TECHNICAL UNIVERSITY OF CLUJ-NAPOCA

ACTA TECHNICA NAPOCENSIS

Series: Applied Mathematics, Mechanics, and Engineering
Vol. 60, Issue IV, November, 2017

ADVANCED NOTIONS IN ANALYTICAL DYNAMICS OF SYSTEMS

Iuliu NEGREAN

Abstract: *The dynamical study of the current and sudden motions of the multibody systems (MBS), as example the mechanical robot structure, and in accordance with differential principles typical to analytical dynamics of systems, is based, among others, on the advanced notions, such as: kinetic energy, acceleration energies of different orders and their absolute time derivatives of higher order. Advanced notions are developed in the direct connection with the generalized variables, also named independent parameters corresponding to holonomic mechanical systems. But, mechanically, the expressions of definition of the advanced notions contain on the one hand kinematical parameters and their differential transformations, corresponding to absolute motions, on the other hand the mass properties, highlighted by mass and mass center, inertial tensors and their generalized laws, as well as pseudoinertial tensors.*
By means of the especially researches of the author, in this paper will be presented reformulations and new formulations concerning the advanced kinematics parameters, as well as advanced notions such as: kinetic energy and acceleration energies of different orders in explicit and matrix form. They are corresponding to the current and sudden motions of MBS. These formulations will also contain the absolute time derivatives of higher order of the advanced notions, according to differential equations of higher order, typically to analytical dynamics of systems.

Key words: *analytical dynamics, mechanics, advanced notions, dynamics equations, robotics.*

1. INTRODUCTION

The advanced dynamics study of the current and sudden motions of the multibody systems (MBS), example in Fig.1 the mechanical robot structure, and in accordance with differential principles typical to analytical dynamics of systems, is based on the advanced notions of dynamics, such as: kinetic energy, acceleration energies of different orders and their absolute time derivatives of higher order [6] – [17]. Advanced notions are developed in the direct connection with the generalized variables, also named independent parameters (d.o.f.). These univocally characterize the absolute motions for any holonomic mechanical systems. But, the expressions of definition of advanced notions of dynamics contain on the one hand the advanced kinematics parameters and their differential transformations, typically to absolute motions, on the other hand mass properties [1] – [20].

Based on especially of the author researches, in the three sections of the paper, reformulations and new formulations on the advanced notions of kinematics and dynamics will be presented considering the researches in [5] – [15] and [17]. So, the first section is devoted to the advanced kinematics parameters typical to MBS with current and sudden motions. In the view of this, matrix exponentials and the time derivatives of higher order will be applied on the kinematical parameters. These expressions will be used in the second section devoted to energies of higher order. So, kinetic energy, acceleration energies of different orders and their absolute time derivatives of higher order will be reformulated and formulated by new expressions. In the third section of the paper, the above advanced notions will be implemented in the dynamics equations of higher order typically to sudden motions.

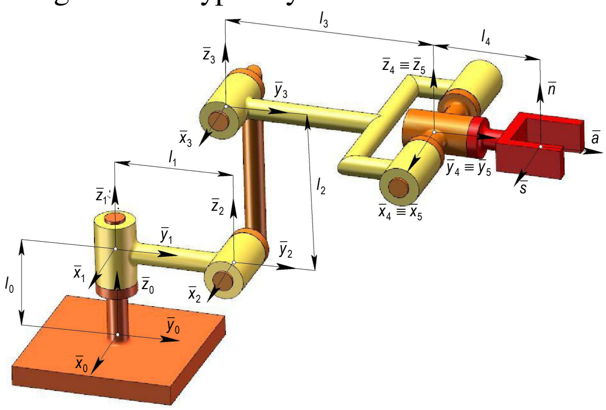

Fig.1 Robot Mechanical Structure (MBS)

2. ADVANCED KINEMATICS NOTIONS

The kinematical and dynamical study from this paper [3], [4], [7] is oriented on mechanical structure with opened kinematical chain, where the kinetic ensembles $i=1\rightarrow n$ are physically linked by driving joints of fifth order. (Example robot mechanical structure, see Fig.1 and Fig.2).

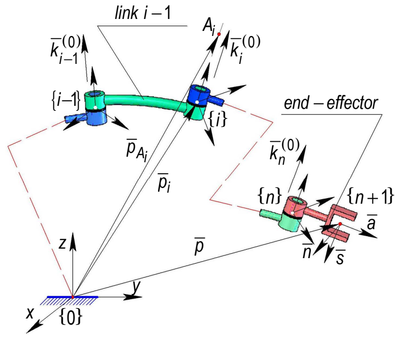

Fig.2 Sequence of Kinetic Ensemble

This is characterized by $(n\ d.o.f.)$, according to:

$$\bar{\theta}\neq\bar{\theta}^{(0)};\quad \bar{\theta}(t)=\left[q_i(t);\quad i=1\rightarrow n\right]^T, \qquad (1)$$

where $q_i(t)$ is the generalized coordinate from every driving axis. But, considering the current and sudden motions, the generalized variables of higher order are developed as follows:

$$\left\{\begin{array}{l}\left\{\bar{\theta}(t);\ \dot{\bar{\theta}}(t);\ \ddot{\bar{\theta}}(t);\cdots;\overset{(m)}{\bar{\theta}}(t)\right\}=\\ =\left\{\begin{array}{c}q_i(t);\ \dot{q}_i(t);\ \ddot{q}_i(t);\cdots;\overset{(m)}{q_i}(t)\\ i=1\rightarrow n,\ m\geq 1\end{array}\right\}\end{array}\right\}, \qquad (2)$$

and (m) represents the time deriving order. The main objective of this section consists in the establishment of the parameters of the advanced kinematics, typical to MBS with current and sudden motions. In the view of this, matrix exponentials and the time derivatives of higher order will be applied on kinematical parameters. These expressions will be used in second section devoted to energies and equations of higher order.

According to researchers [5], [7] and [18], kinematical parameters, are expressed, among others, by means of the matrix exponentials functions. First of all, the position vector and rotation matrix and between $\{i\}$ and $\{0\}$ frames are below expressed with exponentials as:

$$\bar{p}_i=\sum_{j=1}^{i}\left\{exp\left\{\sum_{k=0}^{j-1}\left\{\bar{u}_k^{(0)}\times\right\}\ q_k\cdot\Delta_k\right\}\right\}\cdot\bar{b}_i\ ; \qquad (3)$$

$$\left\{\begin{array}{l}{}^{0}_{i}[R]=\left\{exp\left\{\sum_{j=1}^{i}\left[\bar{u}_j^{(0)}\times\right]q_j\cdot\Delta_j\right\}\right\}\cdot R_{i0}^{(0)}=\\ \quad=\prod_{j=1}^{i}exp\left\{\left[\bar{u}_j^{(0)}\times\right]q_j\cdot\Delta_j\right\}\cdot R_{i0}^{(0)}\end{array}\right\} \qquad (4)$$

where $R_{i0}^{(0)}$ corresponds to initial configuration of the multibody system. The column vector $\bar{b}_i$ of the expression (3), according to [5], [7], [18], is:

$$\left\{\begin{array}{l}\bar{b}_i=\left\{I_3\cdot q_i+\left\{\bar{u}_i^{(0)}\times\right\}\left[1-cos(q_i\cdot\Delta_i)\right]+\right.\\ \quad\left.+\bar{u}_i^{(0)}\cdot\bar{u}_i^{(0)T}\cdot\left[q_i-sin(q_i\cdot\Delta_i)\right]\right\}\cdot\bar{s}_i^{(0)}\end{array}\right\} \qquad (5)$$

In the previous equations, the symbols are used:

$$\bar{u}_i=\{\bar{x}_i;\bar{y}_i;\bar{z}_i\}\ \&\ \bar{s}_i^{(0)}=\left\{\bar{p}_i^{(0)}\times\right\}\bar{u}_i^{(0)}\cdot\Delta_i+(1-\Delta_i)\cdot\bar{u}_i^{(0)}.$$

They expresses the screw parameters also named the homogeneous parameters of the oriented axis $\{i\}$ around of this the generalized coordinates are achieved. According to same papers [5] and [7], expressions of definition for angular velocities, and then angular accelerations of higher order are established on the basis of matrix exponentials:

$$\bar{\omega}_i(t)=\left\{\sum_{j=1}^{i}\left\{exp\left\{\sum_{k=1}^{j-1}\left\{\bar{u}_k^{(0)}\times\right\}q_k\cdot\Delta_k\right\}\right\}\bar{u}_i^{(0)}\cdot\dot{q}_i\cdot\Delta_i\right\} \qquad (6)$$

$$\overset{(k)}{\bar{\omega}}_i(t)=\frac{d^k}{dt^k}\left\{\sum_{j=1}^{i}\left\{exp\left\{\sum_{k=1}^{j-1}\left\{\bar{u}_k^{(0)}\times\right\}q_k\cdot\Delta_k\right\}\right\}\bar{u}_i^{(0)}\cdot\dot{q}_i\cdot\Delta_i\right\} \qquad (7)$$

$k\geq 1;\ k=\{1;2;3;4;5;...\}$ is time derivative order.

According to same [5] and [7], the expressions of definition for the linear velocities of the origin $O_i\in\{i\}$ and then linear accelerations of higher order are also expressed with exponentials as:

$$\bar{v}_i=\frac{d}{dt}\left\{\sum_{j=1}^{i}\left\{exp\left\{\sum_{k=0}^{j-1}\left\{\bar{u}_k^{(0)}\times\right\}\ q_k\cdot\Delta_k\right\}\right\}\cdot\bar{b}_i\right\}= \qquad (8)$$

$$\left\{\begin{array}{l}=\sum_{j=1}^{i}\left\{\sum_{m=1}^{j-1}\left\{exp\left[V_1^*\right]\right\}\cdot exp\left[V^{**}\right]\right\}+\sum_{j=1}^{i}\left\{exp\left[V_3^*\right]\right\}\cdot\dot{\bar{b}}_i\\ \text{where}\quad exp\left[V^{**}\right]=\left\{\bar{u}_m^{(0)}\times\right\}\dot{q}_m\cdot\Delta_m\cdot\left\{exp\left[V_2^*\right]\right\}\end{array}\right\},$$

where $\left\{\begin{array}{l}V_1^*=\sum_{k=0}^{m-1}\left\{\bar{u}_k^{(0)}\times\right\}q_k\cdot\delta_{mk}\cdot\Delta_k\\ V_2^*=\sum_{l=m}^{i}\left\{\bar{u}_l^{(0)}\times\right\}q_l\cdot\Delta_l\\ V_3^*=\sum_{k=0}^{j-1}\left\{\bar{u}_k^{(0)}\times\right\}q_k\cdot\Delta_k\end{array}\right\},$

and $\qquad \delta_{mk}=\{(0;m>j);(1;m\le j)\}$;

$$\overset{(k)}{\bar{v}}_i=\frac{d^k}{dt^k}\left\{\sum_{j=1}^{i}\left\{\exp\left\{\sum_{k=0}^{j-1}\left\{\bar{u}_k^{(0)}\times\right\}\,q_k\cdot\Delta_k\right\}\right\}\cdot\bar{b}_i\right\}; \qquad (9)$$

$$\left\{\begin{aligned}\overset{(k)}{\bar{b}}_i=\frac{d^k}{dt^k}\Big\{\Big\{&I_3\cdot q_i+\left\{\bar{u}_i^{(0)}\times\right\}\left[1-\cos(q_i\cdot\Delta_i)\right]+\\&+\bar{u}_i^{(0)}\cdot\bar{u}_i^{(0)T}\cdot\left[q_i-\sin(q_i\cdot\Delta_i)\right]\Big\}\cdot\bar{s}_i^{(0)}\Big\}\end{aligned}\right\} \qquad (10)$$

The using of matrix exponentials apparently seems to be complicatedly, but these have great advantages of not using reference systems. This observation is visible in the above equations, by the occurrence of homogeneous coordinates (6). These are corresponding to initial configuration.

