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ADVANCED EQUATIONS IN ANALYTICAL DYNAMICS OF SYSTEMS

Iuliu NEGREAN

Abstract: *The dynamical study of the current and sudden motions of the multibody systems (MBS), as example the mechanical robot structure, and in accordance with differential principles typical to analytical dynamics of systems, is based on the advanced notions, such as: generalized forces, kinetic energy, acceleration energies of different orders and their absolute time derivatives of higher order. Advanced notions are developed in the direct connection with the generalized variables, also named independent parameters corresponding to holonomic mechanical systems. But, mechanically, the advanced equations of dynamics contain on the one hand advanced notions from kinematics and their differential transformations, typically to absolute motions, and on the other hand mass properties and generalized forces.*

By means of the especially researches of the author, within of this paper will be presented reformulations and new formulations concerning the advanced kinematics parameters, as well as polynomial interpolating functions of higher order. In the following of the paper a few reformulations on fundamental theorems of dynamics and differential generalized principle of analytical dynamics will be presented. But, the fundamental aspect of this paper will consist in the fact that the author of the paper will propose generalized differential equations of higher order for any sudden and transitory motion. These equations contain acceleration energies of higher orders in generalized mathematical form.

Key words: *analytical dynamics, mechanics, advanced notions, advanced dynamics equations, robotics.*

1. INTRODUCTION

The advanced equations from dynamics of the current and sudden motions of the multibody systems (MBS), example Fig.1 the mechanical robot structure, and according to differential principles from analytical dynamics of systems, are based on the advanced notions of dynamics, as: generalized active and inertia forces, kinetic energy, acceleration energies of various orders and time derivatives of higher orders [6] – [18]. Advanced notions are developed in the direct connection with generalized variables which are univocally characterized for holonomic systems. The advanced equations of dynamics contain on the one hand advanced notions from kinematics and their differential transformations, typically to absolute mechanical motions, and on the other hand mass properties and generalized forces.

Based on especially of the author researches, in the fourth sections of the paper reformulations and new formulations on the advanced notions of kinematics and advanced dynamics equations are presented according to [6] – [15]. As result, first section of paper is devoted to the advanced kinematics for any current and sudden motion. In the view of this, time derivatives of higher order are performed on kinematical parameters. The second section is devoted to reformulations concerning fundamental theorems of dynamics. After a few transformations about theorems, the differential generalized principle of analytical dynamics is obtained. In the third and fourth sections of paper, author presents formulations concerning generalized active and inertia forces. But, the fundamental aspect of the fourth section consist in the fact that the author of the paper proposes generalized differential equations of higher order for sudden and transitory motions. These equations contain acceleration energies of higher orders in generalized mathematical form.

Fig.1 Mechanical Robot Structure (MBS)

2. ADVANCED KINEMATICS OF MBS

The kinematical and dynamical study from this paper [3], [4], [7] is oriented on mechanical structure with opened kinematical chain, where the kinetic ensembles $i = 1 \rightarrow n$ are physically linked by driving joints of fifth order. (Example robot mechanical structure, see Fig.1 and Fig.2).

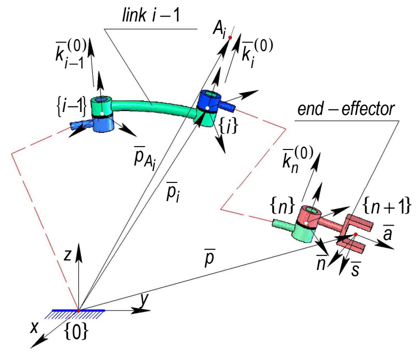

Fig.2 Sequence of Kinetic Ensemble

This is characterized by $(n\ d.o.f.)$, according to:

$$\bar\theta \neq \bar\theta^{(0)};\quad \bar\theta(t) = \left[q_i(t);\quad i = 1 \rightarrow n\right]^T, \tag{1}$$

where $q_i(t)$ is the generalized coordinate from every driving axis. But, considering the current and sudden motions, the generalized variables of higher order are developed as follows:

$$\left\{\begin{array}{l} \left\{\bar\theta(t);\ \dot{\bar\theta}(t);\ \ddot{\bar\theta}(t);\cdots;\overset{(m)}{\bar\theta}(t)\right\} = \\ = \left\{\begin{array}{c} q_i(t);\ \dot q_i(t);\ \ddot q_i(t);\cdots;\overset{(m)}{q_i}(t) \\ i = 1 \rightarrow n,\ m \geq 1 \end{array}\right\} \end{array}\right\}, \tag{2}$$

and (m) represents the time deriving order. The main objective of this section consists in the establishment of the parameters of the advanced kinematics, typical to mechanical system (MBS) characterized by current and sudden motions.

Within of the advanced equations applied in kinematics and analytical dynamics, the time derivatives of higher order for position vectors and rotation matrices must be used as follows:

$$\left\{\begin{array}{l} \frac{d^k}{dt^k}\left[\bar p_i(t)\right] = \overset{(k)}{\bar p_i} = \sum_{j=1}^{i}\left[\frac{\partial \overset{(m)}{\bar p_i}}{\partial \overset{(m)}{q_j}}\cdot \overset{(k)}{q_j}\right] + \\ \sum_{j=1}^{i}\sum_{r=1}^{k-1}\left\{\frac{\prod_{p=1}^{r}(k-p)}{p!}\cdot\left[\frac{p!\cdot m!}{(m+p)!}\cdot\frac{\partial \overset{(m+p)}{\bar p_i}}{\partial \overset{(m)}{q_j}}\cdot \overset{(k-p)}{q_j}\right]\right\} \end{array}\right\} \tag{3}$$

$$\left\{\begin{array}{l} \frac{d^k}{dt^k}\left\{{}^0_i[R](t)\right\} = {}^0_i\overset{(k)}{[R]} = \sum_{j=1}^{i}\left\{\frac{\partial}{\partial \overset{(m)}{q_j}}\left\{{}^0_s\overset{(m)}{[R]}\right\}\cdot\Delta_j\cdot\overset{(k)}{q_j}\right\} + \\ \sum_{j=1}^{i}\sum_{r=1}^{k-1}\left\{\frac{\prod_{p=1}^{r}(k-p)}{p!}\cdot\left\{\frac{d^p}{dt^p}\left\{\frac{\partial}{\partial \overset{(m)}{q_j}}\left\{{}^0_s\overset{(m)}{[R]}\right\}\right\}\cdot\Delta_j\cdot\overset{(k-p)}{q_j}\right\}\right\} = \\ = \sum_{j=1}^{i}\left\{\frac{\partial}{\partial \overset{(m)}{q_j}}\left\{{}^0_s\overset{(m)}{[R]}\right\}\cdot\Delta_j\cdot\overset{(k)}{q_j}\right\} + \\ \sum_{j=1}^{i}\sum_{r=1}^{k-1}\left\{\frac{\prod_{p=1}^{r}(k-p)}{p!}\cdot\left\{\frac{p!\cdot m!}{(m+p)!}\cdot\frac{\partial}{\partial \overset{(m)}{q_j}}\left\{{}^0_s\overset{(m+p)}{[R]}\right\}\cdot\Delta_j\cdot\overset{(k-p)}{q_j}\right\}\right\} \end{array}\right\}$$

and $$\left\{\begin{array}{l} k \geq 1;\ k = \{1;2;3;4;5;.....\} \\ m \geq (k+1);\ m = \{2;3;4;5;.....\} \end{array}\right\}; \tag{4}$$

where the symbols: (k) and (m) are the orders of the time derivatives concerning (3) and (4).

The advanced notions and equations from analytical dynamics [9] – [18] requires angular accelerations of higher order, as well as linear accelerations of higher order, corresponding to mass center for every kinetic ensemble, Fig.3.

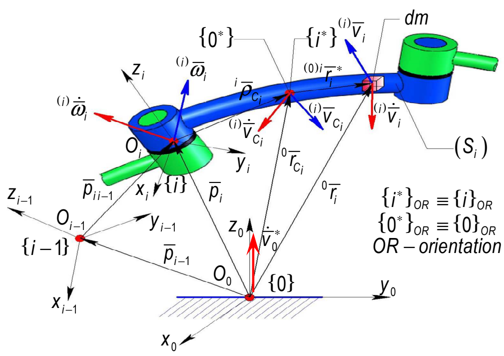

Fig.3 Sequence of Kinetic Ensemble

According to papers [7] and [18], expressions of definition for angular velocities, and then angular accelerations of higher order are established as:

$${}^0\bar\omega_i(t) = {}^0\bar\omega_{i-1}(t) + \Delta_i\cdot\dot q_i(t)\cdot{}^0\bar k_i(t); \tag{5}$$

$${}^0\overset{(k)}{\bar\omega_i}(t) = {}^0\overset{(k)}{\bar\omega}_{i-1}(t) + \Delta_i\cdot\frac{d^k}{dt^k}\left[\dot q_i(t)\cdot{}^0\bar k_i(t)\right]; \tag{6}$$

$k \geq 1;\ k = \{1;2;3;4;5;...\}$ is time derivative order, and ${}^0\bar k_i(t)$ is the unit vector to each driving axis.

