linear and angular transfer matrices for velocities (45), (43) and (47), within of (69) a few transformations are performed as:

$$^{0}\overline{k}_i \cdot \left(1-\Delta_i\right) + {}^{0}\overline{k}_i \times \left({}^{0}\overline{r}_{C_j} - \overline{p}_i\right)\cdot \Delta_i = \frac{\partial\, {}^{0}\overline{r}_{C_j}}{\partial q_i}\;; \quad (70)$$

$$^{0}\overline{r}_{C_j} - \overline{p}_i = \left({}^{0}\overline{r}_{C_j} - \overline{p}_n\right) + \left(\overline{p}_n - \overline{p}_i\right)\;; \quad (71)$$

$$\left\{\begin{matrix} {}^{0}\overline{k}_i \cdot \left(1-\Delta_i\right) + {}^{0}\overline{k}_i \times \left(\overline{p}_n - \overline{p}_i\right)\cdot \Delta_i = \\ = \dfrac{\partial \overline{p}_n}{\partial q_i} \equiv \overline{V}_i \subset {}^{0}J_i \subset {}^{0}J\left[\overline{\theta}(t)\right] \end{matrix}\right\}; \quad (72)$$

$$\left\{\begin{matrix} \overline{g}^T \cdot \left[{}^{0}\overline{k}_i \times \left({}^{0}\overline{r}_{C_j} - \overline{p}_n\right)\right]\cdot \Delta_i = \\ = {}^{0}\overline{k}_i^T \cdot \Delta_i \cdot \left[\left({}^{0}\overline{r}_{C_j} - \overline{p}_n\right)\times \overline{g}\right] \end{matrix}\right\}; \quad (73)$$

$$^{0}\overline{k}_i^T \cdot \Delta_i \equiv \overline{\Omega}_i \subset {}^{0}J_i^T \subset {}^{0}J^T\left[\overline{\theta}(t)\right]. \quad (74)$$

The transformations (70) – (74), in accordance with [5], [7] and [14], are substituted in (69). As a result, the expression of definition is obtained for generalized gravitational force, as follows:

$$Q_g^i = \sum_{j=i}^{n} M_j \cdot \overline{g}^T \cdot \frac{\partial\, {}^{0}\overline{r}_{C_j}}{\partial q_i} = {}^{(n)0}J_i^T \cdot {}^{(n)0}\overline{\ddot{o}}_{X_i}\;; \quad (75)$$

$$\left\{\begin{matrix} \underset{(6\times 1)}{{}^{(n)0}\overline{\ddot{o}}_{X_i}} = \left[{}^{(n)0}\overline{F}_{X_i}^T \quad {}^{(n)0}\overline{N}_{X_i}^T\right]^T = \\ = \begin{bmatrix} \sum_{j=i}^{n} M_j \cdot {}^{(0)n}_{\ n}[R]^T \cdot \overline{g} \\ \cdots\cdots\cdots\cdots \\ \sum_{j=i}^{n} M_j \cdot {}^{(0)n}_{\ n}[R]^T \cdot \left[\left({}^{0}\overline{r}_{C_j} - \overline{p}_n\right)\times \overline{g}\right] \end{bmatrix} \end{matrix}\right\}; \quad (76)$$

$$Q_g\left(\overline{\theta}\right) = \left[Q_g^i = {}^{(n)0}J_i^T \cdot {}^{(n)0}\overline{\ddot{o}}_{X_i}\;;\; i = 1 \to n\right]^T. \quad (77)$$

The column vector (76), expressed with respect to Cartesian space, is mechanically equivalent with reduction torsor of the gravitational forces in $[i;n]$ interval in relation with the $\{n\}$ moving frame, applied in the geometry center of the last driving joint from MBS (see Fig. 4 and Fig.5). Considering (64), (65) and (67), the starting equation for generalized manipulating force is:

$$\left\{\begin{matrix} Q_{SU}^i = \left\{\left\{{}^{\ i}_{n+1}[R]\cdot {}^{n+1}\overline{f}_{n+1}\right\}^T \cdot \left(1-\Delta_i\right) + \right. \\ + \left\{{}^{0}_{i}[R]^T \cdot \left(\overline{p} - \overline{p}_i\right)\times {}^{\ i}_{n+1}[R]\cdot {}^{n+1}\overline{f}_{n+1} + \right. \\ \left. \left. + {}^{\ i}_{n+1}[R]\cdot {}^{n+1}\overline{n}_{n+1}\right\}^T \cdot \Delta_i\right\} \cdot {}^{i}\overline{k}_i \end{matrix}\right\}. \quad (78)$$

Taking into account linear and angular transfer matrices for velocities (45), (43) and (47), within of (78) a few transformations are performed as:

$${}^{\ i}_{n+1}[R] = {}^{0}_{i}[R]^T \cdot {}^{\ 0}_{n+1}[R]\;; \quad (79)$$

$$\left\{{}^{\ i}_{n+1}[R]\cdot {}^{n+1}\overline{f}_{n+1}\right\}^T = \left\{{}^{\ 0}_{n+1}[R]\cdot {}^{n+1}\overline{f}_{n+1}\right\}^T \cdot {}^{0}_{i}[R]\;; \quad (80)$$

$$\left\{{}^{\ i}_{n+1}[R]\cdot {}^{n+1}\overline{n}_{n+1}\right\}^T = \left\{{}^{\ 0}_{n+1}[R]\cdot {}^{n+1}\overline{n}_{n+1}\right\}^T \cdot {}^{0}_{i}[R]\;; \quad (81)$$

$$\left\{\begin{matrix} \left\{{}^{0}_{i}[R]^T \cdot \left(\overline{p} - \overline{p}_i\right)\times {}^{\ i}_{n+1}[R]\cdot {}^{n+1}\overline{f}_{n+1}\right\}^T = \\ = \left\{{}^{\ 0}_{n+1}[R]\cdot {}^{n+1}\overline{f}_{n+1}\right\}^T \cdot \left[\left(\overline{p} - \overline{p}_i\right)\times\right]^T \cdot {}^{0}_{i}[R] \end{matrix}\right\}; \quad (82)$$