In advanced kinematics and dynamics, the time derivatives of higher order for position vectors and rotation matrices must be used as:

$$\left\{\begin{aligned}&\frac{d^k\bar{p}_i(t)}{dt^k}=\overset{(k)}{\bar{p}}_i=\sum_{j=1}^{i}\left[\frac{\partial\overset{(m)}{\bar{p}_i}}{\partial\overset{(m)}{q_j}}\cdot\overset{(k)}{q}_j\right]+\\&\sum_{j=1}^{i}\sum_{r=1}^{k-1}\left\{\frac{\prod_{p=1}^{r}(k-p)}{p!}\cdot\left[\frac{p!\cdot m!}{(m+p)!}\cdot\frac{\partial\overset{(m+p)}{\bar{p}_i}}{\partial\overset{(m)}{q_j}}\cdot\overset{(k-p)}{q}_j\right]\right\}\end{aligned}\right\} \qquad (11)$$

$$\left\{\begin{aligned}&\frac{d^k}{dt^k}\left\{{}_i^0[R](t)\right\}={}_i^0\overset{(k)}{[R]}=\sum_{j=1}^{i}\left\{\frac{\partial}{\partial\overset{(m)}{q_j}}\left\{{}_s^0\overset{(m)}{[R]}\right\}\cdot\Delta_j\cdot\overset{(k)}{q}_j\right\}+\\&\sum_{j=1}^{i}\sum_{r=1}^{k-1}\left\{\frac{\prod_{p=1}^{r}(k-p)}{p!}\cdot\left\{\frac{d^p}{dt^p}\left\{\frac{\partial}{\partial\overset{(m)}{q_j}}\left\{{}_s^0\overset{(m)}{[R]}\right\}\right\}\cdot\Delta_j\cdot\overset{(k-p)}{q}_j\right\}\right\}=\\&=\sum_{j=1}^{i}\left\{\frac{\partial}{\partial\overset{(m)}{q_j}}\left\{{}_s^0\overset{(m)}{[R]}\right\}\cdot\Delta_j\cdot\overset{(k)}{q}_j\right\}+\\&\sum_{j=1}^{i}\sum_{r=1}^{k-1}\left\{\frac{\prod_{p=1}^{r}(k-p)}{p!}\cdot\left\{\frac{p!\cdot m!}{(m+p)!}\cdot\frac{\partial}{\partial\overset{(m)}{q_j}}\left\{{}_s^0\overset{(m+p)}{[R]}\right\}\cdot\Delta_j\cdot\overset{(k-p)}{q}_j\right\}\right\}\end{aligned}\right\}$$

and $\left\{\begin{aligned}k\ge1;\;\;k=\{1;2;3;4;5;.....\}\\m\ge(k+1);\;\;m=\{2;3;4;5;.....\}\end{aligned}\right\}$; $\qquad$ (12)

where the symbols: (k) and (m) are the orders of the time derivatives concerning (11) and (12).

The advanced notions and equations from analytical dynamics [9] – [17] requires angular accelerations of higher order, as well as linear accelerations of higher order corresponding to mass center for every kinetic ensemble of MBS.

According to Fig.3, first of all, the position of the mass center is defined in the classical form, and then on the basis of matrix exponentials as:

$$\bar{r}_{C_i}(t)=\bar{p}_i(t)+{}_i^0[R](t)\cdot{}^i\bar{\rho}_{C_i}; \qquad (13)$$

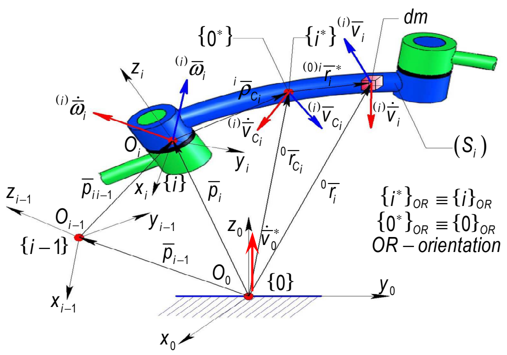

Fig.3 Sequence of Kinetic Ensemble

$$\left\{\begin{aligned}\bar{r}_{C_i}(t)=\sum_{j=1}^{i}&\left\{exp\left\{\sum_{k=0}^{j-1}\left\{\bar{u}_k^{(0)}\times\right\}\,q_k(t)\cdot\Delta_k\right\}\right\}\cdot\bar{b}_i(t)\\&+\left\{exp\left\{\sum_{j=1}^{i}\left[\bar{u}_j^{(0)}\times\right]q_j(t)\cdot\Delta_j\right\}\right\}\cdot\bar{r}_{C_i}^{(0)}\end{aligned}\right\} \qquad (14)$$

Applying the time derivative on (13), the linear velocity of the mass center is obtained, thus:

$$\bar{v}_{C_i}(t)=\bar{v}_i(t)+\bar{\omega}_i(t)\times\bar{\rho}_{C_i}(t); \qquad (15)$$

$${}_i^0\left[\dot{R}\right](t)\cdot{}_i^0[R]^T(t)=\left[\bar{\omega}_i(t)\times\right], \qquad (16)$$

Property (15) is according to papers [7] – [9].
Linear and absolute accelerations of higher order corresponding to mass center are below defined:

$$\left\{\begin{aligned}\overset{(k)}{\bar{v}}_{C_i}=\frac{d^k}{dt^k}&\left\{\sum_{j=1}^{i}\left\{exp\left\{\sum_{k=0}^{j-1}\left\{\bar{u}_k^{(0)}\times\right\}\,q_k\cdot\Delta_k\right\}\right\}\cdot\bar{b}_i\right\}\\&+\frac{d^k}{dt^k}\left\{exp\left\{\sum_{j=1}^{i}\left[\bar{u}_j^{(0)}\times\right]q_j(t)\cdot\Delta_j\right\}\right\}\cdot\bar{r}_{C_i}^{(0)}\end{aligned}\right\} \qquad (17)$$

$$\overset{(k)}{\bar{v}}_{C_i}(t)=\overset{(k)}{\bar{v}}_i(t)+\frac{d^k}{dt^k}\left[\bar{\omega}_i(t)\times\bar{\rho}_{C_i}(t)\right], \qquad (18)$$

where $u=\{\bar{\omega}_i\times;\bar{\rho}_{C_i}\}$; $v=\{\bar{\rho}_{C_i};\bar{\omega}_i\times\}$; $v\ne u$; $X=\{u;v\}$:

$$\left\{\begin{aligned}&\frac{d^k}{dt^k}\left[\bar{\omega}_i(t)\times\bar{\rho}_{C_i}(t)\right]=\sum_{\{u;v\}}\left\{\overset{(k)}{u}\cdot v\right\}+\frac{k}{0!}\cdot\sum_{\{u;v\}}\left\{\overset{(k-1)}{u}\cdot\dot{v}\right\}\\&+(k-\varDelta_k)\cdot\left[k-(j+1-\varDelta_k)\right]\cdot\sum_{\{u;v\}}\left\{\overset{(k-2)}{u}\cdot\ddot{v}\right\}\cdot\delta_k\\&+k\cdot\left[k-(2-\varDelta_k)\right]\cdot\sum_{\{u;v\}}\left\{\overset{(k-2)}{u}\cdot\overset{(3)}{v}\right\}\cdot\delta_{kk}\\&+(k-1)\cdot\left[k-2\cdot(1-\delta_{kk})\right]\cdot\sum_{\{u;v\}}\left\{\overset{(k-j)}{u}\cdot\overset{(k-j)}{v}\right\}\cdot\varDelta_k\end{aligned}\right\}.$$

Position of terms from (18) must be respected, in the case of the time derivative of order (k),

$$\left\{\begin{matrix} and\ 1\le k\le 8;\ \ \delta_{k(k)}=\left\{\left(0;\ k\le 4(6)\right);\left(1;\ k\ge 4(6)\right)\right\} \\ \Delta_k=\left\{\left(1;\ k=2\cdot j\right);\left(0;\ k\ne 2\cdot j\right),\ and\ j\ge 1\right\}\end{matrix}\right\}.$$

According to [10] – [15], angular and linear velocities, as well as the accelerations of higher order (6), (7), (15) – (18) can be also established by means of the following vector time functions:

$$\bar{r}_{C_i}=\bar{r}_{C_i}\left[q_j(t);\quad j=1\rightarrow k^*=n,\ t\right];\quad (19)$$

$$\bar{\bar{\psi}}_i(t)=\left[\alpha_A(t)\quad \beta_B(t)\quad \gamma_C(t)\right]^T;\quad (20)$$

$$\left\{\begin{matrix} {}^0J_{\psi}^i\left[\alpha_A(t)-\beta_B(t)-\gamma_C(t)\right]= \\ \left[{}^0\bar{A}\quad R\left(\bar{A};\alpha_A\right)\cdot\bar{B}\quad R\left(\bar{A};\alpha_A\right)\cdot R\left(\bar{B};\beta_B\right)\cdot\bar{C}\right]\end{matrix}\right\};\ (21)$$

$$\left\{\begin{matrix} \bar{\psi}_i(t)={}^0J_{\psi}\left[\alpha_A(t)-\beta_B(t)-\gamma_C(t)\right]\cdot\bar{\bar{\psi}}_i(t)= \\ =\bar{\psi}_i\left[q_j(t)\cdot\Delta_j;\quad j=1\rightarrow k^*=n,\ t\right]\end{matrix}\right\};\ (22)$$

$$\Delta_j=\left\{\left(0\ for\ q_j-linear\right);\left(1\ for\ q_j-angular\right)\right\}\ (23)$$

where (19) is identical with (13) / (14), and (22) is named the orientation vector for every kinetic ensemble, whose component (21) is known as angular transfer matrix defined as function of set of orientation angles. Considering (19) and (22) it observes that they are functions of generalized variables (2), taking into study the operator (23). Using researches from [10] – [14], on vector functions (19) and (22), differentials properties compulsory applied in advanced kinematics and dynamics have been developed as below follows:

$$\frac{\partial\bar{r}_{C_i}}{\partial q_j}=\frac{\partial\bar{v}_{C_i}}{\partial\dot{q}_j}=\frac{\partial\bar{a}_{C_i}}{\partial\ddot{q}_j}=\frac{\partial\dot{\bar{a}}_{C_i}}{\partial\dddot{q}_j}=\frac{\partial\ddot{\bar{a}}_{C_i}}{\partial\ddddot{q}_j}=...=\frac{\partial\overset{(m)}{\bar{r}}_{C_i}}{\partial\overset{(m)}{q}_j},\ (24)$$

$$\frac{\partial\bar{\psi}_i}{\partial q_j}=\frac{\partial\dot{\bar{\psi}}_i}{\partial\dot{q}_j}=\frac{\partial\bar{\varepsilon}_i}{\partial\ddot{q}_j}=\ldots=\frac{\partial\ddot{\bar{\varepsilon}}_i}{\partial\ddddot{q}_j}=...=\frac{\partial\overset{(m)}{\bar{\psi}}_i}{\partial\overset{(m)}{q}_j},\ (25)$$

$$\left\{\begin{matrix} \frac{d}{dt}\left(\frac{\partial\bar{r}_{C_i}}{\partial q_j}\right)=\frac{\partial\bar{v}_{C_i}}{\partial q_j}=\frac{\partial}{\partial q_j}\left(\sum_{m=1}^{k^*=n}\frac{\partial\bar{r}_{C_i}}{\partial q_m}\cdot\dot{q}_m\right)= \\ =\frac{1}{m+1}\cdot\frac{\partial\overset{(m-1)}{\bar{a}}_{C_i}}{\partial\overset{(m)}{q}_j}=\frac{1}{m+1}\cdot\frac{\partial\overset{(m+1)}{\bar{r}}_{C_i}}{\partial\overset{(m)}{q}_j},\ m\ge 0\end{matrix}\right\},\ (26)$$