The following property is according to [7] – [9]:

$$\left[\bar{\omega}_i(t)\times\right]={}_i^0\left[\dot{R}\right](t)\cdot{}_i^0\left[R\right]^T(t), \qquad (7)$$

According to Fig.3, first of all, the position of the mass center is defined in the classical form:

$$\bar{r}_{C_i}(t)=\bar{p}_i(t)+{}_i^0\left[R\right](t)\cdot{}^i\bar{\rho}_{C_i}; \qquad (8)$$

Applying the time derivative on (6), the linear velocity of the mass center is obtained, thus:

$$\bar{v}_{C_i}(t)=\bar{v}_i(t)+\bar{\omega}_i(t)\times\bar{\rho}_{C_i}(t); \qquad (9)$$

Linear and absolute accelerations of higher order corresponding to mass center are below defined:

$$\overset{(k)}{\bar{v}}_{C_i}(t)=\overset{(k)}{\bar{v}}_i(t)+\frac{d^k}{dt^k}\left[\bar{\omega}_i(t)\times\bar{\rho}_{C_i}(t)\right], \qquad (10)$$

where $u=\left\{\bar{\omega}_i\times;\bar{\rho}_{C_i}\right\};\ v=\left\{\bar{\rho}_{C_i};\bar{\omega}_i\times\right\};\ v\neq u;\ X=\{u;v\}:$

$$\left\{\begin{aligned}&\frac{d^k}{dt^k}\left[\bar{\omega}_i(t)\times\bar{\rho}_{C_i}(t)\right]=\sum_{\{u;v\}}\left\{\overset{(k)}{u}\cdot v\right\}+\frac{k}{0!}\cdot\sum_{\{u;v\}}\left\{\overset{(k-1)}{u}\cdot\dot{v}\right\}\\&+\left(k-\Delta_k\right)\cdot\left[k-\left(j+1-\Delta_k\right)\right]\cdot\sum_{\{u;v\}}\left\{\overset{(k-2)}{u}\cdot\ddot{v}\right\}\cdot\delta_k\\&+k\cdot\left[k-\left(2-\Delta_k\right)\right]\cdot\sum_{\{u;v\}}\left\{\overset{(k-2)}{u}\cdot\overset{(3)}{v}\right\}\cdot\delta_{kk}\\&+(k-1)\cdot\left[k-2\cdot(1-\delta_{kk})\right]\cdot\sum_{\{u;v\}}\left\{\overset{(k-j)}{u}\cdot\overset{(k-j)}{v}\right\}\cdot\Delta_k\end{aligned}\right\}$$

Position of terms from (10) must be respected, in the case of the time derivative of order (k),

$$\left\{\begin{aligned}&\text{and } 1\le k\le 8;\ \ \delta_{k(k)}=\left\{\left(0;\ k\le 4(6)\right);\left(1;\ k\ge 4(6)\right)\right\}\\&\Delta_k=\left\{\left(1;\ k=2\cdot j\right);\left(0;\ k\neq 2\cdot j\right),\ \text{and } j\ge 1\right\}\end{aligned}\right\}.$$

According to [10] – [18], angular and linear velocities, as well as the accelerations of higher order (5), (6), (9) – (10) can be also established by means of the following vector time functions:

$$\bar{r}_{C_i}=\bar{r}_{C_i}\left[q_j(t);\quad j=1\to k^*=n,\ t\right]; \qquad (11)$$

$$\bar{\bar{\psi}}_i(t)=\left[\alpha_A(t)\quad\beta_B(t)\quad\gamma_C(t)\right]^T; \qquad (12)$$

$$\left\{\begin{aligned}&{}^0J_\psi^i\left[\alpha_A(t)-\beta_B(t)-\gamma_C(t)\right]=\\&\left[{}^0\bar{A}\quad R\left(\bar{A};\alpha_A\right)\cdot\bar{B}\quad R\left(\bar{A};\alpha_A\right)\cdot R\left(\bar{B};\beta_B\right)\cdot\bar{C}\right]\end{aligned}\right\}; \qquad (13)$$

$$\left\{\begin{aligned}&\bar{\psi}_i(t)={}^0J_\psi\left[\alpha_A(t)-\beta_B(t)-\gamma_C(t)\right]\cdot\bar{\bar{\psi}}_i(t)=\\&=\bar{\psi}_i\left[q_j(t)\cdot\Delta_j;\quad j=1\to k^*=n,\ t\right]\end{aligned}\right\}; \qquad (14)$$

$$\Delta_j=\left\{\left(0\ \text{for}\ q_j-\text{linear}\right);\left(1\ \text{for}\ q_j-\text{angular}\right)\right\} \qquad (15)$$

where (11) is identical also with (8), and (14) is named the orientation vector for every kinetic ensemble, whose component (13) is known as angular transfer matrix defined as function of set of orientation angles. Considering (11) and (14) it observes that they are functions of generalized variables (2), taking into study the operator (15) devoted to character of the generalized variable. Using researches from [10] – [18], on vector functions (11) and (14), differentials properties compulsory applied in advanced kinematics and dynamics have been developed as below follows:

$$\frac{\partial\bar{r}_{C_i}}{\partial q_j}=\frac{\partial\bar{v}_{C_i}}{\partial\dot{q}_j}=\frac{\partial\bar{a}_{C_i}}{\partial\ddot{q}_j}=\frac{\partial\dot{\bar{a}}_{C_i}}{\partial\dddot{q}_j}=\frac{\partial\ddot{\bar{a}}_{C_i}}{\partial\ddddot{q}_j}=\ldots=\frac{\partial\overset{(m)}{\bar{r}}_{C_i}}{\partial\overset{(m)}{q}_j}, \qquad (16)$$

$$\frac{\partial\bar{\psi}_i}{\partial q_j}=\frac{\partial\dot{\bar{\psi}}_i}{\partial\dot{q}_j}=\frac{\partial\bar{\varepsilon}_i}{\partial\ddot{q}_j}=\ldots=\frac{\partial\ddot{\bar{\varepsilon}}_i}{\partial\ddddot{q}_j}=\ldots=\frac{\partial\overset{(m)}{\bar{\psi}}_i}{\partial\overset{(m)}{q}_j}, \qquad (17)$$

$$\left\{\begin{aligned}&\frac{d}{dt}\left(\frac{\partial\bar{r}_{C_i}}{\partial q_j}\right)=\frac{\partial\bar{v}_{C_i}}{\partial q_j}=\frac{\partial}{\partial q_j}\left(\sum_{m=1}^{k^*=n}\frac{\partial\bar{r}_{C_i}}{\partial q_m}\cdot\dot{q}_m\right)=\\&=\frac{1}{m+1}\cdot\frac{\partial\overset{(m-1)}{\bar{a}}_{C_i}}{\partial\overset{(m)}{q}_j}=\frac{1}{m+1}\cdot\frac{\partial\overset{(m+1)}{\bar{r}}_{C_i}}{\partial\overset{(m)}{q}_j},\ m\ge 0\end{aligned}\right\}, \qquad (18)$$

$$\left\{\begin{aligned}&\frac{d}{dt}\left(\frac{\partial\bar{\psi}_i}{\partial q_j}\right)=\frac{\partial\bar{\omega}_i}{\partial q_j}=\frac{\partial}{\partial q_j}\left(\sum_{m=1}^{k^*=n}\frac{\partial\bar{\psi}_i}{\partial q_m}\cdot\dot{q}_m\right)=\\&=\frac{1}{m+1}\cdot\frac{\partial\overset{(m-1)}{\bar{\varepsilon}}_i}{\partial\overset{(m)}{q}_j}=\frac{1}{m+1}\cdot\frac{\partial\overset{(m+1)}{\bar{\psi}}_i}{\partial\overset{(m)}{q}_j},\ m\ge 0\end{aligned}\right\}, \qquad (19)$$

$$\left\{\frac{d^{k-1}}{dt^{k-1}}\left(\frac{\partial\bar{r}_{C_i}}{\partial q_j}\right)=\frac{(k-1)!\cdot m!}{(m+k-1)!}\cdot\frac{\partial\overset{(m+k-1)}{\bar{r}}_{C_i}}{\partial\overset{(m)}{q}_j}\right\}, \qquad (20)$$

$$\left\{\begin{aligned}&\frac{d^{k-1}}{dt^{k-1}}\left(\frac{\partial\bar{\psi}_i}{\partial q_j}\cdot\Delta_j\right)=\frac{(k-1)!\cdot m!}{(m+k-1)!}\cdot\frac{\partial\overset{(m+k-3)}{\bar{\varepsilon}}_i}{\partial\overset{(m)}{q}_j}\cdot\Delta_j=\\&=\frac{(k-1)!\cdot m!}{(m+k-1)!}\cdot\frac{\partial\overset{(m+k-1)}{\bar{\psi}}_i}{\partial\overset{(m)}{q}_j}\cdot\Delta_j\end{aligned}\right\} \qquad (21)$$

$$\left\{\begin{aligned}&k\ge 1;\ \ k=\{1;2;3;4;5;.....\};\\&m\ge(k+1);\ \ m=\{2;3;4;5;.....\}\end{aligned}\right\}. \qquad (22)$$