Considering (79) – (82), the expression (78) is:

$$\left\{\begin{matrix} Q_{SU}^i = \left\{{}^{\ 0}_{n+1}[R]\cdot {}^{n+1}\overline{f}_{n+1}\right\}^T \cdot \left\{{}^{0}\overline{k}_i \cdot \left(1-\Delta_i\right) + \right. \\ \left. + {}^{0}\overline{k}_i \times \left(\overline{p}_n - \overline{p}_i\right)\cdot \Delta_i\right\} + \\ + \left\{{}^{\ 0}_{n+1}[R]\cdot {}^{n+1}\overline{f}_{n+1}\right\}^T \cdot \left[{}^{0}\overline{k}_i \times \left(\overline{p} - \overline{p}_n\right)\right] + \\ + \left\{{}^{\ 0}_{n+1}[R]\cdot {}^{n+1}\overline{n}_{n+1}\right\}^T \cdot {}^{0}\overline{k}_i \cdot \Delta_i \end{matrix}\right\}; \quad (83)$$

In (83) other transformations are performed as:

$$\left\{\begin{matrix} \left\{{}^{\ 0}_{n+1}[R]\cdot {}^{n+1}\overline{f}_{n+1}\right\}^T \cdot \left\{{}^{0}\overline{k}_i \cdot \left(1-\Delta_i\right) + \right. \\ \left. + {}^{0}\overline{k}_i \times \left(\overline{p}_n - \overline{p}_i\right)\cdot \Delta_i\right\} = \overline{V}_i^T \cdot \left\{{}^{\ 0}_{n+1}[R]\cdot {}^{n+1}\overline{f}_{n+1}\right\} \end{matrix}\right\}; \quad (84)$$

$$\left\{\begin{matrix} \left\{{}^{\ 0}_{n+1}[R]\cdot {}^{n+1}\overline{f}_{n+1}\right\}^T \cdot \left[{}^{0}\overline{k}_i \times \left(\overline{p} - \overline{p}_n\right)\right] + \\ + \left\{{}^{\ 0}_{n+1}[R]\cdot {}^{n+1}\overline{n}_{n+1}\right\}^T \cdot {}^{0}\overline{k}_i \cdot \Delta_i = \\ \overline{\Omega}_i^T \cdot \left\{\left(\overline{p} - \overline{p}_n\right)\times \left\{{}^{\ 0}_{n+1}[R]\cdot {}^{n+1}\overline{f}_{n+1}\right\} + {}^{\ 0}_{n+1}[R]\cdot {}^{n+1}\overline{n}_{n+1}\right\} \end{matrix}\right\} \quad (85)$$

The transformations (79) – (85) are written in keeping with [5], [7] and [14]. So, substituting in (78), generalized manipulating force is obtained:

$$Q_{SU}^i = {}^{(n)0}J_i^T \cdot {}^{(n)0}\overline{\ddot{o}}_X\;; \quad (86)$$

$$\left\{\begin{matrix} \underset{(6\times 1)}{{}^{0}\overline{\ddot{o}}_X} = \left[{}^{0}\overline{F}_X^T \quad {}^{0}\overline{N}_X^T\right]^T = \\ = \begin{bmatrix} {}^{\ 0}_{n+1}[R]\cdot {}^{n+1}\overline{f}_{n+1} \\ \cdots\cdots\cdots\cdots \\ \left\{\overline{p}_{n+1n}\times\right\}\cdot {}^{\ 0}_{n+1}[R]\cdot {}^{n+1}\overline{f}_{n+1} + {}^{\ 0}_{n+1}[R]\cdot {}^{n+1}\overline{n}_{n+1} \end{bmatrix} = \\ \begin{bmatrix} {}^{(n)0}_{\ n+1}[R] & [0] \\ \left\{{}^{(0)n}_{\ n}[R]^T \cdot \overline{p}_{n+1n}\times\right\} {}^{\ n}_{n+1}[R] & {}^{(0)n}_{\ n+1}[R] \end{bmatrix} \cdot \begin{bmatrix} {}^{n+1}\overline{f}_{n+1} \\ {}^{n+1}\overline{n}_{n+1} \end{bmatrix} \end{matrix}\right\} \quad (87)$$

$$\left\{\begin{matrix} Q_{SU}\left(\overline{\theta}\right) = \\ = \left[Q_{SU}^i = {}^{(n)0}J_i^T \cdot {}^{(n)0}\overline{\ddot{o}}_X\;;\; i = 1 \to n\right]^T = \\ = {}^{(n)0}J\left(\overline{\theta}\right)\cdot {}^{(n)0}\overline{\ddot{o}}_X \end{matrix}\right\}. \quad (88)$$

Cartesian column vector (87) is mechanically equivalent [5] – [8] with the reduction torsor of the manipulating load with respect to $\{n\}$ frame.