$$\left\{\begin{matrix} \frac{d}{dt}\left(\frac{\partial\bar{\psi}_i}{\partial q_j}\right)=\frac{\partial\bar{\omega}_i}{\partial q_j}=\frac{\partial}{\partial q_j}\left(\sum_{m=1}^{k^*=n}\frac{\partial\bar{\psi}_i}{\partial q_m}\cdot\dot{q}_m\right)= \\ =\frac{1}{m+1}\cdot\frac{\partial\overset{(m-1)}{\bar{\varepsilon}}_i}{\partial\overset{(m)}{q}_j}=\frac{1}{m+1}\cdot\frac{\partial\overset{(m+1)}{\bar{\psi}}_i}{\partial\overset{(m)}{q}_j},\ m\ge 0\end{matrix}\right\},\ (27)$$

$$\left\{\frac{d^{k-1}}{dt^{k-1}}\left(\frac{\partial\bar{r}_{C_i}}{\partial q_j}\right)=\frac{(k-1)!\cdot m!}{(m+k-1)!}\cdot\frac{\partial\overset{(m+k-1)}{\bar{r}}_{C_i}}{\partial\overset{(m)}{q}_j}\right\},\quad (28)$$

$$\left\{\begin{matrix} \frac{d^{k-1}}{dt^{k-1}}\left(\frac{\partial\bar{\psi}_i}{\partial q_j}\cdot\Delta_j\right)=\frac{(k-1)!\cdot m!}{(m+k-1)!}\cdot\frac{\partial\overset{(m+k-3)}{\bar{\varepsilon}}_i}{\partial\overset{(m)}{q}_j}\cdot\Delta_j= \\ =\frac{(k-1)!\cdot m!}{(m+k-1)!}\cdot\frac{\partial\overset{(m+k-1)}{\bar{\psi}}_i}{\partial\overset{(m)}{q}_j}\cdot\Delta_j\end{matrix}\right\}\ (29)$$

$$\left\{\begin{matrix} k\ge 1;\ \ k=\{1;2;3;4;5;.....\}; \\ m\ge(k+1);\ \ m=\{2;3;4;5;.....\}\end{matrix}\right\}.\quad (30)$$

The symbols (30) highlight time deriving orders. Using (19) – (30), the next expressions become:

$$\bar{v}_{C_i}(t)=\sum_{j=1}^{k^*=n}\frac{\partial\bar{r}_{C_i}(t)}{\partial q_j}\cdot\dot{q}_j(t)=\sum_{j=1}^{k^*=n}\frac{\partial\overset{(m)}{\bar{r}}_{C_i}(t)}{\partial\overset{(m)}{q}_j}\cdot\dot{q}_j(t)\ (31)$$

$$\left\{\begin{matrix} \bar{a}_{C_i}(t)=\dot{\bar{v}}_{C_i}(t)=\ddot{\bar{r}}_{C_i}(t)= \\ =\sum_{j=1}^{k^*=n}\frac{\partial\overset{(m)}{\bar{r}}_{C_i}(t)}{\partial\overset{(m)}{q}_j}\cdot\ddot{q}_j(t)+\sum_{j=1}^{k^*=n}\frac{1}{m+1}\cdot\frac{\partial\overset{(m+1)}{\bar{r}}_{C_i}(t)}{\partial\overset{(m)}{q}_j}\cdot\dot{q}_j(t)\end{matrix}\right\};$$

$$\left\{\begin{matrix} \overset{(k-1)}{\bar{a}}_{C_i}(t)=\overset{(k)}{\bar{v}}_{C_i}(t)=\sum_{j=1}^{k^*=n}\frac{d^{k-1}}{dt^{k-1}}\left[\frac{\partial\overset{(m)}{\bar{r}}_{C_i}(t)}{\partial\overset{(m)}{q}_j}\cdot\ddot{q}_j(t)\right] \\ +\sum_{j=1}^{k^*=n}\frac{d^{k-1}}{dt^{k-1}}\left[\frac{1}{m+1}\cdot\frac{\partial\overset{(m+1)}{\bar{r}}_{C_i}(t)}{\partial\overset{(m)}{q}_j}\cdot\dot{q}_j(t)\right]=\overset{(k+1)}{\bar{r}}_{C_i}(t)\end{matrix}\right\}\ (32)$$

$$\left\{\begin{matrix} \bar{\omega}_i(t)=\sum_{j=1}^{k^*=n}\frac{\partial\bar{\psi}_i(t)}{\partial q_j}\cdot\Delta_j\cdot\dot{q}_j(t)= \\ =\sum_{j=1}^{k^*=n}\frac{\partial\overset{(m)}{\bar{\psi}}_i(t)}{\partial\overset{(m)}{q}_j}\cdot\Delta_j\cdot\dot{q}_j(t)\end{matrix}\right\};\quad (33)$$

$$\left\{\begin{array}{l}\bar{\varepsilon}_i(t)=\dot{\bar{\omega}}_i(t)=\sum\limits_{j=1}^{k^*=n}\dfrac{\partial \overset{(m)}{\bar{\psi}_i}(t)}{\partial \overset{(m)}{q_j}}\cdot\Delta_j\cdot\ddot{q}_j(t)+\\ +\dfrac{1}{m+1}\cdot\sum\limits_{j=1}^{k^*=n}\dfrac{\partial \overset{(m+1)}{\bar{\psi}_i}}{\partial \overset{(m)}{q_j}}\cdot\Delta_j\cdot\dot{q}_j(t)=\ddot{\bar{\psi}}_i(t)\end{array}\right\};\quad(34)$$

$$\left\{\begin{array}{l}\overset{(k-1)}{\bar{\varepsilon}_i}(t)=\overset{(k)}{\bar{\omega}_i}(t)=\sum\limits_{j=1}^{k^*=n}\dfrac{d^{k-1}}{dt^{k-1}}\left[\dfrac{\partial \overset{(m)}{\bar{\psi}_i}(t)}{\partial \overset{(m)}{q_j}}\cdot\Delta_j\cdot\ddot{q}_j(t)\right]\\ +\dfrac{1}{m+1}\cdot\sum\limits_{j=1}^{k^*=n}\dfrac{d^{k-1}}{dt^{k-1}}\left[\dfrac{\partial \overset{(m+1)}{\bar{\psi}_i}}{\partial \overset{(m)}{q_j}}\cdot\Delta_j\cdot\dot{q}_j(t)\right]=\overset{(k+1)}{\bar{\psi}_i}(t)\end{array}\right\}.$$

The expressions (31) and (32) are identical with (15) and respectively (17) / (18), and they are referring to the linear velocity and accelerations of higher order, corresponding to mass center. The others, (34) are identical with (6) and (7) representing the angular velocities, and angular accelerations of higher order of kinetic ensemble.

Analyzing all above parameters of advanced kinematics, it results that they are functions of generalized variables (1) / (2), as well their time derivatives. So, according to author researches they can be developed as time functions, using polynomial interpolating functions [3] and [7]. It proposes following functions of higher order:

$$\left\{\begin{array}{l}\overset{(m-p)}{q_{ji}}(\tau)=(-1)^p\cdot\dfrac{(\tau_i-\tau)^{p+1}}{t_i\cdot(p+1)!}\cdot\overset{(m)}{q}_{ji-1}+\\ +\dfrac{(\tau-\tau_{i-1})^{p+1}}{t_i\cdot(p+1)!}\cdot\overset{(m)}{q}_{ji}+\delta_p\cdot\sum\limits_{k=1}^{p}\dfrac{\tau^{p-k}}{(p-k)!}\cdot a_{jik}\end{array}\right\};\ (35)$$

$$\left\{\begin{array}{c}\text{where}\quad p=0\rightarrow m\\ m-\text{deriving order},\ m\geq 2,\ m=2,3,4,5,\ldots\\ \delta_p=\{(0,\ p=0);(1;p\geq 1)\}\\ j=1\rightarrow n\quad \text{degrees of freedom}-(d.o.f.)\\ i=1\rightarrow s\quad \text{intervals of motion trajectories}\\ \tau-\text{actual time variable}\\ t_i=\tau_i-\tau_{i-1}\ (\text{time to each trajectory interval})\end{array}\right\};\ (36)$$

For every trajectory interval $(i=1\rightarrow s)$, number of unknowns is $(m+1)$, and their significance is:

$$\left\{\begin{array}{l}(a_{jik})\ \text{for}\ k=1\rightarrow m;\ \text{and}\ \left(\overset{(m)}{q}_{ji-1}\right)\ \text{for}\ i=2\rightarrow s\\ \text{where}\ (a_{jik})-\text{integration constants, and}\\ \left(\overset{(m)}{q}_{ji-1}\right)-\text{generalized accelerations of}\ (m)\ \text{order}\end{array}\right\}(37)$$

The determination the unknowns (37) requires, in accordance with [3] – [7], the application of the geometrical and kinematical constraints as:

$$\left\{\begin{array}{l}(\tau_0)\Rightarrow \overset{(m-p)}{q}_{j0},\quad p=0\rightarrow m;\quad (\tau_s)\Rightarrow\left\{\overset{(m)}{q}_{js},\ q_{js}\right\}\\ (\tau_i)\Rightarrow\left\{\begin{array}{c}\overset{(2)}{q}_{ji}-\text{generalized accelerations}\\ \left\{\begin{array}{c}\overset{(m-p)}{q_{ji}}(\tau^+)=\overset{(m-p)}{q}_{ji+1}(\tau^-),\quad p=0\rightarrow m\\ \text{continuity conditions}\end{array}\right\}\\ \text{all conditions are applied to each}\ (\tau_i)\\ \text{where}\quad i=1\rightarrow s-1\end{array}\right\}\end{array}\right\}(38)$$

Finally, the results (35) will be substituted in the advanced notions of kinematics and dynamics.

3. ENERGIES OF HIGHER ORDER

The phrase, "advanced notions" founded in the analytical dynamics, is focused in this paper on the motion energies whose central functions are the accelerations of higher order. They are developing in any sudden and transitory motion of the mechanical systems. Leading to Appell's function, highlighted in 1899, [1] and [2], also named "kinetic energy of accelerations" [19], author has been developed new mathematical formulations on the expressions for acceleration energies of first, second, third and fourth order [6] – [14] and [17]. In this section they will be presented, in explicit and matrix form, and their kinematical parameters of higher order are well defined in the previous section of this paper.