The symbols (22) highlight time deriving orders. Using (11) – (22), the next expressions become:

$$\bar{v}_{C_i}(t)=\sum_{j=1}^{k^*=n}\frac{\partial\bar{r}_{C_i}(t)}{\partial q_j}\cdot\dot{q}_j(t)=\sum_{j=1}^{k^*=n}\frac{\partial\overset{(m)}{\bar{r}}_{C_i}(t)}{\partial\overset{(m)}{q}_j}\cdot\dot{q}_j(t) \qquad (23)$$

$$\left\{\begin{aligned}&\bar{a}_{C_i}(t)=\dot{\bar{v}}_{C_i}(t)=\ddot{\bar{r}}_{C_i}(t)=\\&=\sum_{j=1}^{k^*=n}\frac{\partial\overset{(m)}{\bar{r}}_{C_i}(t)}{\partial\overset{(m)}{q}_j}\cdot\ddot{q}_j(t)+\sum_{j=1}^{k^*=n}\frac{1}{m+1}\cdot\frac{\partial\overset{(m+1)}{\bar{r}}_{C_i}(t)}{\partial\overset{(m)}{q}_j}\cdot\dot{q}_j(t)\end{aligned}\right\};$$

$$\left\{\begin{array}{l} \overset{(k-1)}{\bar a}_{C_i}(t)=\overset{(k)}{\bar v}_{C_i}(t)=\sum\limits_{j=1}^{k^*=n}\frac{d^{k-1}}{dt^{k-1}}\left[\frac{\partial \overset{(m)}{\bar r}_{C_i}(t)}{\partial \overset{(m)}{q}_j}\cdot\ddot q_j(t)\right] \\ +\sum\limits_{j=1}^{k^*=n}\frac{d^{k-1}}{dt^{k-1}}\left[\frac{1}{m+1}\cdot\frac{\partial \overset{(m+1)}{\bar r}_{C_i}(t)}{\partial \overset{(m)}{q}_j}\cdot\dot q_j(t)\right]=\overset{(k+1)}{\bar r}_{C_i}(t)\end{array}\right\} \quad (24)$$

$$\left\{\begin{array}{l} \bar\omega_i(t)=\sum\limits_{j=1}^{k^*=n}\frac{\partial\bar\psi_i(t)}{\partial q_j}\cdot\Delta_j\cdot\dot q_j(t)= \\ =\sum\limits_{j=1}^{k^*=n}\frac{\partial \overset{(m)}{\bar\psi}_i(t)}{\partial \overset{(m)}{q}_j}\cdot\Delta_j\cdot\dot q_j(t)\end{array}\right\}; \quad (25)$$

$$\left\{\begin{array}{l} \bar\varepsilon_i(t)=\dot{\bar\omega}_i(t)=\sum\limits_{j=1}^{k^*=n}\frac{\partial \overset{(m)}{\bar\psi}_i(t)}{\partial \overset{(m)}{q}_j}\cdot\Delta_j\cdot\ddot q_j(t)+ \\ +\frac{1}{m+1}\cdot\sum\limits_{j=1}^{k^*=n}\frac{\partial \overset{(m+1)}{\bar\psi}_i}{\partial \overset{(m)}{q}_j}\cdot\Delta_j\cdot\dot q_j(t)=\ddot{\bar\psi}_i(t)\end{array}\right\}; \quad (26)$$

$$\left\{\begin{array}{l} \overset{(k-1)}{\bar\varepsilon}_i(t)=\overset{(k)}{\bar\omega}_i(t)=\sum\limits_{j=1}^{k^*=n}\frac{d^{k-1}}{dt^{k-1}}\left[\frac{\partial \overset{(m)}{\bar\psi}_i(t)}{\partial \overset{(m)}{q}_j}\cdot\Delta_j\cdot\ddot q_j(t)\right] \\ +\frac{1}{m+1}\cdot\sum\limits_{j=1}^{k^*=n}\frac{d^{k-1}}{dt^{k-1}}\left[\frac{\partial \overset{(m+1)}{\bar\psi}_i}{\partial \overset{(m)}{q}_j}\cdot\Delta_j\cdot\dot q_j(t)\right]=\overset{(k+1)}{\bar\psi}_i(t)\end{array}\right\}.$$

The expressions (23) and (24) are identical with (9) and respectively (10), and they are referring to the linear velocities and linear accelerations of higher order, corresponding to mass center. The others, (25) and (26) are identical with (6) and (7) representing the angular velocities, and accelerations of higher order of kinetic ensemble. When $(i=n)$, the kinematical parameters of the last kinetic ensemble from MBS are obtained. They are operational velocities and accelerations:

$$\left\{\begin{array}{l} \left\{\begin{array}{l} {}^{(n)0}\underset{(6\times 1)}{\overset{(m)}{X}}\left[\bar\theta(t);\dot{\bar\theta}(t);\cdots;\overset{(m)}{\bar\theta}(t)\right]= \\ ={}^{(n)0}R\cdot{}^{(0)n}\overset{(m)}{X}\left[\bar\theta(t);\dot{\bar\theta}(t);\cdots;\overset{(m)}{\bar\theta}(t)\right]\end{array}\right\}= \\ =\left\{\begin{array}{l} {}^{(n)0}\overset{(m-1)}{\bar v}_n\left[\bar\theta(t);\dot{\bar\theta}(t);\cdots;\overset{(m)}{\bar\theta}(t)\right] \\ {}^{(n)0}\overset{(m-1)}{\bar\omega}_n\left[\bar\theta(t);\dot{\bar\theta}(t);\cdots;\overset{(m)}{\bar\theta}(t)\right]\end{array}\right\}\end{array}\right\}; \quad (27)$$

$${}^{(n)0}\underset{(6\times 6)}{R}=\begin{bmatrix} {}^0_n[R]^{(T)} & [0] \\ [0] & {}^0_n[R]^{(T)}\end{bmatrix}; \quad (28)$$

The above expressions (27) represent the linear and angular velocities as well as accelerations of higher order (m) corresponding to motion of the last kinetic ensemble of MBS, with respect to absolute Cartesian frame [7] – [17]. To these, the locating matrix (28), for $(i=n)$, is also added.

According to [4] and [7], Jacobian matrix, also named velocity transfer matrix, sometimes matrix of the partial derivatives of the locating equations is applied in kinematics and dynamics:

$$\left\{{}^0J\left[\bar\theta(t)\right]=\left[{}^0J_i(t)=\begin{pmatrix}\bar V_i(t)\\ \ldots\ldots \\ \bar\Omega_i(t)\end{pmatrix};\ i=1\to n\right]\right\}; \quad (29)$$

$$\begin{pmatrix} \bar V_i(t)=\frac{\partial\bar p_n(t)}{\partial q_i} \\ \ldots\ldots\ldots\ldots \\ \bar\Omega_i(t)={}^0\bar k_i(t)\cdot\Delta_i\end{pmatrix}=\begin{pmatrix} {}^0\bar k_i(t)\times\bar p_{ni}(t)\cdot\Delta_i+(1-\Delta_i)\cdot{}^0\bar k_i(t) \\ \ldots\ldots\ldots\ldots \\ {}^0J_\psi(t)\cdot\frac{\partial}{\partial q_i}\left[\bar\psi(t)\right]\cdot\Delta_i\end{pmatrix}$$

$${}^{(n)0}J\left[\bar\theta(t)\right]={}^{(n)0}R\cdot{}^{(0)n}J\left[\bar\theta(t)\right]; \quad (30)$$

The components from (29) are corresponding to transfer matrices for angular and linear velocities and accelerations respectively, and (30) are the transfer relationships from one to another frame. The inverse of Jacobian matrix is determined as:

$${}^0J(\bar\theta)^{-1}=\left[{}^0J(\bar\theta)^T\cdot{}^0J(\bar\theta)\right]^{-1}\cdot{}^0J(\bar\theta)^T; \quad (31)$$

$${}^0J(\bar\theta)^{-1}={}^0J(\bar\theta)^T\cdot\left[{}^0J(\bar\theta)\cdot{}^0J(\bar\theta)^T\right]^{-1}. \quad (32)$$

When the mechanical robot structure (Fig. 1) is dominated by sudden motions, generalized and operational accelerations of higher order are also developed. Considering researches [7] – [14], in this paper the following expressions become:

$$\left\{\begin{array}{l} {}^0\overset{(m)}{X}(t)={}^0J\left[\bar\theta(t)\right]\cdot\overset{(m)}{\theta}(t)+ \\ +\sum\limits_{k=1}^{m-1}\frac{(m-1)!}{k!(m-k-1)!}\cdot{}^0J\left[\overset{(k)}{\bar\theta}(t)\right]\cdot\overset{[m-k]}{\theta}(t)= \\ =\sum\limits_{k=1}^{m}\frac{(m-1)!}{(k-1)!(m-k)!}\cdot{}^0J\left[\overset{(k-1)}{\bar\theta}(t)\right]\cdot\overset{[m-(k-1)]}{\theta}(t)\end{array}\right\}; \quad (33)$$

where (m) is the order of time derivatives, the symbol ${}^0\overset{(m)}{X}(t)$ represents the column matrix of the operational accelerations of higher orders, while $\overset{(m)}{\theta}(t)$ is the column matrix of generalized accelerations of higher orders, according with:

$$\left\{\begin{array}{l} \overset{(m)}{\theta}(t)={}^0J\left[\bar\theta(t)\right]^{-1}\cdot{}^0\overset{(m)}{X}(t)- \\ -{}^0J\left[\bar\theta(t)\right]^{-1}\cdot\sum\limits_{k=1}^{m-1}\frac{(m-1)!}{k!(m-k-1)!}\cdot{}^0J\left[\overset{(k)}{\bar\theta}(t)\right]\cdot\overset{(m-k)}{\theta}(t)\end{array}\right\} \quad (34)$$

Considering the mathematical models from [7], the Jacobian matrix (29) can be also determined with matrix exponentials. But, the components of (33) and (34) are based on rotation matrices and position vectors, according to (3) and (4). Their time derivatives of higher order show as:

$$ {}^{0}J\underset{(6\times n)}{\left[\overset{(k)}{\bar{\theta}}(t)\right]} \equiv \left[{}^{0}J_i \underset{(6\times 1)}{\left[\overset{(k)}{\bar{\theta}_i}(t)\right]} \text{ where } i=1\rightarrow n \right]; \quad (35) $$