4. GENERALIZED DYNAMICS FORCES

Considering the aspects from [6] – [14], in the case of the dynamical behavior of MBS, in every driving joint, besides the active forces (see previous section), are developing the generalized inertia and driving forces. These constitute the main objective in this section. For beginning, the iterative algorithm of the dynamics equations, in classical form, is applied. By means of outward iterations $(i=1\to n)$, are established following:

$$ {}^{i}\dot{\bar{v}}_{C_i} = {}^{i}\dot{\bar{v}}_i + {}^{i}\dot{\bar{\omega}}_i \times {}^{i}\bar{r}_{C_i} + {}^{i}\bar{\omega}_i \times {}^{i}\bar{\omega}_i \times {}^{i}\bar{r}_{C_i} \;; \quad (89) $$

$$ {}^{i}\bar{F}_i^* = M_i \cdot {}^{i}\dot{\bar{v}}_{C_i} \;; \quad (90) $$

$$ \text{For } i=1,\quad {}^{0}\dot{\bar{v}}_0 \equiv \bar{g} = \tau \cdot g \cdot \bar{k}_0 \;; \quad (91) $$

$$ {}^{i}\bar{N}_i^* = {}^{i}I_i^* \cdot {}^{i}\dot{\bar{\omega}}_i + {}^{i}\bar{\omega}_i \times {}^{i}I_i^* \cdot {}^{i}\bar{\omega}_i \;; \quad (92) $$

$$ {}^{(0)i}I_i^* = \sum_{j=1}^{p_i} \sigma_j \cdot {}^{(0)i}I_j^* = \begin{bmatrix} {}^{(0)i}I_x^* & -{}^{(0)i}I_{xy}^* & -{}^{(0)i}I_{xz}^* \\ -{}^{(0)i}I_{yx}^* & {}^{(0)i}I_y^* & -{}^{(0)i}I_{yz}^* \\ -{}^{(0)i}I_{zx}^* & -{}^{(0)i}I_{zy}^* & {}^{(0)i}I_z^* \end{bmatrix} \quad (93) $$

The above expressions are corresponding to every kinetic ensemble. So, the symbols have the significances: ${}^{i}\dot{\bar{v}}_{C_i}$ is the acceleration of mass center; ${}^{i}\bar{F}_i^*$ and ${}^{i}\bar{N}_i^*$ represent the resultant force and moment of forces, and ${}^{i}I_i^*$ inertial tensor axial and centrifugal with respect to $\{i^*\}$ frame applied in mass center of the kinetic ensemble.

The equations (90) and (92) are fundamental theorems in dynamics of mechanical systems: theorem of the motion of the mass center (90) and theorem of the angular momentum (92).

Applying the inward iterations $(i=n\to 1)$ the following dynamics parameters are determined:

$$ {}^{i}\bar{f}_i = {}^{i}\bar{F}_i^* + {}_{i+1}^{i}[R] \cdot {}^{i+1}\bar{f}_{i+1} \;; \quad (94) $$

$$ \left\{ \begin{array}{c} {}^{i}\bar{n}_i = {}^{i}\bar{r}_{C_i} \times {}^{i}\bar{F}_i^* + {}^{i}\bar{N}_i^* + \\ + {}^{i}\bar{r}_{i+1} \times {}_{i+1}^{i}[R] \cdot {}^{i+1}\bar{f}_{i+1} + {}_{i+1}^{i}[R] \cdot {}^{i+1}\bar{n}_{i+1} \end{array} \right\}; \quad (95) $$

$$ Q_m^i = \left\{ {}^{i}\bar{f}_i^{T} \cdot (1-\Delta_i) + {}^{i}\bar{n}_i^{T} \cdot \Delta_i \right\} \cdot {}^{i}\bar{k}_i \;; \quad (96) $$

The symbols from (94) – (96) have the next significances: ${}^{i}\bar{f}_i$ and ${}^{i}\bar{n}_i$ are the dynamical action (force and moment of force) from $(i-1)$ on (i) ensemble of MBS (Fig.4); and Q_m^i generalized driving force, identical with differential equation corresponding to kinetic ensemble from MBS. According to [5] and [7], (96) is written again:

$$ Q_{mf}^i = (-1)^{\Delta_f} \cdot \frac{1-\Delta_f}{1+3\cdot\Delta_f} \cdot Q_m^i + \Delta_f^2 \cdot Q_{fd}^i \;; \quad (97) $$

The symbol Q_{fd}^i is generalized friction force. Its expression of definition is below presented as:

$$ \left\{ \begin{array}{c} Q_{fd}^i = b_i \cdot \dot{q}_i + \mu_{iT} \cdot (1-\Delta_i) \cdot \left| {}^{i}\bar{k}_i \times {}^{i}\bar{f}_i \right| \cdot sgn(\dot{q}_i \cdot \Delta_\theta) \\ + \mu_{iR} \cdot \frac{d_i}{2} \cdot \Delta_i \cdot \left| {}^{i}\bar{k}_i \times {}^{i}\bar{f}_i \right| \cdot sgn(\dot{q}_i \cdot \Delta_\theta) \end{array} \right\} \quad (98) $$

where symbols: b_i and $\mu_{i(T)R}$ are coefficient of viscous friction and coefficient of dry friction as function of joint type. Within of the (97) and (98) are founded the following operators:

$$ \Delta_f = \left\{ (-1; \Delta_m = -1);\ \left[0; \Delta_m = (0;1)\right];\ \left(1; Q_{fd}^i\right) \right\} \quad (99) $$

$$ \Delta_\theta = \left\{ \left[1; \left\{\dot{\theta}; \ddot{\theta}\right\} \neq 0\right] ; \left[0; \left\{\dot{\theta}; \ddot{\theta}\right\} = 0\right] \right\}; \quad (100) $$

where Δ_f highlights the loads by Δ_m (see (66)) as well as the influence of the complex frictions; Δ_θ shows behavior (0 – statics; 1 – dynamics).