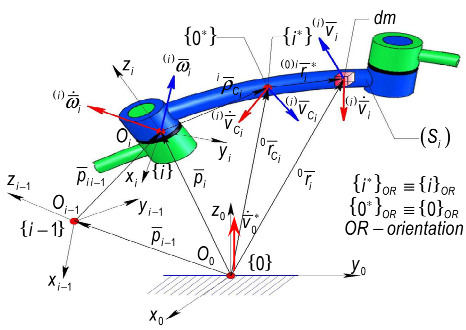

Fig. 4 Kinetic Ensemble from MBS

For understanding the mechanical significances of the energies of higher order, at beginning *the kinetic energy* is defined, according to [3] – [6]. So, König's theorem is devoted to explicit form:

$$\left\{\begin{array}{l}(-1)^{\Delta_M}\cdot\dfrac{1-\Delta_M}{1+3\cdot\Delta_M}\cdot\left\{\dfrac{1}{2}\cdot M_i\cdot{}^i\bar{v}_{C_i}^T\cdot{}^i\bar{v}_{C_i}\right\}+\\ +\Delta_M^2\cdot\dfrac{1}{2}\cdot{}^i\bar{\omega}_i^T\cdot{}^iI_i^*\cdot{}^i\bar{\omega}_i=E_C^i\left[\bar{\theta}(t);\dot{\bar{\theta}}(t)\right]\end{array}\right\}.\ (39)$$

To this, the operator is added with significance:

$\Delta_M=\{(-1;\text{general motion});(0;\text{translation});(1;\text{rotation})\}$

Expression of the kinetic energy (39) contains:

$$ {}^{i}I_{i}^{*}=\int\left[{}^{i}\overline{r}_{i}^{*}\times\right]\cdot\left[{}^{i}\overline{r}_{i}^{*}\times\right]^{T}\cdot dm=\begin{bmatrix} {}^{i}I_{x}^{*} & -{}^{i}I_{xy}^{*} & -{}^{i}I_{xz}^{*} \\ -{}^{i}I_{yx}^{*} & {}^{i}I_{y}^{*} & -{}^{i}I_{yz}^{*} \\ -{}^{i}I_{zx}^{*} & -{}^{i}I_{zy}^{*} & {}^{i}I_{z}^{*} \end{bmatrix} \quad (40) $$

This is inertial tensor axial and centrifugal of the kinetic ensemble (i), in relation with frame $\{i^*\}$ whose origin is the mass center C_i (see Fig.4). Considering the notions from previous section the total kinetic energy of MBS is written as:

$$ \left\{ \begin{array}{c} E_C\left[\overline{\theta}(t);\dot{\overline{\theta}}(t)\right]= \\ =\sum_{i=1}^{n}E_C^{iTR}\left[\overline{\theta}(t);\dot{\overline{\theta}}(t)\right]+\sum_{i=1}^{n}E_C^{iROT}\left[\overline{\theta}(t);\dot{\overline{\theta}}(t)\right] \end{array} \right\}; \quad (41) $$

The translational and rotation components are:

$$ \left\{ \begin{array}{c} \sum_{i=1}^{n}E_C^{iTR}\left[\overline{\theta}(t);\dot{\overline{\theta}}(t)\right]= \\ (-1)^{\Delta_M}\cdot\dfrac{1-\Delta_M}{1+3\cdot\Delta_M}\cdot\dfrac{1}{2}\cdot\sum_{i=1}^{n}M_i\cdot\sum_{j=1}^{k^*=n}\dfrac{1}{m+1}\cdot\dfrac{\partial\ \overset{(m+1)}{\overline{r}_{C_i}}}{\partial\overset{(m)}{q_j}}\cdot\dot{q}_j \end{array} \right\}, $$

$$ \left\{ \begin{array}{c} \sum_{i=1}^{n}E_C^{iROT}\left[\overline{\theta}(t);\dot{\overline{\theta}}(t)\right]= \\ \dfrac{\Delta_M^2}{2}\cdot\sum_{i=1}^{n}\left[\sum_{j=1}^{k^*=n}\dfrac{\partial\overset{(m)}{\overline{\psi}_i}}{\partial\overset{(m)}{q_j}}\cdot\Delta_j\cdot\dot{q}_j\right]\cdot I_S^*\cdot\left[\sum_{p=1}^{k^*=n}\dfrac{\partial\overset{(m)}{\overline{\psi}_i}}{\partial\overset{(m)}{q_p}}\cdot\Delta_p\cdot\dot{q}_p\right] \end{array} \right\}, $$

and $\Delta_j=\left\{\left(0,\ \ q_{j(p)}\in\overline{r}_C\right);\left(1,\ \ q_{j(p)}\in\overline{\psi}\right)\right\}$. (42)

In the dynamic equations of higher order, the kinetic energy is included by means of the time derivative of higher order. It shows as follows:

$$ \left\{ \overset{(k)}{E_C^{iTR}}=\frac{1}{2}\cdot M_i\cdot\sum_{j=1}^{k^*=n}\frac{d^k}{dt^k}\left[\frac{1}{m+1}\cdot\frac{\partial\ \overset{(m+1)}{\overline{r}_{C_i}}}{\partial\overset{(m)}{q_j}}\cdot\dot{q}_j\right] \right\}, \quad (43) $$

$$ \left\{ \begin{array}{c} X=\{u;v\};\ \ v\neq u \\ u=\left\{\dfrac{1}{m+1}\cdot\dfrac{\partial\ \overset{(m+1)}{\overline{r}_{C_i}}}{\partial\overset{(m)}{q_j}};\dot{q}_j\right\};\ v=\left\{\dot{q}_j;\dfrac{1}{m+1}\cdot\dfrac{\partial\ \overset{(m+1)}{\overline{r}_{C_i}}}{\partial\overset{(m)}{q_j}}\right\} \end{array} \right\}, $$

$$ \left\{ \begin{array}{c} \dfrac{d^k}{dt^k}\left[\dfrac{1}{m+1}\cdot\dfrac{\partial\ \overset{(m+1)}{\overline{r}_{C_i}}}{\partial\overset{(m)}{q_j}}\cdot\dot{q}_j\right]= \\ =\sum_{\{u;v\}}\left\{\overset{(k)}{u}\cdot v\right\}+\dfrac{k}{0!}\cdot\sum_{\{u;v\}}\left\{\overset{(k-1)}{u}\cdot\dot{v}\right\}+ \\ +\left(k-\Delta_k\right)\cdot\left[k-\left(j+1-\Delta_k\right)\right]\cdot\sum_{\{u;v\}}\left\{\overset{(k-2)}{u}\cdot\ddot{v}\right\}\cdot\delta_k \\ +k\cdot\left[k-\left(2-\Delta_k\right)\right]\cdot\sum_{\{u;v\}}\left\{\overset{(k-2)}{u}\cdot\overset{(3)}{v}\right\}\cdot\delta_{kk} \\ +(k-1)\cdot\left[k-2\cdot(1-\delta_{kk})\right]\cdot\sum_{\{u;v\}}\left\{\overset{(k-j)}{u}\cdot\overset{(k-j)}{v}\right\}\cdot\Delta_k \end{array} \right\}, \quad (44) $$

$$ \left\{ \begin{array}{c} where\ \ 1\leq k\leq 8;\ \ for\ \ j\geq k\Rightarrow\ \overset{(k-j)}{u}=0;\ \overset{(k-j)}{v}=0 \\ \Delta_k=\left\{(1;\ k=2\cdot j);\ (0;\ k\neq 2\cdot j),\ and\ j=1,2,3,4,\ldots\right\} \\ \delta_{k(k)}=\left\{\left(0;\ k\leq 4(6)\right);\ \left(1;\ k\geq 4(6)\right)\right\} \end{array} \right\}. $$

$$ \left\{ \begin{array}{c} \overset{(k)}{E_C^{iROT}}=\dfrac{1}{2}\cdot{}^{i}\overline{\omega}_i^T\cdot{}^{i}I_i^*\cdot{}^{i}\overline{\omega}_i= \\ \dfrac{1}{2}\cdot\dfrac{d^k}{dt^k}\left\{\left[\sum_{j=1}^{k^*=n}\dfrac{\partial\overset{(m)}{\overline{\psi}_i}}{\partial\overset{(m)}{q_j}}\cdot\Delta_j\cdot\dot{q}_j\right]\cdot I_i^*\cdot\left[\sum_{p=1}^{k^*=n}\dfrac{\partial\overset{(m)}{\overline{\psi}_i}}{\partial\overset{(m)}{q_p}}\cdot\Delta_p\cdot\dot{q}_p\right]\right\} \end{array} \right\} \quad (45) $$

$$ \left\{ \begin{array}{c} \left\{ \begin{array}{c} X=\{u;v;w\} \\ u=\{a;b;c\} \\ v=\{b;c;a\} \\ w=\{c;b;a\} \\ v\neq u;w\neq v;u\neq v \end{array} \right\};\ a=\sum_{j=1}^{k}\dfrac{\partial\overset{(m)}{\overline{\psi}_i}}{\partial\overset{(m)}{q_j}}\cdot\Delta_j\cdot\dot{q}_j \\ b=I_i^*={}_{i}^{0}[R]\cdot{}^{i}I_i^*\cdot{}_{i}^{0}[R]^T;\ \ c=\sum_{p=1}^{k}\dfrac{\partial\overset{(m)}{\overline{\psi}_i}}{\partial\overset{(m)}{q_p}}\cdot\Delta_p\cdot\dot{q}_p \end{array} \right\}, $$

$$ \left\{ \begin{array}{c} \overset{(k)}{E_C^{iROT}}=\dfrac{1}{2}\cdot{}^{i}\overline{\omega}_i^T\cdot{}^{i}I_i^*\cdot{}^{i}\overline{\omega}_i=\dfrac{1}{2}\cdot\sum_{\{u;v;w\}}\left\{\overset{(k)}{u}\cdot v\cdot w\right\}+ \\ +\dfrac{1}{2}\cdot\dfrac{k}{(4-k)!}\cdot\sum_{\{u;v;w\}}\left\{\overset{(k-1)}{u}\cdot\left(\dot{v}\cdot w+v\cdot\dot{w}\right)\right\}+ \\ +\dfrac{1}{2}\cdot\dfrac{k!}{4\cdot(k-4)!}\cdot\sum_{\{u;v;w\}}\left\{\overset{(k-2)}{u}\cdot\overset{(k-2)}{v}\cdot w\right\}+ \\ +\dfrac{1}{2}\cdot\dfrac{k!}{(6-k)\cdot(k-3)!}\cdot\sum_{\{u;v;w\}}\left\{\overset{(k-2)}{u}\cdot\dot{v}\cdot\dot{w}\right\} \end{array} \right\}, \quad (46) $$

where $k\leq 4$, and position of terms from (45) must be respected, referring to time derivative in (46). The matrix form of the kinetic energy shows as:

$$ \left\{ \begin{array}{c} E_C\left[\overline{\theta}(t);\dot{\overline{\theta}}(t)\right]=\dfrac{1}{2}\cdot\dot{\overline{\theta}}^T(t)\cdot M\left[\overline{\theta}(t)\right]\cdot\dot{\overline{\theta}}(t) \\ =\dfrac{1}{2}\cdot{}^{0}\dot{X}^T(t)\cdot M_X\left[\overline{\theta}(t)\right]\cdot{}^{0}\dot{X}(t) \end{array} \right\} \quad (47) $$

The mass matrix (inertia matrix) from (47) is:

$$ M\left[\overline{\theta}(t)\right]=\underset{(n\times n)}{Matrix}\left\{M_{ij}\quad \begin{array}{c} i=1\rightarrow n \\ j=1\rightarrow n \end{array}\right\}, \quad (48) $$

Its components are determined with exponentials:

$$ M_{ij}=\sum_{k=\max(i;j)}^{n}Trace\left[A_{ki}\cdot{}^{k}I_{psk}\cdot A_{kj}^T\right]; \quad (49) $$

$$ \underset{(4\times 4)}{A_{ki(j)}}=\left[\begin{array}{ccc|c} & A_{ki(j)}(R) & & A_{ki(j)}(\overline{p}) \\ \hline 0 & 0 & 0 & 0 \end{array}\right], $$

$$\left\{\begin{array}{c} A_{ki}(R)=\dfrac{\partial}{\partial q_i}\left\{{}^{0}_{k}[R]\right\}= \\ =\left\{exp\left\{\sum\limits_{j=0}^{i-1}\left(\overline{k}_j^{(0)}\times\right)\cdot q_j\cdot\Delta_j\right\}\right\}\cdot A^*_{ki} \end{array}\right\}, \quad (50)$$

$$A^*_{ij}=\left(\overline{k}_j^{(0)}\times\right)\cdot\Delta_j\cdot\exp\left\{\sum_{l=j}^{i}\left(\overline{k}_l^{(0)}\times\right)\cdot q_l\cdot\Delta_l\right\}\cdot R^{(0)}_{i0};$$

$$\left\{\begin{array}{c} \underset{(3\times 1)}{A_{ki}(\overline{p})}=\dfrac{\partial\,\overline{p}_k}{\partial q_i}= \\ =\left\{exp\left[\sum\limits_{j=0}^{i-1}\left(\overline{k}_j^{(0)}\times\right)\cdot q_j\cdot\Delta_j\right]\right\}\cdot X_i+ \\ +exp\left[\sum\limits_{l=i}^{k}\left(\overline{k}_l^{(0)}\times\right)\cdot q_l\cdot\Delta_l\right]\cdot\overline{p}_k^{(0)}+A^*_{ki}(\overline{p}) \end{array}\right\}, \quad (51)$$

$$while\;\; X_i=\left(\overline{p}_i^{(0)}\times\right)\cdot\overline{k}_i^{(0)}\cdot\Delta_i+\left(1-\Delta_i\right)\cdot\overline{k}_i^{(0)},$$

$$\left\{\begin{array}{l} and\; A^*_{ki}(\overline{p})=\Delta_i\cdot exp\left[\sum\limits_{j=0}^{i-1}\left(\overline{k}_j^{(0)}\times\right)\cdot q_j\cdot\Delta_j\right]\cdot A^{**}_{ki}(\overline{p}) \\ A^{**}_{ki}(\overline{p})=\sum\limits_{l=i}^{k}\left\{exp\left[\sum\limits_{m=i-1}^{l-1}\left(\overline{k}_m^{(0)}\times\right)\cdot q_m\cdot\Delta_m\cdot\delta_m\right]\right\}\cdot\overline{b}_l \end{array}\right\};$$

$$\underset{(4\times 4)}{{}^{k}I_{psk}}=\sum_{j=1}^{p_i}\sigma_j\cdot{}^{k}I_{psj}=\left[\begin{array}{c|c} {}^{k}I_{pk} & M_k\cdot{}^{k}\overline{r}_{C_k} \\ \hline M_k\cdot{}^{k}\overline{r}^{T}_{C_k} & M_k \end{array}\right]; \quad (52)$$

where (52) is known as pseudoinertial tensor.