Analyzing all above parameters of advanced kinematics, it results that they are functions of generalized variables (1) / (2), as well their time derivatives. So, according to author researches they can be developed as time functions, using polynomial interpolating functions [3] and [7]. It proposes following functions of higher order:

$$ \left\{ \begin{array}{l} \overset{(m-p)}{q_{ji}}(\tau) = (-1)^p \cdot \dfrac{(\tau_i-\tau)^{p+1}}{t_i\cdot(p+1)!}\cdot \overset{(m)}{q}_{ji-1} + \\ + \dfrac{(\tau-\tau_{i-1})^{p+1}}{t_i\cdot(p+1)!}\cdot \overset{(m)}{q}_{ji} + \delta_p \cdot \sum\limits_{k=1}^{p} \dfrac{\tau^{p-k}}{(p-k)!}\cdot a_{jik} \end{array} \right\}; \quad (36) $$

$$ \left\{ \begin{array}{c} \text{where } p=0\rightarrow m \\ m-\text{deriving order},\ m\geq 2,\ m=2,3,4,5,\ldots \\ \delta_p = \{(0,\ p=0);(1;p\geq 1)\} \\ j=1\rightarrow n \ \text{ degrees of freedom} -(d.o.f.) \\ i=1\rightarrow s \ \text{ intervals of motion trajectories} \\ \tau - \text{actual time variable} \\ t_i = \tau_i - \tau_{i-1} \ (\text{time to each trajectory interval}) \end{array} \right\}; \quad (37) $$

$$ \left\{ \begin{array}{l} (a_{jik}) \text{ for } k=1\rightarrow m; \text{and } \left(\overset{(m)}{q}_{ji-1}\right) \text{ for } i=2\rightarrow s \\ \text{where } (a_{jik}) - \text{integration constants, and} \\ \left(\overset{(m)}{q}_{ji-1}\right) - \text{generalized accelerations of } (m) \text{ order} \end{array} \right\} (38) $$

$$ \left\{ \begin{array}{l} (\tau_0) \Rightarrow \overset{(m-p)}{q}_{j0},\ \ p=0\rightarrow m;\ \ (\tau_s)\Rightarrow \left\{\overset{(m)}{q}_{js},\ q_{js}\right\} \\ (\tau_i) \Rightarrow \left\{ \begin{array}{c} \overset{(2)}{q}_{ji} - \text{generalized accelerations} \\ \left\{ \begin{array}{c} \overset{(m-p)}{q}_{ji}(\tau^+) = \overset{(m-p)}{q}_{ji+1}(\tau^-),\ \ p=0\rightarrow m \\ \text{continuity conditions} \end{array} \right\} \\ \text{all conditions are applied to each } (\tau_i) \\ \text{where } \ i=1\rightarrow s-1 \end{array} \right\} \end{array} \right\} (39) $$

For every trajectory interval $(i=1\rightarrow s)$, number of unknowns is $(m+1)$, and significance is (38). The determination the unknowns (38) requires, in accordance with [3] – [7], application of the geometrical and kinematical constraints (39). Finally, the results (36) will be substituted in the advanced notions of kinematics and dynamics.

3. ADVANCED DYNAMICS THEOREMS

The fundamental theorems, corresponding to Newtonian Dynamics, are: motion theorem of the mass center (momentum theorem), theorem of the angular momentum and theorem of the kinetic energy in differential and integral form. These are according to scientific literature, for example [7] and [8]. Applying the notions from advanced kinematics, see previous section, the main objective of this section consists in a few reformulations of the fundamental theorems, in consonance with the general motion of MBS, and considering the mechanical aspects from Fig.3.

So, *the motion theorem of the mass center* is characterized by means of the next equation:

$$ M_i\cdot \bar{a}_{C_i} = M_i\cdot \dot{\bar{v}}_{C_i} = M_i\cdot \ddot{\bar{r}}_{C_i} = \bar{F}_i^*, \quad (40) $$

where $\bar{F}_i^*$ is the resultant vector of active forces applied on the kinetic ensemble (i), see Fig.3. Substituting the linear acceleration of the mass center with (23), the theorem (40) is changed as:

$$ M_i\cdot \sum_{j=1}^{k^*=n}\left[\frac{\partial \overset{(m)}{\bar{r}}_{C_i}}{\partial \overset{(m)}{q}_j}\cdot \ddot{q}_j + \frac{1}{m+1}\cdot \frac{\partial \overset{(m+1)}{\bar{r}}_{C_i}}{\partial \overset{(m)}{q}_j}\cdot \dot{q}_j \right] = \bar{F}_i^*. \quad (41) $$

The theorem of the angular momentum, relative to mass center (Euler's equation) is defined by:

$$ I_i^*\cdot \bar{\varepsilon}_i + \frac{d}{dt}\left(I_i^*\right)\cdot \bar{\omega}_i = I_i^*\cdot \bar{\varepsilon}_i + \bar{\omega}_i\times I_i^*\cdot \bar{\omega}_i = \bar{N}_i^*. \quad (42) $$

Substituting angular velocity and acceleration with (25) and (26), the theorem (42) is changed:

$$ \left\{ \begin{array}{l} I_i^*\cdot \sum\limits_{j=1}^{k^*=n}\left[\dfrac{\partial \overset{(m)}{\bar{\psi}}_i}{\partial \overset{(m)}{q}_j}\cdot \Delta_j\cdot \ddot{q}_j + \dfrac{1}{m+1}\cdot \dfrac{\partial \overset{(m+1)}{\bar{\psi}}_i}{\partial \overset{(m)}{q}_j}\cdot \Delta_j\cdot \dot{q}_j \right] + \\ \sum\limits_{j=1}^{k^*=n}\sum\limits_{p=1}^{k^*=n}\left[\dfrac{\partial \overset{(m)}{\bar{\psi}}_i}{\partial \overset{(m)}{q}_j} \right]\times\left[I_i^*\cdot \dfrac{\partial \overset{(m)}{\bar{\psi}}_i}{\partial \overset{(m)}{q}_p} \right]\cdot \Delta_j\cdot \Delta_p\cdot \dot{q}_j\cdot \dot{q}_p = \bar{N}_i^* \end{array} \right\} (43) $$

$$ \text{where } \Delta_{j(p)} = \left\{\left(0,\ q_{j(p)}\in \bar{r}_{C_i}\right);\left(1,\ q_{j(p)}\in \bar{\psi}_i\right)\right\}, \quad (44) $$

and $\bar{N}_i^*$ is the resultant moment of active forces, while I_i^* is inertia tensor axial and centrifugal, the both are in relation with the mass center.

The theorem of the kinetic energy in differential form is considered the most general theorem of dynamics. Its equation of definition is written as:

$$dE_C^i = dL^i\,, \qquad \sum_{i=1}^{n} dE_C^i = \sum_{i=1}^{n} dL^i\,. \tag{45}$$

The components from (44) have the expressions:

$$E_C^i = \frac{1}{2}\cdot M_i\cdot \bar{v}_{C_i}^T\cdot \bar{v}_{C_i} + \frac{1}{2}\cdot \bar{\omega}_i\cdot I_i^*\cdot \bar{\omega}_i\,, \tag{46}$$

$$\left\{\begin{array}{c} dE_C^i = \displaystyle\sum_{j=1}^{k^*=n} M_i\cdot \bar{a}_{C_i}^T\cdot \frac{\partial \bar{r}_{C_i}}{\partial q_j}\cdot dq_j + \\ +\displaystyle\sum_{j=1}^{k^*=n}\left(I_i^*\cdot\bar{\varepsilon}_i + \bar{\omega}_i\times I_i^*\cdot\bar{\omega}_i\right)^T\cdot\frac{\partial\bar{\psi}_i}{\partial q_j}\cdot dq_j\cdot\Delta_j \end{array}\right\}; \tag{47}$$

$$\left\{\begin{array}{c} dL^i = \bar{F}_i^{*T}\cdot d\bar{r}_{C_i} + \bar{N}_i^{*T}\cdot d\bar{\psi}_i = \\ \displaystyle\sum_{j=1}^{k^*=n}\bar{F}_i^{*T}\cdot\frac{\partial\bar{r}_{C_i}}{\partial q_j}\cdot dq_j + \sum_{j=1}^{k^*=n}\bar{N}_i^{*T}\cdot\frac{\partial\bar{\psi}_i}{\partial q_j}\cdot dq_j\cdot\Delta_j \end{array}\right\}; \tag{48}$$

where (46) and (47) are König's theorem and the differential expression of the kinetic energy, and (48) elementary work. Expressions (46) – (48) are corresponding to general motion of MBS. Substituting (16) and (17) in (47) and (48), and left member from (41) and (43) in (47), theorem of the kinetic energy, under the differential form (45) finally is mathematically reformulated.

In the case of the multibody systems (MBS), with holonomic mechanical significance, a few conditions are applied on (45), (47) and (48):

$$\left\{\begin{array}{c} q_j\neq 0,\ dq_j\neq 0,\ j=1\rightarrow n \\ q_i=0,\ dq_i=0,\ i=1\rightarrow n,\ i\neq j \end{array}\right\}. \tag{49}$$

They are referring to independent parameters in in the both finite and elementary displacements. After a few transformations on the differential of the theorem of the kinetic energy it obtains:

$$\left\{\begin{array}{c} 0=\displaystyle\sum_{i=1}^{n}\left\{\left[\bar{F}_i^{*T} - M_i\cdot\bar{a}_{C_i}^T\right]\cdot\frac{\partial\bar{r}_{C_i}}{\partial q_j} + \right. \\ \left. +\displaystyle\sum_{i=1}^{n}\left[\bar{N}_i^{*T} - \left(I_i^*\cdot\bar{\varepsilon}_i + \bar{\omega}_i\times I_i^*\cdot\bar{\omega}_i\right)^T\right]\cdot\frac{\partial\bar{\psi}_i}{\partial q_j}\cdot\Delta_j\right\} \end{array}\right\} \tag{50}$$

According to author researches [7] – [18], for the multibody rigid system, expression (50) is considered the differential generalized principle (generalization of the D'Alembert — Lagrange principle) in analytical dynamics of systems.