The classical approach (94) – (96) is based on the D'Alembert principle. It observes that Q_m^i is a function, in exclusivity, of gravitational and manipulating loads, as well as of inertia forces situated in the mechanical interval $[i; n+1]$. So, according to [5] – [7], similarly with (64), (65), and considering the operators (66) and (100), dynamical actions (94) and (95) are rewritten as:

$$ \left\{ \begin{array}{c} {}^{i}\bar{f}_i = \Delta_m^2 \cdot \left\{ \sum_{j=i}^{n} {}_{j}^{i}[R] \cdot {}^{j}\bar{F}_j^* + \sum_{j=i}^{n} M_j \cdot {}_{j}^{0}[R]^T \cdot \bar{g} \right\} + \\ + (-1)^{\Delta_m} \cdot \frac{1-\Delta_m}{1+3\cdot\Delta_m} \cdot {}_{n+1}^{i}[R] \cdot {}^{n+1}\bar{f}_{n+1} \end{array} \right\} \quad (101) $$

$$ \left\{ \begin{array}{c} {}^{i}\bar{n}_i = \Delta_m^2 \cdot \left\{ \Delta_\theta \cdot \left\{ \sum_{j=i}^{n} {}_{i}^{0}[R]^T \cdot \left({}^{0}\bar{r}_{C_j} - \bar{p}_i\right) \times {}_{j}^{i}[R] \cdot {}^{j}\bar{F}_j^* \right. \right. \\ \left. \left. + {}_{j}^{i}[R] \cdot {}^{j}\bar{N}_j^* \right\} + \sum_{j=i}^{n} M_j \cdot {}_{i}^{0}[R]^T \cdot \left[\left({}^{0}\bar{r}_{C_j} - \bar{p}_i\right) \times \bar{g} \right] \right\} + \\ (-1)^{\Delta_m} \cdot \frac{1-\Delta_m}{1+\Delta_m} \cdot \left\{ {}_{i}^{0}[R]^T \cdot (\bar{p} - \bar{p}_i) \times {}_{n+1}^{i}[R] \cdot {}^{n+1}\bar{f}_{n+1} \right. \\ \left. + {}_{n+1}^{i}[R] \cdot {}^{n+1}\bar{n}_{n+1} \right\} \end{array} \right\} \quad (102) $$

Substituting (101) and (102) in (96), it obtains:

$$ \left\{ \begin{array}{c} Q_m^i(t) = \Delta_m^2 \cdot \left[\Delta_\theta \cdot Q_{iö}^i(t) + Q_g^i(t) \right] + \\ + (-1)^{\Delta_m} \cdot \frac{1-\Delta_m}{1+3\cdot\Delta_m} \cdot Q_{SU}^i(t) \end{array} \right\}; \quad (103) $$

$Q_g^i(t)$ and $Q_{SU}^i(t)$ are generalized active forces (previous section, expressions: (75) and (86)).

Considering (101), (102), as well (90) and (92), the generalized inertia force $Q^i_{iö}\ (t)$, included in (103), has the starting equation the following:

$$\begin{cases} Q^i_{iö} = \sum_{j=i}^{n}\Big\{\big\{{}^i_j[R]\cdot{}^j\bar{F}^*_j\big\}^T\cdot(1-\Delta_i)+ \\ +\Delta_i\cdot\big\{{}^0_i[R]^T\cdot({}^0\bar{r}_{C_j}-\bar{p}_i)\times{}^i_j[R]\cdot{}^j\bar{F}^*_j\big\}^T+ \\ +\Delta_i\cdot\big[{}^i_j[R]\cdot{}^j\bar{N}^*_j\big]^T\Big\}\cdot{}^i\bar{k}_i \end{cases}. \quad (104)$$

Taking into account linear and angular transfer matrices for velocities (45), (43) and (47), in the (104) a few transformations are performed as:

$${}^i_j[R] = {}^0_i[R]^T\cdot{}^0_j[R]; \quad (105)$$

$$\big\{{}^i_j[R]\cdot{}^j\bar{F}^*_j\big\}^T = \big\{{}^0_j[R]\cdot{}^j\bar{F}^*_j\big\}^T\cdot{}^0_i[R]; \quad (106)$$

$$\big\{{}^i_j[R]\cdot{}^j\bar{N}^*_j\big\}^T = \big\{{}^0_j[R]\cdot{}^j\bar{N}^*_j\big\}^T\cdot{}^0_i[R]; \quad (107)$$

$$\begin{cases} Q^i_{iö} = \sum_{j=i}^{n}\Big\{\big\{{}^0_j[R]\cdot{}^j\bar{F}^*_j\big\}^T\cdot\big[{}^0\bar{k}_i\cdot(1-\Delta_i)+ \\ +{}^0\bar{k}_i\times(\bar{p}_n-\bar{p}_i)\big]+ \\ +\Delta_i\cdot\big\{{}^0_j[R]\cdot{}^j\bar{F}^*_j\big\}^T\cdot\big[{}^0\bar{k}_i\times\big({}^0\bar{r}_{C_j}-\bar{p}_n\big)\big]+ \\ +\Delta_i\cdot\big\{{}^0_j[R]\cdot{}^j\bar{N}^*_j\big\}^T\cdot{}^0\bar{k}_i\Big\} \end{cases}. \quad (108)$$

Similarly with (82) – (85), finally the expression of the generalized inertia force becomes thus:

$$Q^i_{iö} = {}^{(n)0}J_i^T\cdot{}^{(n)0}\bar{ö}^*_{X_i}; \quad (109)$$

$$\begin{cases} {}^0\bar{ö}^*_{X_i} = \big[{}^0\bar{F}^{*T}_{X_i}\quad{}^0\bar{N}^{*T}_{X_i}\big]^T = \\ = \begin{bmatrix} \sum_{j=i}^{n}{}^0_j[R]\cdot{}^j\bar{F}^*_j \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \sum_{j=i}^{n}\big\{\big({}^0\bar{r}_{C_j}-\bar{p}_n\big)\times{}^0_j[R]\cdot{}^j\bar{F}^*_j+{}^0_j[R]\cdot{}^j\bar{N}^*_j\big\} \end{bmatrix} \end{cases}; \quad (110)$$

$$\begin{cases} \underset{(n\times1)}{Q_{iö}}\big[\bar{\theta}(t);\dot{\bar{\theta}}(t);\ddot{\bar{\theta}}(t)\big] = \\ = \big[Q^i_{iö}\ (t) = {}^0J_i^T(t)\cdot{}^0\bar{ö}^*_{X_i}(t);\ \ i=1\to n\big]^T \end{cases}. \quad (111)$$

Column vector (110), expressed with respect to Cartesian space, is mechanically equivalent with reduction torsor of the inertia forces from $[i;n]$ interval in relation with the $\{n\}$ moving frame, applied in the geometry center of the last driving joint from MBS (see Fig.4 and Fig.5).