Considering papers [6] – [14], in following *the acceleration energies of order* $(p\geq 1)$ will be defined. The starting equation shows as follows:

$$\overset{(k-1)}{E_A^{(p)}}\left[\overline{\theta}(t);\dot{\overline{\theta}}(t);\cdots;\overset{(p+k)}{\overline{\theta}}(t)\right]= \quad (53)$$

$$\left\{\begin{array}{c} =\dfrac{1}{2}\cdot\sum\limits_{i=1}^{n}Trace\left\{\overset{(p+k)}{{}^{0}_{i}[R]}\cdot\left[\int{}^{i}\overline{r}^{*}_{i}\cdot{}^{i}\overline{r}^{*T}_{i}\cdot dm+\right.\right. \\ \left.\left.+{}^{i}\overline{r}_{C_i}\cdot{}^{i}\overline{r}^{T}_{C_i}\cdot\int dm\right]\cdot\overset{(p+k)}{{}^{0}_{i}[R]^{T}}\right\}+ \\ +\dfrac{1}{2}\cdot\sum\limits_{i=1}^{n}Trace\left\{\dfrac{d^{k-1}}{dt^{k-1}}\left[\overset{(p+1)}{\overline{p}_i}\cdot\overset{(p+1)}{\overline{p}_i^{T}}\right]\right\}\cdot\int dm= \\ \dfrac{1}{2}\cdot\sum\limits_{i=1}^{n}Trace\dfrac{d^{k-1}}{dt^{k-1}}\left\{\overset{(p+1)}{{}^{0}_{i}[R]}\cdot\left[{}^{i}I^{*}_{pi}+M_i\cdot{}^{i}\overline{r}_{C_i}\cdot{}^{i}\overline{r}^{T}_{C_i}\right]\cdot\overset{(p+1)}{{}^{0}_{i}[R]^{T}}\right\} \\ +\dfrac{1}{2}\cdot\sum\limits_{i=1}^{n}Trace\left\{\dfrac{d^{k-1}}{dt^{k-1}}\left[\overset{(p+1)}{\overline{p}_i}\cdot\overset{(p+1)}{\overline{p}_i^{T}}\right]\right\}\cdot M_i \end{array}\right\}$$

$$\left\{\begin{array}{c} where\; p\geq 1,\;\; k\geq 1,\; \{p;\,k\}=\{1;2;3;4;5;.....\}, \\ and\;\; \overset{(0)}{E_A^{(p)}}\left[\overline{\theta}(t);\dot{\overline{\theta}}(t);\cdots;\overset{(p+k)}{\overline{\theta}}(t)\right]= \\ =E_A^{(p)}\left[\overline{\theta}(t);\dot{\overline{\theta}}(t);\cdots;\overset{(p+1)}{\overline{\theta}}(t)\right] \end{array}\right\}. \quad (54)$$

The expression (53) includes the inertia tensor planar and centrifugal, relative to the frame $\left\{i^*\right\}$:

$${}^{i}I^{*}_{pi}=\int{}^{i}\overline{r}^{*}_{i}\cdot{}^{i}\overline{r}^{*T}_{i}\cdot dm=\begin{bmatrix} {}^{i}I^{*}_{xx} & {}^{i}I^{*}_{xy} & {}^{i}I^{*}_{xz} \\ {}^{i}I^{*}_{yx} & {}^{i}I^{*}_{yy} & {}^{i}I^{*}_{yz} \\ {}^{i}I^{*}_{zx} & {}^{i}I^{*}_{zy} & {}^{i}I^{*}_{zz} \end{bmatrix}. \quad (55)$$

According to the scientific literature [1], [2], [6]-[14], [19] and [20] in 1879, Gibbs defines the differential equations of motion. In 1899, Paul Appell performs a detailed study on these equations. As a result, Gibbs-Appell equations were deduced. They are applied for holonomic and nonholonomic systems, where the kinetic energy was substituted through the acceleration energy or “kinetic energy of acceleration”, also known as Appell’s function. Unlike the studies, above mentioned, in the papers [6] – [14] the author was established the acceleration energy in a generalized form, corresponding to a MBS founded in the general motion. This was named acceleration energy of first order, according to:

$$\left\{\begin{array}{c} E_A^{(1)}\left[\overline{\theta}(t);\dot{\overline{\theta}}(t);\ddot{\overline{\theta}}(t)\right]= \\ =(-1)^{\Delta_M}\cdot\dfrac{1-\Delta_M}{1+3\cdot\Delta_M}\sum\limits_{i=1}^{n}\left[\dfrac{1}{2}\cdot M_i\cdot{}^{(i)}\dot{\overline{v}}^{T}_{C_i}\cdot{}^{(i)}\dot{\overline{v}}_{C_i}\right] \\ +\Delta_M^2\cdot\sum\limits_{l=1}^{n}\dfrac{1}{2}\cdot{}^{(i)}\dot{\overline{\omega}}^{T}_{i}\cdot{}^{(i)}I^{*}_{i}\cdot{}^{(i)}\dot{\overline{\omega}}_{i}+ \\ +\Delta_M^2\cdot\sum\limits_{i=1}^{n}\left[{}^{(i)}\dot{\overline{\omega}}^{T}_{i}\cdot\left({}^{(i)}\overline{\omega}_{i}\times{}^{(i)}I^{*}_{i}\cdot{}^{(i)}\overline{\omega}_{i}\right)\right]+ \\ +E_A^{(1)}\left[\overline{\theta}(t);\dot{\overline{\theta}}^{4}(t)\right] \end{array}\right\} \quad (56)$$

$$\left\{\begin{array}{c} E_A^{(1)}\left[\overline{\theta}(t);\dot{\overline{\theta}}^{4}(t)\right]= \\ =\Delta_M^2\cdot\sum\limits_{i=1}^{n}\left\{\dfrac{1}{2}\cdot{}^{(i)}\overline{\omega}^{T}_{i}\cdot\left[{}^{(i)}\overline{\omega}^{T}_{i}\cdot Trace\left({}^{(i)}I^{*}_{pi}\right)\cdot{}^{(i)}\overline{\omega}_{i}\right.\right. \\ \left.\left.-{}^{(i)}\overline{\omega}^{T}_{i}\cdot{}^{(i)}I^{*}_{pi}\cdot{}^{(i)}\overline{\omega}_{i}\right]\cdot{}^{(i)}\overline{\omega}_{i}\right\} \end{array}\right\} \quad (57)$$

Considering the notions from previous section the two components (translational and rotation) of acceleration energy of first order show as:

$$\left\{\begin{array}{c} \left\{E_A^{(1)TR}=\dfrac{1}{2}\cdot\sum\limits_{i=1}^{n}M_i\cdot a^2_{C_i}\right\}= \\ =\dfrac{1}{2}\cdot\sum\limits_{i=1}^{n}M_i\cdot\left\{\sum\limits_{j=1}^{k^*=n}\sum\limits_{p=1}^{k^*=n}\left[\dfrac{\partial^2\,\overset{(m)}{\overline{r}_{C_i}}}{\partial\overset{(m)}{q_j}\cdot\partial\overset{(m)}{q_p}}\cdot\ddot{q}_j\cdot\ddot{q}_p+\right.\right. \\ +\dfrac{1}{(m+1)^2}\cdot\dfrac{\partial^2\,\overset{(m+1)}{\overline{r}_{C_i}}}{\partial\overset{(m)}{q_j}\cdot\partial\overset{(m)}{q_p}}\cdot\dot{q}_j\cdot\dot{q}_p+ \\ \left.\left.+\dfrac{1}{m+1}\cdot\dfrac{\partial\,\overset{(m)}{\overline{r}_{C_i}}}{\partial\overset{(m)}{q_j}}\cdot\dfrac{\partial\,\overset{(m+1)}{\overline{r}_{C_i}}}{\partial\overset{(m)}{q_p}}\cdot\ddot{q}_j\cdot\dot{q}_p\right]\right\} \end{array}\right\}; \quad (58)$$

$$\left\{E_A^{(1)ROT\varepsilon} = \frac{1}{2}\cdot\sum_{i=1}^{n}\bar{\varepsilon}_i^T\cdot I_i^*\cdot\bar{\varepsilon}_i\right\} = \tag{59}$$

$$\left\{\frac{1}{2}\cdot\sum_{i=1}^{n}\sum_{j=1}^{k^*=n}\left[\frac{\partial\overset{(m)}{\bar{\psi}_i}}{\partial\overset{(m)}{q_j}}\cdot\Delta_j\cdot\ddot{q}_j+\frac{1}{m+1}\cdot\frac{\partial\overset{(m+1)}{\bar{\psi}_i}}{\partial\overset{(m)}{q_j}}\cdot\Delta_j\cdot\dot{q}_j\right]^T\cdot I_i^*\cdot\bar{\varepsilon}_i\right\},$$

$$I_i^*\cdot\bar{\varepsilon} = I_i^*\cdot\sum_{j=1}^{k^*}\left[\frac{\partial\overset{(m)}{\bar{\psi}_i}}{\partial\overset{(m)}{q_j}}\cdot\Delta_j\cdot\ddot{q}_j+\frac{1}{m+1}\cdot\frac{\partial\overset{(m+1)}{\bar{\psi}_i}}{\partial\overset{(m)}{q_j}}\cdot\Delta_j\cdot\dot{q}_j\right];$$

$$\left\{E_A^{(1)ROT\omega\varepsilon} = \sum_{i=1}^{n}\bar{\varepsilon}_i^T\cdot\left(\bar{\omega}_i\times I_i^*\cdot\bar{\omega}_i\right)\right\} = \tag{60}$$

$$\left\{\sum_{i=1}^{n}\sum_{j=1}^{k^*=n}\left[\frac{\partial\overset{(m)}{\bar{\psi}_i}}{\partial\overset{(m)}{q_j}}\cdot\Delta_j\cdot\ddot{q}_j+\frac{1}{m+1}\cdot\frac{\partial\overset{(m+1)}{\bar{\psi}_i}}{\partial\overset{(m)}{q_j}}\cdot\Delta_j\cdot\dot{q}_j\right]^T\cdot E_A^{(1)ROT\omega\omega}\right\};$$

$$E_A^{(1)ROT\omega\omega} = \left(\bar{\omega}_i\times I_i^*\cdot\bar{\omega}_i\right) = \tag{61}$$

$$\left\{\sum_{i=1}^{n}\sum_{j=1}^{k^*=n}\sum_{p=1}^{k^*=n}\left[\frac{\partial\overset{(m)}{\bar{\psi}_i}}{\partial\overset{(m)}{q_j}}\right]\times\left[I_i^*\cdot\frac{\partial\overset{(m)}{\bar{\psi}_i}}{\partial\overset{(m)}{q_p}}\right]\cdot\Delta_j\cdot\Delta_p\cdot\dot{q}_j\cdot\dot{q}_p\right\}.$$