4. GENERALIZED DYNAMICS FORCES

According to [3], [4], [7] and [17] on every kinetic ensemble $(i=1\rightarrow n)$, belonging to the mechanical robot structure, as integrated part from MBS, are especially applied a system of external and active forces, manipulating loads, as well as complex friction forces, see Fig.4.

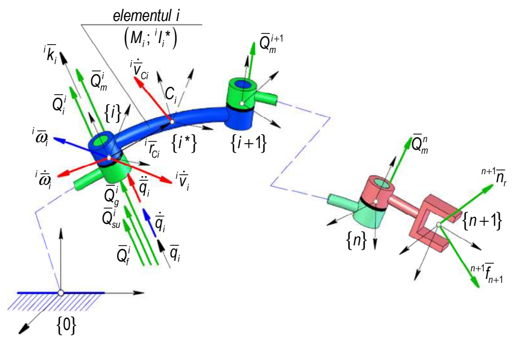

Fig.4 Generalized Forces in MBS

In function of (static or dynamic) behavior in every physical link (driving joint of fifth order) generalized static or driving force is developed.

Using, among of these, the author researches [4] – [17], the main objective of this section consists in the presentation of the expressions of definition for the generalized forces due to gravitation and manipulating loads, as well as due to inertia property typically to MBS (Fig.4). So, considering [17], the expression of definition for generalized gravitational force shows thus:

$$Q_g\left(\bar{\theta}\right) = \left[Q_g^i = {}^{(n)0}J_i^T\cdot{}^{(n)0}\bar{\ddot{o}}_{X_i};\ i=1\rightarrow n\right]^T. \tag{51}$$

$$\underset{(n\times n)}{Diag}\left[Q_g^i\left(\bar{\theta}\right)\right] = {}^0J\left(\bar{\theta}\right)^T\cdot\underset{(6\times n)}{Matrix}\left[{}^0\bar{\ddot{o}}_{X_i};\ i=1\rightarrow n\right];$$

$$\left\{\begin{array}{c} where\quad {}^{(n)0}\bar{F}_{X_i} = \displaystyle\sum_{j=i}^{n} M_j\cdot{}^{(0)n}_{\ \ n}[R]^T\cdot\bar{g} \\ {}^{(n)0}\bar{N}_{X_i} = \displaystyle\sum_{j=i}^{n} M_j\cdot{}^{(0)n}_{\ \ n}[R]^T\cdot\left[\left({}^0\bar{r}_{C_j} - \bar{p}_n\right)\times\bar{g}\right] \\ \underset{(6\times 1)}{{}^{(n)0}\bar{\ddot{o}}_{X_i}} = \left[{}^{(n)0}\bar{F}_{X_i}^T\quad {}^{(n)0}\bar{N}_{X_i}^T\right]^T \end{array}\right\}; \tag{52}$$

$$\left\{\begin{array}{c} and\qquad \bar{g}=\tau\cdot g\cdot\bar{k}_0,\quad \tau=\mp\bar{k}_0^T\cdot\bar{k}_g \\ \bar{k}_g = {}^0\bar{g}/\left|{}^0\bar{g}\right|,\ \ \bar{k}_0 - vertical\ unit\ vector\in\{0\} \end{array}\right\}. \tag{53}$$

The column vector (52), expressed with respect to Cartesian space, is mechanically equivalent with reduction torsor of the gravitational forces in $[i;n]$ interval in relation with the $\{n\}$ frame. This is applied in the geometry center of the last driving joint belonging to MBS (Fig. 4).

In keeping with the same paper [17] generalized manipulating force is below characterized with:

$$\left\{\begin{array}{l} Q_{SU}\left(\bar{\theta}\right)=\left[Q_{SU}^{i}={}^{(n)0}J_i^T\cdot{}^{(n)0}\bar{\text{ö}}_X;\ i=1\rightarrow n\right]^T \\ ={}^{(n)0}J\left(\bar{\theta}\right)\cdot{}^{(n)0}\bar{\text{ö}}_X \end{array}\right\} \quad (54)$$

$$\left\{\begin{array}{l} where\ \underset{(6\times1)}{{}^{0}\bar{\text{ö}}_X}=\left[{}^{0}\bar{F}_X^T\quad {}^{0}\bar{N}_X^T\right]^T= \\ \left[\begin{array}{cc} {}^{(n)0}_{n+1}[R] & \underset{(3\times3)}{[0]} \\ \left\{{}^{(0)n}_{n}[R]^T\cdot\bar{p}_{n+1n}\times\right\}{}^{n}_{n+1}[R] & {}^{(0)n}_{n+1}[R] \end{array}\right]\cdot\left[\begin{array}{c} {}^{n+1}\bar{f}_{n+1} \\ {}^{n+1}\bar{n}_{n+1} \end{array}\right] \end{array}\right\} \quad (55)$$

Cartesian column vector (55) is mechanically equivalent [4] – [8] with the reduction torsor of the manipulating load with respect to $\{n\}$ frame.

Considering the aspects from [6] – [18], in the case of the dynamical behavior of MBS, in every driving joint, besides the active forces (see above expressions), are developing generalized inertia and driving forces. Finally, the expression of the generalized inertia force becomes thus:

$$\left\{\begin{array}{l} \underset{(n\times1)}{Q_{i\text{ö}}}\left[\bar{\theta}(t);\dot{\bar{\theta}}(t);\ddot{\bar{\theta}}(t)\right]= \\ =\left[Q_{i\text{ö}}^{i}(t)={}^{0}J_i^T(t)\cdot{}^{0}\bar{\text{ö}}^{*}_{X_i}(t);\quad i=1\rightarrow n\right]^T \end{array}\right\}; \quad (56)$$

$$\underset{(n\times n)}{Diag}\left[Q_{i\text{ö}}\left(\bar{\theta}\right)\right]={}^{0}J\left(\bar{\theta}\right)^T\cdot\underset{(6\times n)}{Matrix}\left[{}^{0}\bar{\text{ö}}^{*}_{X_i};\ i=1\rightarrow n\right];$$

$$\left\{\begin{array}{l} where\ {}^{0}\bar{\text{ö}}^{*}_{X_i}=\left[{}^{0}\bar{F}_{X_i}^{*T}\quad {}^{0}\bar{N}_{X_i}^{*T}\right]^T \\ {}^{0}\bar{F}^{*}_{X_i}=\sum_{j=i}^{n}{}^{0}_{j}[R]\cdot{}^{j}\bar{F}_j^{*} \\ {}^{0}\bar{N}^{*}_{X_i}=\sum_{j=i}^{n}\left\{\left({}^{0}\bar{r}_{C_j}-\bar{p}_n\right)\times{}^{0}_{j}[R]\cdot{}^{j}\bar{F}_j^{*}+{}^{0}_{j}[R]\cdot{}^{j}\bar{N}_j^{*}\right\} \end{array}\right\}. \quad (57)$$

Column vector (57), expressed with respect to Cartesian space, is mechanically equivalent with reduction torsor of the inertia forces from $[i;n]$ interval in relation with the $\{n\}$ moving frame, applied in the geometry center of the last driving joint from MBS (see Fig.4).

Comparing (51), (54) and (56) it observes that they have unique character. The generalized active and inertia forces are mathematically identical as form of expression. This aspect has important advantage, in the establishment of the dynamics equations of motion, corresponding to every kinetic ensemble of mechanical system.

The generalized driving force from every driving axis from MBS is finally obtained thus:

$$\left\{\begin{array}{l} Q_m^i(t)=\Delta_m^2\cdot\left[\Delta_\theta\cdot Q_{i\text{ö}}^{i}(t)+Q_g^i(t)\right]+ \\ +(-1)^{\Delta_m}\cdot\dfrac{1-\Delta_m}{1+3\cdot\Delta_m}\cdot Q_{SU}^{i}(t) \end{array}\right\}; \quad (58)$$

$$\left\{\begin{array}{l} Q_m^i(t)={}^{(n)0}J_i^T(t)\cdot\left\{\Delta_m^2\cdot\left[\Delta_\theta\cdot{}^{(n)0}\bar{\text{ö}}^{*}_{X_i}(t)+\right.\right. \\ \left.\left.+{}^{(n)0}\bar{\text{ö}}_{X_i}(t)\right]+(-1)^{\Delta_m}\cdot\dfrac{1-\Delta_m}{1+3\cdot\Delta_m}\cdot{}^{(n)0}\bar{\text{ö}}_X(t)\right\} \end{array}\right\}; \quad (59)$$

$$\left\{\begin{array}{l} \Delta_m=\left\{\left[-1;(SU;M_i)\right];(0;SU);(1;M_i)\right\} \\ \Delta_\theta=\left\{\left[1;\left\{\dot{\bar{\theta}};\ddot{\bar{\theta}}\right\}\neq0\right];\left[0;\left\{\dot{\bar{\theta}};\ddot{\bar{\theta}}\right\}=0\right]\right\} \end{array}\right\} \quad (60)$$

$$\Delta_f=\left\{\left(-1;\Delta_m=-1\right);\left[0;\Delta_m=(0;1)\right];\left(1;Q_{fd}^i\right)\right\} \quad (61)$$

$$Q_{mf}^i(t)=(-1)^{\Delta_f}\cdot\frac{1-\Delta_f}{1+3\cdot\Delta_f}\cdot Q_m^i(t)+\Delta_f^2\cdot Q_{fd}^i(t) \quad (62)$$

where Δ_m highlights: gravitational loads by (M_i) and manipulating loads by the symbol (SU); Δ_f highlights the loads by Δ_m and the influence of the complex frictions; Δ_θ shows the behavior of mechanical system (0 – statics; 1 – dynamics). As a result, (62) for $(i=1\rightarrow n)$ constitutes the system of (n) generalized driving forces. They are identical with dynamics equations of MBS, in which the both generalized active and inertia forces, and the complex frictions are founded.