The column matrix of the generalized inertia forces (111) can be written as matrix diagonal:

$$\underset{(n\times n)}{Diag}\big[Q_{iö}\ \ (\bar{\theta})\big] = {}^0J(\bar{\theta})^T\cdot\underset{(6\times n)}{Matrix}\big[{}^0\bar{ö}^*_{X_i};\ i=1\to n\big],$$

$$\begin{cases} \begin{bmatrix} Q^1_{iö}\ (t) & \cdots & 0 & \cdots & 0 \\ \vdots & & \cdots & & \vdots \\ 0 & \cdots & Q^i_{iö}\ (t) & \cdots & 0 \\ \vdots & & \cdots & & \vdots \\ 0 & \cdots & 0 & \cdots & Q^n_{iö}\ (t) \end{bmatrix} = \\ = {}^{(n)0}J\big[\bar{\theta}(t)\big]^T\cdot\underset{(6\times n)}{Matrix}\big\{{}^{(n)0}\bar{ö}^*_{X_i}(t);\ i=1\to n\big\} \end{cases}. \quad (112)$$

Substituting (75) and (86), as well as (109) in (103), the generalized driving force from every driving axis from MBS is finally obtained thus:

$$\begin{cases} Q^i_m(t) = {}^{(n)0}J_i^T(t)\cdot\big\{\Delta_m^2\cdot\big[\Delta_\theta\cdot{}^{(n)0}\bar{ö}^*_{X_i}(t)+ \\ +{}^{(n)0}\bar{ö}_{X_i}(t)\big]+(-1)^{\Delta_m}\cdot\frac{1-\Delta_m}{1+3\cdot\Delta_m}\cdot{}^{(n)0}\bar{ö}_X(t)\big\} \end{cases}; \quad (113)$$

Δ_m and Δ_θ have significances (66) and (100).

Comparing (75), (86) and (109) it observes that they have unique character. So, the generalized active and inertia forces are mathematically identical as form of expression. This aspect has important advantage, in the establishment of the dynamics equations (113), corresponding to every kinetic ensemble from MBS (see Fig.5).

In consonance with above mathematical aspects, the resultant force vector included in generalized friction forces (98) is determined as follows:

$${}^i\bar{f}_i = {}^0_i[R]^T\cdot\big\{{}^0\bar{F}_{X_i}+{}^0\bar{F}^*_{X_i}+{}^0\bar{F}_X\big\}\subset Q^i_{fd}\ . \quad (114)$$

So, considering (97) and (98), (113) is changed:

$$Q^i_{mf}(t) = (-1)^{\Delta_f}\cdot\frac{1-\Delta_f}{1+3\cdot\Delta_f}\cdot Q^i_m(t)+\Delta_f^2\cdot Q^i_{fd}(t) \quad (115)$$

As a result, (115) for $(i=1\to n)$ constitutes the system of (n) generalized driving forces. They are identical with dynamics equations of MBS, in which the both generalized active and inertia forces, and the complex frictions are founded.

5. HIGHER ORDER DYNAMICS EQUATIONS

When the mechanical systems (MBS) are dominated by sudden motions, as well as by the transitory motions, on the basis of the author's researches [6] – [14] it demonstrates theoretical and experimental existing of the accelerations energy of higher order. They are included in the dynamics equations of higher order. As a result, time variations of generalized forces are obvious.

In the case of the mechanical robot structure (Fig.5) characterized by the sudden motions, generalized accelerations and forces of higher order in the dynamical behavior are developed. Considering the researches [6] – [14], they are:

$$\left\{\begin{array}{l} \overset{(k)}{Q^{i}_{i\ddot{o}}}(t) = {}^{0}J_{i}\left[\overline{\theta}(t)\right]\cdot \overset{(k)}{{}^{0}\overline{\ddot{o}}}{}^{*}_{X_i} + \\ +\sum_{m=1}^{k-1}\frac{(k-1)!}{m!(k-m-1)!}\cdot {}^{0}\overset{(m)}{J_{i}}\left[\overline{\theta}(t)\right]\cdot \overset{[k-m]}{{}^{0}\overline{\ddot{o}}}{}^{*}_{X_i} = \\ = \sum_{m=1}^{k}\frac{(k-1)!}{(m-1)!(k-m)!}\cdot {}^{0}\overset{(m-1)}{J_{i}}\left[\overline{\theta}(t)\right]\cdot \overset{[k-(m-1)]}{{}^{0}\overline{\ddot{o}}}{}^{*}_{X_i} \end{array}\right\}; \quad (116)$$

$$\left\{\begin{array}{l} \overset{(k)}{Q^{i}_{g}}(t) = {}^{0}J_{i}\left[\overline{\theta}(t)\right]\cdot \overset{(k)}{{}^{0}\overline{\ddot{o}}}_{X_i} + \\ +\sum_{m=1}^{k-1}\frac{(k-1)!}{m!(k-m-1)!}\cdot {}^{0}\overset{(m)}{J_{i}}\left[\overline{\theta}(t)\right]\cdot \overset{[k-m]}{{}^{0}\overline{\ddot{o}}}_{X_i} = \\ = \sum_{m=1}^{k}\frac{(k-1)!}{(m-1)!(k-m)!}\cdot {}^{0}\overset{(m-1)}{J_{i}}\left[\overline{\theta}(t)\right]\cdot \overset{[k-(m-1)]}{{}^{0}\overline{\ddot{o}}}_{X_i} \end{array}\right\}; \quad (117)$$