According to [6] – [14], besides of the explicit form (56), for acceleration energy of first order has been demonstrated the matrix form, that is:

$$E_A^{(1)}\left[\bar{\theta}(t);\dot{\bar{\theta}}(t);\ddot{\bar{\theta}}(t)\right] = \tag{62}$$

$$\left\{\begin{array}{c} =\frac{1}{2}\cdot\left\{\ddot{\bar{\theta}}^T(t)\cdot M[\bar{\theta}(t)]\cdot\ddot{\bar{\theta}}(t)+\right. \\ \left.+\ddot{\bar{\theta}}^T(t)\cdot V\left[\bar{\theta}(t);\dot{\bar{\theta}}^2(t)\right]+E_A^{(1)}\left[\bar{\theta}(t);\dot{\bar{\theta}}^4(t)\right]\right\}\end{array}\right\}$$

$$E_A^{(1)}\left[\bar{\theta}(t);\dot{\bar{\theta}}^4(t)\right] = \left[\dot{\bar{\theta}}^T(t)\cdot D\left[\bar{\theta}(t);\dot{\bar{\theta}}^2(t)\right]\cdot\dot{\bar{\theta}}(t)\right].$$

Besides (48), in (62) is also included the matrix of centrifugal and Coriolis terms, according to:

$$V\left[\bar{\theta}(t);\dot{\bar{\theta}}^2(t)\right] = \underset{(n\times 1)}{Matrix}\left[V_i\left[\bar{\theta}(t);\dot{\bar{\theta}}^2(t)\right];\ i=1\to n\right],$$

$$V_i\left[\bar{\theta}(t);\dot{\bar{\theta}}^2(t)\right] = \dot{\bar{\theta}}^T\cdot\left[V_{ijm}\quad\begin{matrix}j=1\to n\\ m=1\to n\end{matrix}\right]\cdot\dot{\bar{\theta}}, \tag{63}$$

where $V_{ijm} = \sum_{k=\max(j;m)}^{n} Trace\left[A_{ki}\cdot{}^kI_{psk}\cdot A_{kjm}^T\right]$, (64)

and $A_{kjm}(R) = \left[\begin{array}{ccc|c} & A_{kjm}(R) & & A_{kjm}(\bar{p}) \\ \hline 0 & 0 & 0 & 0\end{array}\right]$, (65)

$$A_{kjm}(R) = \frac{\partial^2}{\partial q_j\cdot\partial q_m}\left\{{}_k^0[R]\right\} = \frac{\partial}{\partial q_m}\left[A_{kj}(R)\right],$$

$$A_{kjm}(\bar{p}) = \frac{\partial^2\bar{p}_k}{\partial q_j\cdot\partial q_m} = \frac{\partial}{\partial q_m}\left[A_{kj}(\bar{p})\right].$$

According to the author researches, [6] – [14], the sudden motion of MBS, the transient motion phases, as well as mechanical systems subjected to the action of a system of external forces, with a time variation law, are characterized by linear and angular accelerations of higher order (see previous section of paper). So, the acceleration energy of second order has been also developed. First of all, its explicit form is below shown as:

$$E_A^{(2)}\left[\bar{\theta}(t);\dot{\bar{\theta}}(t);\ddot{\bar{\theta}}(t);\dddot{\bar{\theta}}(t)\right] = \tag{66}$$

$$\left\{\begin{array}{c} =(-1)^{\Delta_M}\cdot\frac{1-\Delta_M}{1+3\cdot\Delta_M}\cdot\sum_{i=1}^{n}\left\{\frac{1}{2}\cdot M_i\cdot{}^i\dddot{\bar{v}}_{C_i}^T\cdot{}^i\dddot{\bar{v}}_{C_i}\right\}+ \\ +\Delta_M^2\cdot\sum_{i=1}^{n}\left\{\frac{1}{2}\cdot{}^i\ddot{\bar{\omega}}_i^T\cdot{}^iI_i^*\cdot{}^i\ddot{\bar{\omega}}_i+2\cdot{}^i\bar{\omega}_i^T\cdot\left({}^i\ddot{\bar{\omega}}_i\times{}^iI_{pi}^*\cdot{}^i\dot{\bar{\omega}}_i\right)+\right. \\ \left.+{}^i\dot{\bar{\omega}}_i^T\cdot\left({}^i\ddot{\bar{\omega}}_i\times{}^iI_{pi}^*\cdot{}^i\bar{\omega}_i\right)-{}^i\bar{\omega}_i^T\cdot\left({}^i\ddot{\bar{\omega}}_i^T\cdot{}^iI_i^*\cdot{}^i\bar{\omega}_i\right)\cdot{}^i\bar{\omega}_i\right\}+ \\ +E_A^{(2)}\left[\bar{\theta}(t);\dot{\bar{\theta}}(t);\ddot{\bar{\theta}}^2(t)\right]\end{array}\right\}$$

where $E_A^{(2)}\left[\bar{\theta}(t);\dot{\bar{\theta}}(t);\ddot{\bar{\theta}}^2(t)\right] =$ (67)

$$\left\{\begin{array}{c} =\Delta_M^2\cdot\sum_{i=1}^{n}\left\{2\cdot{}^i\dot{\bar{\omega}}_i^T\cdot\left({}^i\bar{\omega}_i^T\cdot{}^iI_i^*\cdot{}^i\dot{\bar{\omega}}_i\right)\cdot{}^i\bar{\omega}_i+\right. \\ +2\cdot{}^i\bar{\omega}_i^T\cdot\left[{}^i\dot{\bar{\omega}}_i^T\cdot{}^iI_{pi}^*\cdot{}^i\dot{\bar{\omega}}_i\right]\cdot{}^i\bar{\omega}_i- \\ -5\cdot\left({}^i\dot{\bar{\omega}}_i^T\cdot{}^iI_{pi}^*\right)\cdot\left({}^i\dot{\bar{\omega}}_i^T\cdot{}^i\bar{\omega}_i\right)\cdot{}^i\bar{\omega}_i+ \\ +\frac{5}{2}\cdot\left({}^i\dot{\bar{\omega}}_i^T\cdot{}^i\bar{\omega}_i\right)\cdot Trace\left({}^iI_{pi}^*\right)\cdot\left({}^i\dot{\bar{\omega}}_i^T\cdot{}^i\bar{\omega}_i\right) \\ \left.+\frac{1}{2}\cdot{}^i\dot{\bar{\omega}}_i^T\cdot\left[{}^i\bar{\omega}_i^T\cdot{}^iI_{pi}^*\cdot{}^i\bar{\omega}_i\right]\cdot{}^i\dot{\bar{\omega}}_i\right\}+ \\ +E_A^{(2)}\left[\bar{\theta}(t);\dot{\bar{\theta}}^6(t);\ddot{\bar{\theta}}(t)\right]\end{array}\right\},$$

and $E_A^{(2)}\left[\bar{\theta}(t);\dot{\bar{\theta}}^6(t);\ddot{\bar{\theta}}(t)\right] =$ (68)

$$\left\{\begin{array}{c} =\Delta_M^2\cdot\left\{\sum_{i=1}^{n}{}^i\bar{\omega}_i^T\cdot\left[{}^i\bar{\omega}_i^T\cdot\left({}^i\dot{\bar{\omega}}_i\times{}^iI_{pi}^*\cdot{}^i\bar{\omega}_i\right)\right]\cdot{}^i\bar{\omega}_i+\right. \\ \left.\frac{1}{2}\cdot{}^i\bar{\omega}_i^T\cdot\left[{}^i\bar{\omega}_i^T\cdot\left({}^i\bar{\omega}_i^T\cdot{}^iI_i^*\cdot{}^i\bar{\omega}_i\right)\cdot{}^i\bar{\omega}_i\right]\cdot{}^i\bar{\omega}_i\right\}\end{array}\right\}.$$

This can be also considered a generalization of König's theorem of second order. According to [8] – [14], the matrix expression of acceleration energy of second order is also below defined:

$$E_A^{(2)}\left[\bar{\theta}(t);\dot{\bar{\theta}}(t);\ddot{\bar{\theta}}(t);\dddot{\bar{\theta}}(t)\right] =$$

$$\left\{\begin{array}{c} =\frac{1}{2}\cdot\dddot{\bar{\theta}}^T(t)\cdot M[\bar{\theta}(t)]\cdot\dddot{\bar{\theta}}(t)+ \\ +3\cdot\dddot{\bar{\theta}}^T(t)\cdot V\left[\bar{\theta}(t);\dot{\bar{\theta}}(t);\ddot{\bar{\theta}}(t)\right]+ \\ +\dddot{\bar{\theta}}^T(t)\cdot H\left[\bar{\theta}(t);\dot{\bar{\theta}}^2(t)\right]\cdot\dot{\bar{\theta}}(t)+ \\ +E_A^{(2)}\left[\bar{\theta}(t);\dot{\bar{\theta}}(t);\ddot{\bar{\theta}}^2(t)\right]\end{array}\right\}, \tag{69}$$

$$E_A^{(2)}\left[\bar\theta(t);\dot{\bar\theta}(t);\ddot{\bar\theta}^2(t)\right]=$$

$$\left\{\begin{array}{c}=\dfrac{9}{2}\cdot\ddot{\bar\theta}^T(t)\cdot D\left[\bar\theta(t);\dot{\bar\theta}(t);\ddot{\bar\theta}(t)\right]\cdot\dot{\bar\theta}(t)+\\ +3\cdot\ddot{\bar\theta}^T(t)\cdot K\left[\bar\theta(t);\dot{\bar\theta}^4(t)\right]+\\ +\dfrac{1}{2}\cdot\dot{\bar\theta}^T(t)\cdot N\left[\bar\theta(t);\dot{\bar\theta}^4(t)\right]\cdot\dot{\bar\theta}(t)\end{array}\right\}. \quad (70)$$

Besides (48) and (63), within (69) and (70) the others dynamics matrices are also included as:

$$H\left[\bar\theta(t);\dot{\bar\theta}^2(t)\right]=\underset{(n\times n)}{Matrix}\left\{H_{ij}\quad\begin{array}{c}i=1\to n\\ j=1\to n\end{array}\right\}, \quad (71)$$

$$H_{ij}\left[\bar\theta(t);\dot{\bar\theta}^2(t)\right]=\dot{\bar\theta}^T\cdot\left\{H_{ijlm}\quad\begin{array}{c}l=1\to n\\ m=1\to n\end{array}\right\}\cdot\dot{\bar\theta},$$

where $H_{ijlm}=\sum_{k=\max(i;j;l;m)}^{n}Tr\left[A_{ki}\cdot{}^kI_{psk}\cdot A_{kjlm}^T\right]$, (72)

$$A_{kjlm}(R)=\left[\begin{array}{ccc|c}&A_{kjlm}(R)&&A_{kjlm}(\bar p)\\ \hline 0&0&0&0\end{array}\right], \quad (73)$$

$$A_{kjlm}(R)=\frac{\partial^3}{\partial q_j\cdot\partial q_m\cdot\partial q_l}\left\{{}_k^0[R]\right\}=\frac{\partial}{\partial q_l}\left[A_{kjm}(R)\right],$$

$$A_{kjlm}(\bar p)=\frac{\partial}{\partial q_l}\left[A_{kjm}(\bar p)\right]=\frac{\partial^2}{\partial q_m\cdot\partial q_l}\left[A_{kj}(\bar p)\right];$$

$$D\left[\bar\theta(t);\dot{\bar\theta}(t);\ddot{\bar\theta}(t)\right]=\underset{(n\times n)}{Matrix}\left\{D_{ij}\quad\begin{array}{c}i=1\to n\\ j=1\to n\end{array}\right\}, \quad (74)$$

$$D_{ij}\left[\bar\theta(t);\dot{\bar\theta}(t);\ddot{\bar\theta}(t)\right]=\ddot{\bar\theta}^T\cdot\left[D_{ijlm}\quad\begin{array}{c}l=1\to n\\ m=1\to n\end{array}\right]\cdot\dot{\bar\theta},$$

where $D_{ijlm}=\sum_{k=\max(i;j;l;m)}^{n}Trace\left[A_{kij}\cdot{}^kI_{psk}\cdot A_{klm}^T\right]$. (75)