5. ADVANCED EQUATIONS OF DYNAMICS

In the case of mechanical systems (MBS) dominated by sudden motions, as well as by the transitory motions, on the basis of the author's researches [6] – [18] it demonstrates theoretical and experimental existing of the accelerations energy of higher order. They are substituted in the advanced equations of higher order from analytical dynamics. As a result, time variations of generalized forces, presented in the previous section, are obvious. So, considering the aspects from [17], generalized forces are time derived:

$$\left\{\begin{array}{l} \overset{(k)}{Q_{i\text{ö}}^{i}}(t)={}^{0}J_i\left[\bar{\theta}(t)\right]\cdot{}^{0}\overset{(k)}{\bar{\text{ö}}}{}^{*}_{X_i}+ \\ +\sum_{m=1}^{k-1}\dfrac{(k-1)!}{m!(k-m-1)!}\cdot{}^{0}J_i\left[\overset{(m)}{\bar{\theta}(t)}\right]\cdot{}^{0}\overset{[k-m]}{\bar{\text{ö}}}{}^{*}_{X_i}= \\ =\sum_{m=1}^{k}\dfrac{(k-1)!}{(m-1)!(k-m)!}\cdot{}^{0}J_i\left[\overset{(m-1)}{\bar{\theta}(t)}\right]\cdot{}^{0}\overset{[k-(m-1)]}{\bar{\text{ö}}}{}^{*}_{X_i} \end{array}\right\}; \quad (63)$$

$$\left\{\begin{array}{l} \overset{(k)}{Q}{}^i_g(t) = {}^0J_i\left[\bar\theta(t)\right]\cdot {}^0\overset{(k)}{\bar{\ddot{o}}}_{X_i} + \\ +\sum_{m=1}^{k-1}\frac{(k-1)!}{m!(k-m-1)!}\cdot {}^0J_i\left[\overset{(m)}{\bar\theta}(t)\right]\cdot {}^0\overset{[k-m]}{\bar{\ddot{o}}}_{X_i} = \\ = \sum_{m=1}^{k}\frac{(k-1)!}{(m-1)!(k-m)!}\cdot {}^0J_i\left[\overset{(m-1)}{\bar\theta}(t)\right]\cdot {}^0\overset{[k-(m-1)]}{\bar{\ddot{o}}}_{X_i} \end{array}\right\}; \quad (64)$$

$$\left\{\begin{array}{l} Q_{SU}\left[\overset{(k)}{\bar\theta}(t)\right] = {}^0J\left[\bar\theta(t)\right]\cdot {}^0\overset{(k)}{\bar{\ddot{o}}}_{X} + \\ +\sum_{m=1}^{k-1}\frac{(k-1)!}{m!(k-m-1)!}\cdot {}^0J\left[\overset{(m)}{\bar\theta}(t)\right]\cdot {}^0\overset{[k-m]}{\bar{\ddot{o}}}_{X} = \\ = \sum_{m=1}^{k}\frac{(k-1)!}{(m-1)!(k-m)!}\cdot {}^0J\left[\overset{(m-1)}{\bar\theta}(t)\right]\cdot {}^0\overset{[k-(m-1)]}{\bar{\ddot{o}}}_{X} \end{array}\right\}. \quad (65)$$

The significance of the terms from (63) – (65) is well defined in the previous sections of this paper, and $(k \geq 1)$ is the time deriving order. But considering dynamical equations, instead of (k) is written $(k-1)$. When $(k=1)$, then (63) – (65) are degenerated in: (56), (54), and last in (51).

Taking into study the differential generalized principle (50), generalized inertia and driving forces (56) and (58), dynamics equations are:

$$\left\{\begin{array}{l} Q^i_{i\ddot{o}}\ (t) = \sum_{i=1}^{n} M_i\cdot \bar a^T_{C_i}\cdot\frac{\partial \bar r_{C_i}}{\partial q_j} + \\ +\sum_{i=1}^{n}\left(I^*_i\cdot\bar\varepsilon_i + \bar\omega_i\times I^*_i\cdot\bar\omega_i\right)^T\cdot\frac{\partial\bar\psi_i}{\partial q_j}\cdot\Delta_j = \\ = \sum_{i=1}^{n}\bar F^{*T}_i\cdot\frac{\partial \bar r_{C_i}}{\partial q_j} + \sum_{i=1}^{n}\bar N^{*T}_i\cdot\frac{\partial\bar\psi_i}{\partial q_j}\cdot\Delta_j \end{array}\right\}; \quad (66)$$

$$\left\{\begin{array}{l} \text{where}\ \ \sum_{i=1}^{n}\bar F^{*T}_i\cdot\frac{\partial \bar r_{C_i}}{\partial q_j} + \sum_{i=1}^{n}\bar N^{*T}_i\cdot\frac{\partial\bar\psi_i}{\partial q_j}\cdot\Delta_j = \\ Q^i_m(t) - \Delta^2_m\cdot Q^i_g(t) - (-1)^{\Delta_m}\cdot\frac{1-\Delta_m}{1+3\cdot\Delta_m}\cdot Q^i_{SU}(t) \end{array}\right\} \quad (67)$$

According to Lagrange's equations of second kind, generalized inertia forces are identical as:

$$\frac{d}{dt}\left(\frac{\partial E_C}{\partial \dot q_j}\right) - \frac{\partial E_C}{\partial q_j} = Q^j_{i\ddot{o}}\ (t); \quad (68)$$

$$\frac{1}{m}\cdot\left[\frac{\partial \overset{(m)}{E_C}}{\partial \overset{(m)}{q_j}} - (m+1)\cdot\frac{\partial E_C}{\partial q_j}\right] = Q^j_{i\ddot{o}}\ (t). \quad (69)$$

Expression (69) represents Tsenov – Mangeron formulation, and (m) is the time deriving order. Considering acceleration energy of first order [1] – [20] and time derivatives of higher order, generalized inertia forces are also identical with:

$$\left\{\begin{array}{l} \frac{\partial}{\partial \overset{(m)}{q_j}}\left\{\overset{(m-2)}{E^{(1)}_A}\left[\bar\theta(t);\dot{\bar\theta}(t);\ddot{\bar\theta}(t);\cdots;\overset{(m)}{\bar\theta}(t)\right]\right\} = \\ = Q^j_{i\ddot{o}}\left[\bar\theta(t);\dot{\bar\theta}(t);\overset{(2)}{\bar\theta}(t)\right] \end{array}\right\}; \quad (70)$$

$$\left\{\begin{array}{l} \text{where}\ \ \overset{(0)}{E^{(1)}_A} = E^{(1)}_A \quad j = 1\to n, \quad k=1 \\ m \geq \left[(k+1)=2\right], \text{and}\,(k)\ \text{are time deriving orders} \end{array}\right\};$$

where according to [1] – [20], (70) is named generalization of Gibbs – Appell's equations.

Using the author researches from [4] – [18] in the following of this section are presented the motion differential equations of various order. At beginning, time derivatives of first, second and third orders are applied on expression (66) and (67). After transformations they become:

$$\overset{(1)}{Q}{}^j_{i\ddot{o}}\left[\bar\theta(t);\dot{\bar\theta}(t);\cdots;\overset{(m)}{\bar\theta}(t)\right] = \quad (71)$$

$$\left\{\begin{array}{l} = \frac{d}{dt}\left[\sum_{i=1}^{n} M_i\cdot\bar a^T_{C_i}\cdot\frac{\partial\bar r_{C_i}}{\partial q_j} + \right. \\ \left. +\sum_{i=1}^{n}\left(I^*_i\cdot\bar\varepsilon_i + \bar\omega_i\times I^*_i\cdot\bar\omega_i\right)^T\cdot\frac{\partial\bar\psi_i}{\partial q_j}\cdot\Delta_j\right] \end{array}\right\};$$

$$\overset{(1)}{Q}{}^j_{i\ddot{o}}\left[\bar\theta(t);\dot{\bar\theta}(t);\cdots;\overset{(m)}{\bar\theta}(t)\right] = \quad (72)$$

$$\left\{\begin{array}{l} = \frac{d}{dt}\left\{\frac{\partial \overset{(m-2)}{E^{(1)}_A}}{\partial \overset{(m)}{q_j}}\right\} = \frac{\partial E^{(2)}_A}{\partial \overset{(3)}{q_j}} + \frac{1}{2}\cdot\frac{\partial E^{(1)}_A}{\partial \overset{(1)}{q_j}} = \\ = \frac{1}{m+1}\cdot\frac{\partial}{\partial \overset{(m)}{q_j}}\left[2\cdot\overset{(m-3)}{E^{(2)}_A} + \overset{(m-1)}{E^{(1)}_A}\right] \end{array}\right\};$$

$$\left\{\begin{array}{l} \overset{(1)}{Q}{}^i_m(t) = \Delta^2_m\cdot\left[\Delta_\theta\cdot\overset{(1)}{Q}{}^i_{i\ddot{o}}\ (t) + \overset{(1)}{Q}{}^i_g(t)\right] + \\ +(-1)^{\Delta_m}\cdot\frac{1-\Delta_m}{1+3\cdot\Delta_m}\cdot\overset{(1)}{Q}{}^i_{SU}(t) \end{array}\right\}; \quad (73)$$

$$\overset{(2)}{Q}{}^j_{i\ddot{o}}\left[\bar\theta(t);\dot{\bar\theta}(t);\cdots;\overset{(m)}{\bar\theta}(t)\right] = \quad (74)$$