$$\left\{\begin{array}{l} \overset{(k)}{Q_{SU}}\left[\overline{\theta}(t)\right] = {}^{0}J\left[\overline{\theta}(t)\right]\cdot \overset{(k)}{{}^{0}\overline{\ddot{o}}}_{X} + \\ +\sum_{m=1}^{k-1}\frac{(k-1)!}{m!(k-m-1)!}\cdot {}^{0}\overset{(m)}{J}\left[\overline{\theta}(t)\right]\cdot \overset{[k-m]}{{}^{0}\overline{\ddot{o}}}_{X} = \\ = \sum_{m=1}^{k}\frac{(k-1)!}{(m-1)!(k-m)!}\cdot {}^{0}\overset{(m-1)}{J}\left[\overline{\theta}(t)\right]\cdot \overset{[k-(m-1)]}{{}^{0}\overline{\ddot{o}}}_{X} \end{array}\right\}. \quad (118)$$

The significance of the terms from (116) – (118) is well defined in the previous sections of this paper, and $(k \geq 1)$ is the time deriving order. But considering dynamical equations, instead of (k) is written $(k-1)$. When $(k=1)$, then (116) – (118) are degenerated in: (109), (75), and last in (86).

According to Lagrange's equations of second kind, generalized inertia forces are identical as:

$$\frac{d}{dt}\left(\frac{\partial E_C}{\partial \dot{q}_j}\right) - \frac{\partial E_C}{\partial q_j} = Q^{j}_{i\ddot{o}}\ (t); \quad (119)$$

$$\frac{1}{m}\cdot\left[\frac{\partial \overset{(m)}{E_C}}{\partial \overset{(m)}{q_j}} - (m+1)\cdot\frac{\partial E_C}{\partial q_j}\right] = Q^{j}_{i\ddot{o}}\ (t). \quad (120)$$

The symbol E_C is kinetic energy, expression (120) is Tsenov – Mangeron formulation, and (m) is time deriving order. But, considering the acceleration energy of first order [1] – [15], as well as its time derivative of higher order, the generalized inertia forces are also identical with:

$$\left\{\begin{array}{l} \frac{\partial}{\partial \overset{(m)}{q_j}}\left\{\overset{(m-2)}{E^{(1)}_{A}}\left[\overline{\theta}(t);\dot{\overline{\theta}}(t);\ddot{\overline{\theta}}(t);\cdots;\overset{(m)}{\overline{\theta}}(t)\right]\right\} = \\ = Q^{j}_{i\ddot{o}}\left[\overline{\theta}(t);\dot{\overline{\theta}}(t);\overset{(2)}{\overline{\theta}}(t)\right] \end{array}\right\}; \quad (121)$$

$$\left\{\begin{array}{l} where\ \ \overset{(0)}{E^{(1)}_{A}} = E^{(1)}_{A}\quad j=1\rightarrow n,\quad k=1 \\ m \geq \left[(k+1)=2\right],\ and\,(k)\ are\ time\ deriving\ orders \end{array}\right\};$$

where according to [1] – [15], (121) is named generalization of Gibbs – Appell's equations. Following the application of time derivatives of higher order (m) and (k), the equations (120) and (121), become new differential expressions:

$$\left\{\begin{array}{l} \frac{1}{m}\cdot\left[\frac{\partial \overset{(m+k-1)}{E_C}}{\partial \overset{(m)}{q_j}} - (m+1)\cdot\frac{\partial \overset{(k-1)}{E_C}}{\partial q_j}\right] = \\ = \overset{(k-1)}{Q^{j}_{i\ddot{o}}}\left[\overline{\theta}(t);\dot{\overline{\theta}}(t);\cdots;\overset{(m)}{\overline{\theta}}(t)\right] \end{array}\right\}; \quad (122)$$

$$\frac{\partial \overset{(m+k-3)}{E^{(1)}_{A}}}{\partial \overset{(m)}{q_j}} = \overset{(k-1)}{Q^{j}_{i\ddot{o}}}\left[\overline{\theta}(t);\dot{\overline{\theta}}(t);\cdots;\overset{(m)}{\overline{\theta}}(t)\right]. \quad (123)$$

According to [9] – [14], sudden and transitory motions of MBS are represented by dynamics equations, and the central function is highlighted through acceleration energies of higher order. As result, considering acceleration energies of first, second and third order, and applying the time derivatives of higher order (m) and (k), see [12] and [14], the differential dynamics equations are:

$$\left\{\begin{array}{l} \frac{1}{m+1}\cdot\frac{\partial}{\partial \overset{(m)}{q_j}}\left[2\cdot \overset{(m+k-5)}{E^{(2)}_{A}} + \overset{(m+k-3)}{E^{(1)}_{A}}\right] = \\ = \overset{(k-1)}{Q^{j}_{i\ddot{o}}}\left[\overline{\theta}(t);\dot{\overline{\theta}}(t);\cdots;\overset{(m)}{\overline{\theta}}(t)\right] \end{array}\right\}; \quad (124)$$

$$\overset{(0)}{E^{(2)}_{A}} = E^{(2)}_{A},\ j=1\rightarrow n,\ \ k=2,\ m \geq \left[(k+1)=3\right];$$

$$\left\{\begin{array}{l} \frac{2}{(m+1)\cdot(m+2)}\cdot\frac{\partial}{\partial \overset{(m)}{q_j}}\left[5\cdot \overset{(m+k-5)}{E^{(3)}_{A}} + 2\cdot \overset{(m+k-3)}{E^{(2)}_{A}} + \right. \\ \left. + \overset{(m+k-1)}{E^{(1)}_{A}}\right] = \overset{(k-1)}{Q^{j}_{i\ddot{o}}}\left[\overline{\theta}(t);\dot{\overline{\theta}}(t);\cdots;\overset{(m)}{\overline{\theta}}(t)\right] \end{array}\right\}; \quad (125)$$

$$\left\{\begin{array}{l} where\quad \overset{(0)}{E^{(3)}_{A}} = E^{(3)}_{A},\ \ j=1\rightarrow n \\ k=3,\ m \geq \left[(k+1)=4\right],\quad m=4,5,6,\ldots \end{array}\right\}.$$

The acceleration energy of first, second and third order is defined in papers of author [10] – [14].