The study of advanced dynamics is extended on acceleration energy of third order. According to [10] – [14], author proposes explicit equation of the acceleration energy of third order thus:

$$E_A^{(3)}\left[\bar\theta(t);\dot{\bar\theta}(t);\ddot{\bar\theta}(t);\dddot{\bar\theta}(t);\ddddot{\bar\theta}(t)\right]= \quad (76)$$

$$\left\{\begin{array}{c}=(-1)^{\Delta_M}\cdot\dfrac{1-\Delta_M}{1+3\cdot\Delta_M}\cdot\sum_{i=1}^{n}\left\{\dfrac{1}{2}\cdot M_i\cdot{}^i\dddot{\bar v}_{C_i}^T\cdot{}^i\dddot{\bar v}_{C_i}\right\}+\\ +\Delta_M^2\cdot\sum_{i=1}^{n}\left\{\dfrac{1}{2}\cdot{}^i\dddot{\bar\omega}_i^T\cdot{}^iI_i^*\cdot{}^i\dddot{\bar\omega}_i+3\cdot\bar\omega_i^T\cdot\left(\dddot{\bar\omega}_i\times I_{pi}^*\cdot\ddot{\bar\omega}_i\right)+\right.\\ +3\cdot\dot{\bar\omega}_i^T\cdot\left(\dddot{\bar\omega}_i\times I_{pi}^*\cdot\dot{\bar\omega}_i\right)+3\cdot\ddot{\bar\omega}_i^T\cdot\left(\dddot{\bar\omega}_i\times I_{pi}^*\cdot\bar\omega_i\right)+\\ +2\cdot\left(\bar\omega_i\times\dddot{\bar\omega}_i\right)^T\cdot I_{pi}^*\cdot\left(\dot{\bar\omega}_i\times\bar\omega_i\right)-\\ -5\cdot\dot{\bar\omega}_i^T\cdot\left[\bar\omega_i^T\cdot I_i^*\cdot\dddot{\bar\omega}_i\right]\cdot\bar\omega_i-\bar\omega_i^T\cdot\left[\dot{\bar\omega}_i^T\cdot I_i^*\cdot\dddot{\bar\omega}_i\right]\cdot\bar\omega_i+\\ \left.+\bar\omega_i^T\cdot\left[\bar\omega_i^T\cdot I_{pi}^*\cdot\left(\dddot{\bar\omega}_i\times\bar\omega_i\right)\right]\cdot\bar\omega_i\right\}+\\ +E_A^{(3)}\left[\bar\theta(t);\dot{\bar\theta}(t);\ddot{\bar\theta}(t);\dddot{\bar\theta}^2(t)\right].\end{array}\right.$$

Similarly with the first two types of acceleration energies, it can also observe an extension of the generalization of König's theorem regarding the acceleration energy of third order. According to [10] – [14], matrix expression of the acceleration energy of third order is defined as below follows:

$$E_A^{(3)}\left[\bar\theta(t);\dot{\bar\theta}(t);\ddot{\bar\theta}(t);\dddot{\bar\theta}(t);\ddddot{\bar\theta}(t)\right]= \quad (77)$$

$$\left\{\begin{array}{c}=\dfrac{1}{2}\cdot\ddddot{\bar\theta}^T(t)\cdot M[\bar\theta(t)]\cdot\ddddot{\bar\theta}(t)+\\ +4\cdot\ddddot{\bar\theta}^T(t)\cdot V\left[\bar\theta(t);\dot{\bar\theta}(t);\dddot{\bar\theta}(t)\right]+\\ +3\cdot\ddddot{\bar\theta}^T(t)\cdot V^*\left[\bar\theta(t);\ddot{\bar\theta}^2(t)\right]+\\ +6\cdot\ddddot{\bar\theta}^T(t)\cdot H^*\left[\bar\theta(t);\dot{\bar\theta}^2(t)\right]\cdot\ddot{\bar\theta}(t)+\\ +\ddddot{\bar\theta}^T(t)\cdot K^*\left[\bar\theta(t);\dot{\bar\theta}^4(t)\right]+\\ +E_A^{(3)}\left[\bar\theta(t);\dot{\bar\theta}(t);\ddot{\bar\theta}(t);\dddot{\bar\theta}^2(t)\right]\end{array}\right\}.$$

The dynamics matrices of third order, included in the equations (70) and (77), are defined as:

$$N\left[\bar\theta(t);\dot{\bar\theta}^4(t)\right]=\underset{(n\times n)}{Matrix}\left\{N_{ij}\quad\begin{array}{c}i=1\to n\\ j=1\to n\end{array}\right\}, \quad (78)$$

$$N_{ij}\left[\bar\theta(t);\dot{\bar\theta}^4(t)\right]=\dot{\bar\theta}^T\cdot\left\{\begin{array}{c}\dot{\bar\theta}^T\cdot\left[N_{ijlmpr}\quad\begin{array}{c}p=1\to n\\ r=1\to n\end{array}\right]\cdot\dot{\bar\theta}\\ l=1\to n\quad m=1\to n\end{array}\right\}\cdot\dot{\bar\theta}$$

where $N_{ijlmpr}=\sum_{k=\max(i;j;l;m;p;r)}^{n}Tr\left[A_{kijl}\cdot{}^kI_{psk}\cdot A_{kmpr}^T\right]$;(79)

$$K^*\left[\bar\theta(t);\dot{\bar\theta}^4(t)\right]=\underset{(n\times 1)}{Matrix}\left\{K_i^*,\ i=1\to n\right\}^T, \quad (80)$$

$$K_i^*\left[\bar\theta(t);\dot{\bar\theta}^4(t)\right]=\dot{\bar\theta}^T\cdot\left\{\begin{array}{c}\dot{\bar\theta}^T\cdot\left[K_{ijlmp}\quad\begin{array}{c}m=1\to n\\ p=1\to n\end{array}\right]\cdot\dot{\bar\theta}\\ j=1\to n\ ;\ l=1\to n\end{array}\right\}\cdot\dot{\bar\theta}$$

where $K_{ijlmp}=\sum_{k=\max(i;j;l;m;p)}^{n}Tr\left[A_{ki}\cdot{}^kI_{psk}\cdot A_{kjlmp}^T\right]$ (81)

$$A_{kjlmp}(R)=\left[\begin{array}{ccc|c}&A_{kjlmp}(R)&&A_{kjlmp}(\bar p)\\ \hline 0&0&0&0\end{array}\right], \quad (82)$$

$$A_{kjlmp}(R)=\frac{\partial^4\ \left\{{}_k^0[R]\right\}}{\partial q_j\cdot\partial q_m\cdot\partial q_l\cdot\partial q_p}=\frac{\partial}{\partial q_p}\left[A_{kjlm}(R)\right],$$

$$A_{kjlmp}(\bar p)=\frac{\partial}{\partial q_p}\left[A_{kjlm}(\bar p)\right]=\frac{\partial^2}{\partial q_l\cdot\partial q_p}\left[A_{kjm}(\bar p)\right].$$

Considering (50) and (51), it observes that the components of the differential matrices (65), (73) and (82) included in (64), (72), (75), (79) and (81) have been also determined by means of the matrix exponentials, developed within of the previous section, according to [10] – [14].

On the basis of the starting equation (53) and conditions (54), the author researches have been extended [14] on acceleration energy of fourth order. The starting equation is shown below:

$$E_A^{(4)}\left[\bar{\theta}(t);\dot{\bar{\theta}}(t);\ddot{\bar{\theta}}(t);\cdots;\overset{(5)}{\bar{\theta}}(t)\right]= \tag{83}$$

$$\left\{\begin{array}{c}\dfrac{1}{2}\cdot\displaystyle\sum_{i=1}^{n} Trace\left\{{}_{i}^{0}\overset{(5)}{[R]}\cdot\left[{}^{i}I_{pi}^{*}+M_i\cdot{}^{i}\bar{r}_{C_i}\cdot{}^{i}\bar{r}_{C_i}^{T}\right]\cdot{}_{i}^{0}\overset{(5)}{[R]}^{T}\right\}+\\ +\dfrac{1}{2}\cdot\displaystyle\sum_{i=1}^{n} Trace\left[\overset{(5)}{\bar{p}_i}\cdot\overset{(5)}{\bar{p}_i^{T}}\right]\cdot M_i\end{array}\right\}.$$

The absolute time derivatives of fifth order for rotation matrix and position vector, included in the above equation, are determined with (11) and (12), where: $k=5, and\ m\geq 6$. But, the same result is obtained with the differential equation:

$$\left\{\begin{array}{c}\left\{{}_{i}^{0}\overset{(5)}{[R]}^{(T)};\overset{(5)}{\bar{p}_i}^{(T)}\right\}=\dfrac{d}{dt}\left\{\displaystyle\sum_{j=1}^{i}A_{ij}^{(T)}\left[(R);(\bar{p})\right]\cdot\dddot{q}_j+\right.\\ +4\cdot\displaystyle\sum_{j=1}^{i}\sum_{k=1}^{i}A_{ijk}^{(T)}\left[(R);(\bar{p})\right]\dddot{q}_j\cdot\dot{q}_k+\\ \left.+3\cdot\displaystyle\sum_{j=1}^{i}\sum_{k=1}^{i}A_{ijk}^{(T)}\left[(R);(\bar{p})\right]\ddot{q}_j\cdot\ddot{q}_k\right\}+\\ \dfrac{d}{dt}\left\{6\cdot\displaystyle\sum_{j=1}^{i}\sum_{k=1}^{i}\sum_{m=1}^{i}A_{ijkm}^{(T)}\left[(R);(\bar{p})\right]\cdot\ddot{q}_j\cdot\dot{q}_k\cdot\dot{q}_m+\right.\\ \left.+\displaystyle\sum_{j=1}^{i}\sum_{k=1}^{i}\sum_{m=1}^{i}\sum_{p=1}^{i}A_{ijkmp}^{(T)}\left[(R);(\bar{p})\right]\cdot\dot{q}_j\cdot\dot{q}_k\cdot\dot{q}_m\cdot\dot{q}_p\right\}\end{array}\right\}. \tag{84}$$

Equation (84) is exclusiveness considered, first of all, (R) rotation, then $(\bar{p})$ position function.

The differential matrices of various orders, included in (84), are determined in accordance with (65), (73) and (82), by means of the same matrix exponentials and their time derivatives.