$$\left\{\begin{array}{l} = \frac{d^2}{dt^2}\left[\sum_{i=1}^n M_i \cdot \bar{a}_{C_i}^T \cdot \frac{\partial \bar{r}_{C_i}}{\partial q_j} + \right. \\ \left. + \sum_{i=1}^n \left(I_i^* \cdot \bar{\varepsilon}_i + \bar{\omega}_i \times I_i^* \cdot \bar{\omega}_i\right)^T \cdot \frac{\partial \bar{\psi}_i}{\partial q_j} \cdot \Delta_j \right]\end{array}\right\};$$

$$\overset{(2)}{Q_{i\ddot{o}}^j}\left[\bar{\theta}(t);\dot{\bar{\theta}}(t);\cdots;\overset{(m)}{\bar{\theta}}(t)\right] = \tag{75}$$

$$\left\{\begin{array}{l} = \frac{d^2}{dt^2}\left\{\frac{\partial \overset{(m-2)}{E_A^{(1)}}}{\partial \overset{(m)}{q_j}}\right\} = \frac{\partial E_A^{(3)}}{\partial \overset{(4)}{q_j}} + \frac{1}{3}\cdot\frac{\partial E_A^{(2)}}{\partial \overset{(2)}{q_j}} + \frac{\partial E_A^{(1)}}{\partial q_j} = \\ = \frac{2}{(m+1)\cdot(m+2)}\cdot\frac{\partial}{\partial \overset{(m)}{q_j}}\left[5\cdot \overset{(m-4)}{E_A^{(3)}} + 2\cdot \overset{(m-2)}{E_A^{(2)}} + \overset{(m)}{E_A^{(1)}}\right] \end{array}\right\};$$

$$\left\{\begin{array}{l} where \quad j = 1 \to n, \quad k = 3,\ m \geq \left[(k+1) = 4\right] \\ m = 4,5,6,\ldots \quad and\ \overset{(0)}{E_A^{(3)}} = E_A^{(3)} \end{array}\right\};$$

$$\left\{\begin{array}{l} \overset{(2)}{Q_m^i}(t) = \Delta_m^2 \cdot \left[\Delta_\theta \cdot \overset{(2)}{Q_{i\ddot{o}}^i}(t) + \overset{(2)}{Q_g^i}(t)\right] + \\ \quad + (-1)^{\Delta_m} \cdot \frac{1-\Delta_m}{1+3\cdot\Delta_m} \cdot \overset{(2)}{Q_{SU}^i}(t) \end{array}\right\}; \tag{76}$$

$$\overset{(3)}{Q_{i\ddot{o}}^j}\left[\bar{\theta}(t);\dot{\bar{\theta}}(t);\cdots;\overset{(m)}{\bar{\theta}}(t)\right] = \tag{77}$$

$$\left\{\begin{array}{l} = \frac{d^3}{dt^3}\left[\sum_{i=1}^n M_i \cdot \bar{a}_{C_i}^T \cdot \frac{\partial \bar{r}_{C_i}}{\partial q_j} + \right. \\ \left. + \sum_{i=1}^n \left(I_i^* \cdot \bar{\varepsilon}_i + \bar{\omega}_i \times I_i^* \cdot \bar{\omega}_i\right)^T \cdot \frac{\partial \bar{\psi}_i}{\partial q_j} \cdot \Delta_j \right]\end{array}\right\};$$

$$\left\{\begin{array}{l} \overset{(3)}{Q_{i\ddot{o}}^j}\left[\bar{\theta}(t);\dot{\bar{\theta}}(t);\cdots;\overset{(m)}{\bar{\theta}}(t)\right] = \frac{d^3}{dt^3}\left\{\frac{\partial \overset{(m-2)}{E_A^{(1)}}}{\partial \overset{(m)}{q_j}}\right\} = \\ = \delta_{Q\ddot{o}} \cdot \frac{\partial}{\partial \overset{(m)}{q_j}}\left[9\cdot \overset{(m-5)}{E_A^{(4)}} + 5\cdot \overset{(m-3)}{E_A^{(3)}} + 2\cdot \overset{(m-1)}{E_A^{(2)}} + \overset{(m+1)}{E_A^{(1)}}\right] \end{array}\right\}; \tag{78}$$

$$\delta_{Q\ddot{o}} = \frac{2\cdot 3}{(m+1)\cdot(m+2)\cdot(m+3)} = \frac{(k-1)!\,m!}{(m+k-1)!}, \tag{79}$$

$$\left\{\begin{array}{l} and\ j = 1 \to n, \quad k = 4, \quad m \geq \left[(k+1) = 5\right] \\ m = 5,6,7,\ldots \quad and\ \overset{(0)}{E_A^{(4)}} = E_A^{(4)} \end{array}\right\};$$

$$\left\{\begin{array}{l} \overset{(3)}{Q_m^i}(t) = \Delta_m^2 \cdot \left[\Delta_\theta \cdot \overset{(3)}{Q_{i\ddot{o}}^i}(t) + \overset{(3)}{Q_g^i}(t)\right] + \\ \quad + (-1)^{\Delta_m} \cdot \frac{1-\Delta_m}{1+3\cdot\Delta_m} \cdot \overset{(3)}{Q_{SU}^i}(t) \end{array}\right\}. \tag{80}$$

The dynamics equations of order $(m \geq 2)$: (72), (75) and (78) contain the acceleration energies of higher order $(1 \leq p \leq 4)$, whose expressions of definition are presented in the papers [6] – [18]. Following the application of time derivatives of higher order (m) and (k), the equations (69) and (70) are changed in new differential expressions:

$$\left\{\begin{array}{l} \frac{1}{m}\cdot\frac{d^{k-1}}{dt^{k-1}}\left[\frac{\partial \overset{(m)}{E_C}}{\partial \overset{(m)}{q_j}} - (m+1)\cdot\frac{\partial E_C}{\partial q_j}\right] = \\ = \overset{(k-1)}{Q_{i\ddot{o}}^j}\left[\bar{\theta}(t);\dot{\bar{\theta}}(t);\cdots;\overset{(m)}{\bar{\theta}}(t)\right] \end{array}\right\}; \tag{81}$$

$$\frac{d^{k-1}}{dt^{k-1}}\left\{\frac{\partial \overset{(m-2)}{E_A^{(1)}}}{\partial \overset{(m)}{q_j}}\right\} = \overset{(k-1)}{Q_{i\ddot{o}}^j}\left[\bar{\theta}(t);\dot{\bar{\theta}}(t);\cdots;\overset{(m)}{\bar{\theta}}(t)\right]. \tag{82}$$

According to [9] – [14], sudden and transitory motions of MBS are represented by dynamics equations, and the central function is highlighted through acceleration energies of higher order. As result, considering acceleration energies of first, second, third and fourth order, and applying the time derivatives of higher order (m) and (k), on (72), (75) and (78), the dynamics equations are:

$$\left\{\begin{array}{l} \frac{1}{m+1}\cdot\frac{d^{k-1}}{dt^{k-1}}\frac{\partial}{\partial \overset{(m)}{q_j}}\left[2\cdot \overset{(m-3)}{E_A^{(2)}} + \overset{(m-1)}{E_A^{(1)}}\right] = \\ = \overset{(k-1)}{Q_{i\ddot{o}}^j}\left[\bar{\theta}(t);\dot{\bar{\theta}}(t);\cdots;\overset{(m)}{\bar{\theta}}(t)\right] \end{array}\right\}, \tag{83}$$

$$\left\{\overset{(0)}{E_A^{(2)}} = E_A^{(2)},\ j = 1 \to n,\ k = 2,\ m \geq \left[(k+1) = 3\right]\right\};$$

$$\left\{\delta_{Q\ddot{o}} \cdot \frac{d^{k-1}}{dt^{k-1}}\frac{\partial}{\partial \overset{(m)}{q_j}}\left[5\cdot \overset{(m-4)}{E_A^{(3)}} + 2\cdot \overset{(m-2)}{E_A^{(2)}} + \overset{(m)}{E_A^{(1)}}\right] = \right\}$$

$$= \overset{(k-1)}{Q_{i\ddot{o}}^j}\left[\bar{\theta}(t);\dot{\bar{\theta}}(t);\cdots;\overset{(m)}{\bar{\theta}}(t)\right], \tag{84}$$

$$\left\{\begin{array}{l} where\ \delta_{Q\ddot{o}}\ (see\ (79)),\ \overset{(0)}{E_A^{(3)}} = E_A^{(3)},\ \ j = 1 \to n \\ k = 3,\ m \geq \left[(k+1) = 4\right], \quad m = 4,5,6,\ldots \end{array}\right\};$$

$$\left\{\delta_{Q\ddot{o}}\cdot\frac{d^{k-1}}{dt^{k-1}}\frac{\partial}{\partial \overset{(m)}{q_j}}\left[9\cdot \overset{(m-5)}{E_A^{(4)}}+5\cdot \overset{(m-3)}{E_A^{(3)}}+2\cdot \overset{(m-1)}{E_A^{(2)}}+\overset{(m+1)}{E_A^{(1)}}\right]=\right\}$$

$$=\overset{(k-1)}{Q^j_{i\ddot{o}}}\left[\bar\theta(t);\dot{\bar\theta}(t);\ddot{\bar\theta}(t);\cdots;\overset{(m)}{\bar\theta}(t)\right]; \qquad (85)$$

$$\text{where}\quad \delta_{Q\ddot{o}}=\frac{(k-1)!m!}{(m+k-1)!}\ (\text{see (79)}),$$

$$\left\{\begin{array}{c}\text{and } j=1\to n,\ k=4,\ m\ge\left[(k+1)=5\right]\\ m=5,6,7,\dots\quad \text{and } \overset{(0)}{E_A^{(4)}}=E_A^{(4)}\end{array}\right\}.$$