Author has proposed [13] – [14], the generalized differential equations of higher order in the case of the mechanical systems (MBS), dynamically characterized by sudden and transitory motions:

$$\left\{\begin{array}{c} \dfrac{(k-1)!\cdot m!}{(m+k-1)!}\cdot\dfrac{\partial}{\partial \overset{(m)}{q_j}}\left\{\left(\sum_{p=1}^{k}\Delta_p\right)\cdot \overset{(m+k)-(2\cdot p+1)}{E_A^{(p)}}\right\} \\ =\overset{(k-1)}{Q_{iö}^{j}}\left[\bar{\theta}(t);\dot{\bar{\theta}}(t);\cdots;\overset{(m)}{\bar{\theta}}(t)\right] \end{array}\right\};(126)$$

$$\left\{\begin{array}{l} where\quad E_A^{(p)}=E_A^{(p)}\left[\bar{\theta}(t);\dot{\bar{\theta}}(t);\cdots;\overset{(p+1)}{\bar{\theta}}(t)\right] \\ and\ \sum_{p=1}^{k}\Delta_p=\sum_{p=1}^{k}\left[\dfrac{p\cdot(p+1)}{2}-\delta_p\right] \end{array}\right\};(127)$$

The necessary conditions in (126) are following:

$$\left\{\begin{array}{l} p=1\rightarrow k\ ;\ \ \delta_p=\{\{0;p=1\};\{1;p>1\}\} \\ and\ \ k\geq 1;\quad k=\{1;2;3;4;5;.....\} \\ m\geq(k+1);\quad m=\{2;3;4;5;.....\} \end{array}\right\}.\ (128)$$

Generalized differential equations (126) contain acceleration energies of order $(p=1\rightarrow k)$. Using the aspects from Fig.4, the starting equation is:

$$\left\{\begin{array}{c} E_A^{(p)}\left[\bar{\theta}(t);\dot{\bar{\theta}}(t);\cdots;\overset{(p+1)}{\bar{\theta}}(t)\right]= \\ =\dfrac{1}{2}\cdot\sum_{i=1}^{n}Trace\left\{\overset{(p+1)}{{}_i^0[R]}\cdot\left[\int {}^i\bar{r}_i^{*}\cdot{}^i\bar{r}_i^{*T}\cdot dm+\right.\right. \\ \left.\left.+{}^i\bar{r}_{C_i}\cdot{}^i\bar{r}_{C_i}^{T}\cdot\int dm\right]\cdot\overset{(p+1)}{{}_i^0[R]^T}\right\}+ \\ +\dfrac{1}{2}\cdot\sum_{i=1}^{n}Trace\left[\overset{(p+1)}{\bar{p}_i}\cdot\overset{(p+1)}{\bar{p}_i^{T}}\right]\cdot\int dm= \\ \dfrac{1}{2}\cdot\sum_{i=1}^{n}Trace\left\{\overset{(p+1)}{{}_i^0[R]}\cdot\left[{}^iI_{pi}^{*}+M_i\cdot{}^i\bar{r}_{C_i}\cdot{}^i\bar{r}_{C_i}^{T}\right]\cdot\overset{(p+1)}{{}_i^0[R]^T}\right\} \\ +\dfrac{1}{2}\cdot\sum_{i=1}^{n}Trace\left[\overset{(p+1)}{\bar{p}_i}\cdot\overset{(p+1)}{\bar{p}_i^{T}}\right]\cdot M_i \end{array}\right\}\ (129)$$

$${}^{(0)i}I_{pi}^{*}=\sum_{j=1}^{p_i}\sigma_j\cdot{}^{(0)i}I_{pj}^{*}=\begin{bmatrix} {}^{(0)i}I_{xx}^{*} & {}^{(0)i}I_{xy}^{*} & {}^{(0)i}I_{xz}^{*} \\ {}^{(0)i}I_{yx}^{*} & {}^{(0)i}I_{yy}^{*} & {}^{(0)i}I_{yz}^{*} \\ {}^{(0)i}I_{zx}^{*} & {}^{(0)i}I_{zy}^{*} & {}^{(0)i}I_{zz}^{*} \end{bmatrix},\ (130)$$

(130) is inertia tensor planar and centrifugal of the kinetic ensemble (i) with respect to frames $\{\{i^{*}\};\ \{0^{*}\}\}$, having the origin in mass center C_i. The author has demonstrated in various papers, as example [10]-[14], expressions of definition in the explicit and matrix form, for acceleration energies of first, second, third and fourth order.

6. CONCLUSIONS

The currently paper was devoted, especially, to presentation a few essential new formulations about kinematical transformations concerning the locating matrices, the generalized active and dynamics forces, as well dynamics equations of higher order compulsory included in analytical dynamics of the multibody systems (MBS). So, in the case of (MBS), for example, mechanical robot structure, unlike the classical models the author describes in the first section of this paper new formulations on the locating matrices and their time derivatives. As result, the kinematical parameters of the absolute motions for every kinetic ensemble have been implemented. In the same section the author presents the expressions of the operational and generalized accelerations of higher order, answerable to sudden motions.