4. ADVANCED DYNAMICS EQUATIONS

When the mechanical systems (MBS) are dominated by sudden motions, as well as by the transitory motions, on the basis of the author's researches, in the previous section of the paper, it was demonstrated existing of acceleration energies of higher order. They are compulsory included in the dynamics equations of higher order, regarding the mechanical motion of MBS. First of all, Tsenov – Mangeron formulation on Lagrange's equations of second kind are studied. Following the application of time derivatives of higher order (m) and (k), the motion differential equations, become new differential expressions:

$$\left\{\begin{array}{c}\dfrac{1}{m}\cdot\dfrac{d^{k-1}}{dt^{k-1}}\left[\dfrac{\partial \overset{(m)}{E_C}}{\partial \overset{(m)}{q_j}}-(m+1)\cdot\dfrac{\partial E_C}{\partial q_j}\right]=\\ =\overset{(k-1)}{Q_{i\ddot{o}}^{j}}\left[\bar{\theta}(t);\dot{\bar{\theta}}(t);\cdots;\overset{(m)}{\bar{\theta}}(t)\right]\end{array}\right\}. \tag{85}$$

But, considering the acceleration energy of first order (56) – (62) and time derivatives of higher order (m) and (k) applied on the generalization of Gibbs – Appell, these equations are changed:

$$\dfrac{d^{k-1}}{dt^{k-1}}\left\{\dfrac{\partial \overset{(m-2)}{E_A^{(1)}}}{\partial \overset{(m)}{q_j}}\right\}=\overset{(k-1)}{Q_{i\ddot{o}}^{j}}\left[\bar{\theta}(t);\dot{\bar{\theta}}(t);\cdots;\overset{(m)}{\bar{\theta}}(t)\right]. \tag{86}$$

Author has proposed [13] – [14], the generalized differential equations of higher order in the case of the mechanical systems (MBS), dynamically characterized by sudden and transitory motions:

$$\left\{\begin{array}{c}\dfrac{(k-1)!\cdot m!}{(m+k-1)!}\cdot\dfrac{\partial}{\partial \overset{(m)}{q_j}}\left\{\left(\displaystyle\sum_{p=1}^{k}\Delta_p\right)\cdot\overset{(m+k)-(2\cdot p+1)}{E_A^{(p)}}\right\}\\ =\overset{(k-1)}{Q_{i\ddot{o}}^{j}}\left[\bar{\theta}(t);\dot{\bar{\theta}}(t);\cdots;\overset{(m)}{\bar{\theta}}(t)\right]\end{array}\right\}; \tag{87}$$

$$\left\{\begin{array}{c}where\quad E_A^{(p)}=E_A^{(p)}\left[\bar{\theta}(t);\dot{\bar{\theta}}(t);\cdots;\overset{(p+1)}{\bar{\theta}}(t)\right]\\ and\ \displaystyle\sum_{p=1}^{k}\Delta_p=\sum_{p=1}^{k}\left[\dfrac{p\cdot(p+1)}{2}-\delta_p\right]\end{array}\right\}; \tag{88}$$

The necessary conditions in (87) are following:

$$\left\{\begin{array}{c}p=1\rightarrow k\ ;\ \ \delta_p=\{\{0;p=1\};\{1;p>1\}\}\\ and\quad k\geq 1;\quad k=\{1;2;3;4;5;.....\}\\ m\geq(k+1);\quad m=\{2;3;4;5;.....\}\end{array}\right\}. \tag{89}$$

Generalized differential equations (87) contain acceleration energies of order $(p=1\rightarrow 4)$, whose expressions of definition, in explicit and matrix form, are detailed presented in previous section. The symbol $Q_{i\ddot{o}}^{i}\ (t)$ from equations (85) – (87) is the generalized inertia force. This is included in the equations of generalized driving forces:

$$\left\{\begin{array}{l} \overset{(k-1)}{Q_m^i}\left[\overline{\theta}(t);\dot{\overline{\theta}}(t);\cdots;\overset{(m)}{\overline{\theta}}(t)\right]= \\ =\overset{(k-1)}{Q_{i\ddot{o}}^i}(t)+\overset{(k-1)}{Q_g^i}(t)+\overset{(k-1)}{Q_{SU}^i}(t)(t) \end{array}\right\}; \qquad (90)$$

where $Q_g^i(t)$ and $Q_{SU}^i(t)$ are the generalized forces due to gravitation and manipulating from MBS.

5. CONCLUSIONS

The currently paper was devoted especially to presentation a few essential reformulations and new formulations concerning some notions from advanced kinematics and dynamics. These are compulsory included in dynamics equations of higher order, corresponding to the current and sudden motions in the case of multibody systems (MBS), for example mechanical robot structure.

So, unlike the classical models the author has presented in first section of paper reformulations and new formulations regarding the parameters of the advanced kinematics. In the view of this, matrix exponentials and the time derivatives of higher order have been applied, concerning the linear and angular accelerations of higher order. In this section, important differential properties have been developed concerning position of the mass center and orientation vector. They are also used for determine the same linear and angular accelerations of higher order above mentioned. According to author researches, the parameters of advanced kinematics have been developed as time functions with the polynomial interpolating functions of higher order, defined in this section. These are required in the advanced notions and equations from analytical dynamics of systems.

The phrase, “advanced notions” founded in the analytical dynamics, is focused in this paper on the motion energies whose central functions are the accelerations of higher order. They are developing in any sudden and transitory motion of the mechanical systems. Leading to Appell’s function, also named “the kinetic energy of the accelerations” the author has been developed in the second section of paper new mathematical formulations on the expressions for acceleration energies of first, second, third and fourth order. For understanding the mechanical significances of the energies of higher order, at beginning the König’s theorem for kinetic energy has been presented in explicit and then matrix form. In this section kinetic energy and acceleration energy of first order have been also defined by means of the differential properties concerning position of the mass center and orientation vector well defined in the first section. The second section was also devoted to expressions of the acceleration energies of first, second, third and fourth order, in explicit and matrix form.

When the mechanical systems (MBS) are dominated by sudden motions, as well as by the transitory motions, on the basis of the author's researches, in the previous section of the paper, it was demonstrated existing of the acceleration energies of higher order. They are compulsory included in the dynamics equations of higher order, regarding the mechanical motion of MBS. So, in the last section of the paper author has proposed the generalized differential equations of higher order, corresponding to dynamical behavior of mechanical systems characterized by the current, sudden and transitory motions.

6. REFERENCES

[1] Appell, P., *Sur une forme générale des equations de la dynamique*, Paris, 1899.

[2] Appell, P., *Traité de mécanique rationnelle*, Paris, 1903.

[3] Fu, K., et al, *Robotics: Control, Sensing, Vision and Intelligence*, McGraw-Hill Book Co., International Edition, 1987

[4] Negrean, I., *Robotics - Kinematic and Dynamic Modelling*, EDP-R.A., Bucureşti, ISBN 973-30-5958-7, 1998.

[5] Negrean, I., Negrean, D. C., *Matrix Exponentials to Robot Kinematics,* 17th International Conference on CAD/CAM, Robotics and Factories of the Future, CARS&FOF 2001, Durban South Africa, 2001, Vol.2, pp. 1250-1257, 32 rel., 4 ref.

[6] Negrean, I., Negrean, D. C., *The Acceleration Energy to Robot Dynamics,* International Conference on Automation, Quality and Testing, Robotics, AQTR 2002, May 23-25, Cluj-Napoca, Tome II, pp. 59-64

[7] Negrean I., *Mecanică avansată în Robotică*, ISBN 978-973-662-420-9, UT Press, Cluj-Napoca, 2008.

[8] Negrean I., *Mecanică, Teorie şi Aplicaţii*, Editura UT PRESS, Cluj – Napoca, ISBN 978-973-662-523-7, 2012.

[9] Negrean, I., *Formulations in Advanced Dynamics of Mechanical Systems*, The 11 th IFTOMM International Symposium on Science of Mechanisms and Machines, Mechanisms and Machine Science 18, DOI:10.1007/978-3-319-01845-4-19 Springer International Publishing Switzerland, pp. 185-195.

[10] Negrean, I., *Energies of Acceleration in Advanced Robotics Dynamics,* Applied Mechanics and Materials, ISSN: 1662-7482, vol 762, pp 67-73 Submitted: 2014-08-05 ©(2015)TransTech Publications Switzerland DOI:10.4028/www.scientific.net/AMM.762.67, Revised:2014-11-16.

[11] Negrean I., *New Formulations on Acceleration Energy in Analytical Dynamics,* Applied Mechanics and Materials, vol. 823 pp 43-48 © (2016) TransTech Publications Switzerland Revised: 2015-09, Doi: 10.4028 /www.scientific.net/AMM.823.43.

[12] Negrean I., *New Formulations on Motion Equations in Analytical Dynamics,* Applied Mechanics and Materials, vol. 823 (2016), pp 49-54 © (2016) TransTech Publications Switzerland Revised: 2015-09-09, DOI: 10.4028/www.scientific.net/AMM.823.49.

[13] Negrean I., *Formulations on the Advanced Notions in analytical Dynamics of Mechanical Systems,* published in IJERM, ISSN: 2349-2058, Volume-03, Issue-04, pp 123-130, 2016.

[14] Negrean I., *Advanced Notions and Differential Principles of Motion in Analytical Dynamics,* Journal of Engineering Sciences and Innovation, Technical Sciences Academy of Romania, Vol. 1/2016, Issue 1, pp. 49-72, ISSN 2537-320X.

[15] Negrean I., *New Formulations in Analytical Dynamics of Systems*, Acta Technica Napocensis, Series: Applied Mathematics, Mechanics, and Engineering, Vol.60, Issue I, March 2017, pp. 49 – 56.

[16] Negrean I., *Mass Distribution in Analytical Dynamics of Systems*, Acta Technica Napocensis, Series: Applied Mathematics, Mechanics, and Engineering, Vol.60, Issue II, June 2017, pp. 175 – 184.

[17] Negrean I., *Generalized Forces in Analytical Dynamics of Systems*, Acta Technica Napocensis, Series: Applied Mathematics, Mechanics, and Engineering, Vol.60, Issue III, September 2017, pp.

[18] Park, F.C., *Computational Aspects of the Product-of-Exponentials Formula for Robot Kinematics*, IEEE Transaction on Automatic Control, Vol. 39, No. 3, 1994

[19] Pars L.A., *A Treatise on Analytical Dynamics*, Heinemann, London, (2007), Volume I, pp. 1-122.

[20] Ursu – Fischer, N., *Elemente de Mecanică Analitică*, Casa Cărții de Știință, 2015.

NOȚIUNI AVANSATE ÎN DINAMICA ANALITICĂ A SISTEMELOR

Studiul dinamic al mișcărilor curente și rapide ale sistemelor mecanice multicorp (MBS), spre exemplu structurile mecanice de roboți seriali și în conformitate cu principiile diferențiale specifice dinamicii analitice a sistemelor, se bazează, printre altele, pe noțiunile avansate, cum sunt: energia cinetică, energiile de accelerații de diferite ordine și derivatele absolute în raport cu timpul a acestora de ordin superior. Noțiunile avansate sunt dezvoltate în conexiune directă cu variabilele generalizate, de asemenea, denumite parametrii independenți corespunzători sistemelor mecanice olonome. Dar, sub aspect mecanic, expresiile de definiție ale noțiunilor avansate conțin pe de o parte parametrii cinematici și transformările lor diferențiale corespunzătoare mișcării absolute, iar pe de altă parte proprietățile maselor, evidențiate prin masa și centrul maselor, tensorii inerțiali și legea de variație generalizată a acestora, precum și tensorii pseudoinerțiali. Cu ajutorul, în special, cercetărilor autorului în această lucrare se vor prezenta reformulări și formulări noi cu privire la parametrii cinematicii avansate, precum și noțiunile avansate cum sunt: energia cinetică, și energiile de accelerații de diferite ordine în forma explicită și matriceală. Acestea corespund mișcărilor curente și rapide ale MBS. Aceste formulări vor conține, de asemenea, derivatele absolute în raport cu timpul de ordin superior ale noțiunilor avansate, conform cu ecuațiile diferențiale de ordin superior, specifice dinamicii analitice a sistemelor.

Iuliu NEGREAN Professor Ph.D., Member of the Academy of Technical Sciences of Romania, Director of Department of Mechanical Systems Engineering, Technical University of Cluj-Napoca, iuliu.negrean@mep.utcluj.ro, http://users.utcluj.ro/~inegrean, Office Phone 0264/401616.