Unlike the classical approaches, in the second and third sections of this paper, the author presents formulations concerning generalized active and inertia forces based in exclusivity on linear and angular transfer matrices, also described in the first section of this paper. It observed that they have unique character. So, the generalized active and inertia forces are mathematically identical as form of expression. In these formulations are also included complex frictions and generalized driving forces, identical with dynamics equations of MBS. The above aspects have an important advantage, in the establishment of the dynamics equations, corresponding to every kinetic ensemble MBS.

When the mechanical systems (MBS) are dominated by sudden motions, as well as by the transitory motions, on the basis of the author's researches was demonstrated theoretical and experimental existing of the accelerations energy of higher order. They are included in the dynamics equations of higher order. As a result, time variations of generalized forces are obvious. So, in the fourth section of the paper, the author presented the expressions of definition for the generalized forces of higher order. They are included in the dynamics equations, where the central functions are acceleration energies of higher order. In the last part of the fourth section author has proposed the generalized differential equations of higher order mechanical systems characterized by sudden and transitory motions.

7. REFERENCES

[1] Appell, P., *Sur une forme générale des equations de la dynamique*, Paris, 1899.

[2] Appell, P., *Traité de mécanique rationnelle*, Paris, 1903.

[3] Craig, J.J., *Introduction to Robotics*, Addison-Wesley, Amsterdam, 1989.

[4] Fu, K., Gonzales, R. and Lee, C., *Robotics: Control, Sensing, Vision and Intelligence*, McGraw-Hill Book Co., International Edition, 1987

[5] Negrean, I., *Robotics - Kinematic and Dynamic Modelling*, Editura Didactică şi Pedagogică R.A., Bucureşti, ISBN 973-30-5958-7, 1998.

[6] Negrean, I., Negrean, D. C., *The Acceleration Energy to Robot Dynamics,* International Conference on Automation, Quality and Testing, Robotics, AQTR 2002, May 23-25, Cluj-Napoca, Tome II, pp. 59-64

[7] Negrean I., *Mecanică avansată în Robotică*, ISBN 978-973-662-420-9, UT Press, Cluj-Napoca, 2008.

[8] Negrean I., *Mecanică, Teorie şi Aplicaţii*, Editura UT PRESS, Cluj – Napoca, ISBN 978-973-662-523-7, 2012.

[9] Negrean, I., *Formulations in Advanced Dynamics of Mechanical Systems*, The 11 th IFTOMM International Symposium on Science of Mechanisms and Machines, Mechanisms and Machine Science 18, DOI:10.1007/978-3-319-01845-4-19 Springer International Publishing Switzerland, pp. 185-195.

[10] Negrean, I., *Energies of Acceleration in Advanced Robotics Dynamics,* Applied Mechanics and Materials, ISSN: 1662-7482, vol 762, pp 67-73 Submitted: 2014-08-05 ©(2015)TransTech Publications Switzerland DOI:10.4028/www.scientific.net/AMM.762.67, Revised:2014-11-16.

[11] Negrean I., *New Formulations on Acceleration Energy in Analytical Dynamics,* Applied Mechanics and Materials, vol. 823 pp 43-48 © (2016) TransTech Publications Switzerland Revised: 2015-09, Doi: 10.4028 /www.scientific.net/AMM.823.43.

[12] Negrean I., *New Formulations on Motion Equations in Analytical Dynamics,* Applied Mechanics and Materials, vol. 823 (2016), pp 49-54 © (2016) TransTech Publications Switzerland Revised: 2015-09-09, DOI: 10.4028/www.scientific.net/AMM.823.49.

[13] Negrean I., *Formulations on the Advanced Notions in analytical Dynamics of Mechanical Systems,* published in IJERM, ISSN: 2349-2058, Volume-03, Issue-04, pp 123-130, 2016.

[14] Negrean I., *Advanced Notions and Differential Principles of Motion in Analytical Dynamics,* Journal of Engineering Sciences and Innovation, Technical Sciences Academy of Romania, Vol. 1/2016, Issue 1, pp. 49-72, ISSN 2537-320X.

[15] Pars L.A., *A Treatise on Analytical Dynamics*, Heinemann, London, (2007), Volume I, pp. 1-122.

Forțele Generalizate în Dinamica Analitică a Sistemelor

În cazul sistemelor mecanice multicorp (MBS), spre exemplu structura mecanică a robotului și în conformitate cu principiile diferențiale specific dinamicii analitice a sistemelor, studiul comportamentului dinamic se bazează pe forțele generalizate. Ele se dezvoltă în conexiune directă cu variabilele generalizate, cunoscute de asemenea ca parametrii independenți ai sistemelor olonome și neolonome. Dar, sub aspect mecanic, forțele generalizate se datorează: surselor de acționare, forțelor gravitaționale, sarcinilor de manipulare, precum și frecărilor complexe din legăturile fizice dintre elementele cinetice, aparținând MBS. Expresiile de definiție ale forțelor generalizate conțin pe de o parte parametrii cinematici corespunzători mișcării absolute, iar pe de altă parte proprietățile maselor. Acestea din urmă se evidențiază prin masa și poziția centrului maselor, tensorii inerțiali și pseudoinerțiali. În special pe baza cercetărilor autorului, în această lucrare se vor prezenta formulări noi cu privire la parametrii cinematici, forțele generalizate și ecuațiile dinamice ale mișcărilor curente și rapide. Studiul dinamic va conține, de asemenea, energia accelerațiilor și derivatele ei în raport cu timpul, conform cu ecuațiile diferențiale de ordin superior specifice dinamicii analitice a sistemelor.

Iuliu NEGREAN Professor Ph.D., Member of the Academy of Technical Sciences of Romania, Director of Department of Mechanical Systems Engineering, Technical University of Cluj-Napoca, iuliu.negrean@mep.utcluj.ro, http://users.utcluj.ro/~inegrean, Office Phone 0264/401616.