Acceleration energies of first, second, third and fourth order are defined in papers [10] – [18]. Author has proposed [13] – [14], the generalized differential equations of higher order in the case of the mechanical systems (MBS), dynamically characterized by sudden and transitory motions:

$$\left\{\begin{array}{c}\overset{(k-1)}{Q^j_{i\ddot{o}}}\left[\bar\theta(t);\dot{\bar\theta}(t);\cdots;\overset{(m)}{\bar\theta}(t)\right]=\dfrac{d^{k-1}}{dt^{k-1}}\left[\displaystyle\sum_{i=1}^{n}M_i\cdot\bar a_{C_i}^T\cdot\frac{\partial\bar r_{C_i}}{\partial q_j}\right.\\ \left.+\displaystyle\sum_{i=1}^{n}\left(I_i^*\cdot\bar\varepsilon_i+\bar\omega_i\times I_i^*\cdot\bar\omega_i\right)^T\cdot\frac{\partial\bar\psi_i}{\partial q_j}\cdot\Delta_j\right]=\\ =\dfrac{(k-1)!\cdot m!}{(m+k-1)!}\cdot\dfrac{\partial}{\partial\overset{(m)}{q_j}}\left\{\left(\displaystyle\sum_{p=1}^{k}\Delta_p\right)\cdot\overset{(m+k)-(2\cdot p+1)}{E_A^{(p)}}\right\}\end{array}\right\},$$

$$\left\{\begin{array}{c}\dfrac{(k-1)!\cdot m!}{(m+k-1)!}\cdot\dfrac{\partial}{\partial\overset{(m)}{q_j}}\left\{\left(\displaystyle\sum_{p=1}^{k}\Delta_p\right)\cdot\overset{(m+k)-(2\cdot p+1)}{E_A^{(p)}}\right\}\\ =\overset{(k-1)}{Q^j_{i\ddot{o}}}\left[\bar\theta(t);\dot{\bar\theta}(t);\cdots;\overset{(m)}{\bar\theta}(t)\right]\end{array}\right\}, \qquad (86)$$

$$\left\{\begin{array}{l}\text{where}\quad E_A^{(p)}=E_A^{(p)}\left[\bar\theta(t);\dot{\bar\theta}(t);\cdots;\overset{(p+1)}{\bar\theta}(t)\right]\\ \text{and}\quad\left(\displaystyle\sum_{p=1}^{k}\Delta_p\right)=\displaystyle\sum_{p=1}^{k}\left[\frac{p\cdot(p+1)}{2}-\delta_p\right]\end{array}\right\}. \qquad (87)$$

The necessary conditions in (86) are following:

$$\left\{\begin{array}{l}p=1\to k\ ;\ \delta_p=\{\{0;p=1\};\{1;p>1\}\}\\ \text{and}\quad k\ge1;\quad k=\{1;2;3;4;5;\dots\}\\ m\ge(k+1);\quad m=\{2;3;4;5;\dots\}\end{array}\right\}. \qquad (88)$$

Considering (58), (73), (76) and (80), besides generalized inertia forces (72), (75), (78), (86), dynamics equations of higher order also contain the generalized gravitational and manipulating forces (64) and (65). According to [6] – [17], generalized gravitational forces of higher order are also determined with the next expression:

$$\overset{(k-1)}{Q^j_g}(t)=\sum_{m=1}^{k}\frac{(k-2)!}{(m-1)!\left[k-(m-1)\right]!}\cdot{}^0J_i\left[\overset{(m-1)}{\bar\theta}(t)\right]\cdot{}^0\overset{[k-m]}{\ddot{o}}{}_{x_i}=$$

$$\left\{\begin{array}{c}=\dfrac{(k-1)!\cdot m!}{(m+k-1)!}\cdot\dfrac{\partial}{\partial\overset{(m)}{q_j}}\displaystyle\sum_{i=1}^{n}\left[M_i\cdot\bar g^T\cdot\overset{(m+k-1)}{\bar r_{C_i}}+\right.\\ \left.+\bar r_{C_i}\times M_i\cdot\bar g^T\cdot\overset{(m+k-1)}{\bar\psi_i}\cdot\Delta_j\right]+\displaystyle\sum_{i=1}^{n}\overset{(k-1)}{\bar r_{C_i}}\times M_i\cdot\bar g^T\cdot\left[\frac{\partial\overset{(m)}{\bar\psi_i}}{\partial\overset{(m)}{q_j}}\cdot\Delta_j\right]+\end{array}\right\}$$

$$+\delta_{kk}\cdot\left\{\delta_{Qg}\cdot\frac{\partial}{\partial\overset{(m)}{q_j}}\sum_{i=1}^{n}\left[\overset{(k-2)}{\bar r_{C_i}}\times M_i\cdot\bar g^T\cdot\overset{(m+k-2)}{\bar\psi_i}\cdot\Delta_j\right]\right\}, \qquad (89)$$

$$\text{where}\quad\delta_{Qg}=(k-1)\cdot\frac{(k-2)!\cdot m!}{(m+k-2)!},$$

$$\left\{\begin{array}{c}\text{where } p\ge1,\ k\ge1,\ \{p;k\}=\{1;2;3;4;5;\dots\},\\ m\ge(k+1);\quad m=\{2;3;4;5;\dots\}\\ p\ge1;\ \delta_{kk}=\left[(0,\text{ for } k\le2);\ (1,\text{ for } k>2)\right]\\ \bar g=\tau\cdot g\cdot\bar k_0,\quad \tau=\mp\bar k_0^T\cdot\bar k_g\\ \text{and } \bar k_g={}^0\bar g/\left|{}^0\bar g\right|,\ \bar k_0-\text{vertical unit vector}\in\{0\}\end{array}\right\}.$$

As a result, the advanced dynamics equations of higher order, defined by generalized driving forces from MBS are written as below follows:

$$\left\{\begin{array}{c}\overset{(k-1)}{Q^i_m}\left[\bar\theta(t);\dot{\bar\theta}(t);\cdots;\overset{(m)}{\bar\theta}(t)\right]=\\ =\Delta_m^2\cdot\left[\Delta_\theta\cdot\overset{(k-1)}{Q^i_{i\ddot{o}}}(t)+\overset{(k-1)}{Q^i_g}(t)\right]+\\ +(-1)^{\Delta_m}\cdot\dfrac{1-\Delta_m}{1+3\cdot\Delta_m}\cdot\overset{(k-1)}{Q^i_{SU}}(t)\end{array}\right\}. \qquad (90)$$

Generalized differential equations (86) contain acceleration energies of order $(p\ge1)$. Using the aspects from Fig.3, starting equation shows as:

$$\overset{(k-1)}{E_A^{(p)}}\left[\bar\theta(t);\dot{\bar\theta}(t);\cdots;\overset{(p+k)}{\bar\theta}(t)\right]= \qquad (91)$$

$$\left\{\begin{array}{c}=\dfrac{1}{2}\cdot\displaystyle\sum_{i=1}^{n}Trace\left\{{}^0_i\overset{(p+k)}{[R]}\cdot\left[\int{}^i\bar r_i^*\cdot{}^i\bar r_i^{*T}\cdot dm+\right.\right.\\ \left.\left.+{}^i\bar r_{C_i}\cdot{}^i\bar r_{C_i}^T\cdot\int dm\right]\cdot{}^0_i\overset{(p+k)}{[R]}{}^T\right\}+\\ +\dfrac{1}{2}\cdot\displaystyle\sum_{i=1}^{n}Trace\left\{\frac{d^{k-1}}{dt^{k-1}}\left[\overset{(p+1)}{\bar p_i}\cdot\overset{(p+1)}{\bar p_i^T}\right]\right\}\cdot\int dm=\\ \dfrac{1}{2}\cdot\displaystyle\sum_{i=1}^{n}Trace\frac{d^{k-1}}{dt^{k-1}}\left\{{}^0_i\overset{(p+1)}{[R]}\cdot\left[{}^iI_{pi}^*+M_i\cdot{}^i\bar r_{C_i}\cdot{}^i\bar r_{C_i}^T\right]\cdot{}^0_i\overset{(p+1)}{[R]}{}^T\right\}\\ +\dfrac{1}{2}\cdot\displaystyle\sum_{i=1}^{n}Trace\left\{\frac{d^{k-1}}{dt^{k-1}}\left[\overset{(p+1)}{\bar p_i}\cdot\overset{(p+1)}{\bar p_i^T}\right]\right\}\cdot M_i\end{array}\right\},$$

$$\left\{\begin{array}{l} \text{where } p \geq 1,\ \ k \geq 1,\ \{p;\ k\} = \{1;2;3;4;5;.....\}, \\ \text{and } \ \overset{(0)}{E}{}_A^{(p)}\left[\bar{\theta}(t);\dot{\bar{\theta}}(t);\cdots;\overset{(p+k)}{\bar{\theta}}(t)\right] = \\ = E_A^{(p)}\left[\bar{\theta}(t);\dot{\bar{\theta}}(t);\cdots;\overset{(p+1)}{\bar{\theta}}(t)\right] \end{array}\right\}. \quad (92)$$

The author has demonstrated in various papers, for example [10]-[18], generalized expressions in the explicit and matrix form, for acceleration energies of various orders, considering general motion of the multibody rigid systems (MBS).

6. CONCLUSIONS

The currently paper was devoted especially to presentation a few essential reformulations and new formulations concerning some notions from advanced kinematics and dynamics. These are compulsory included in dynamics equations of higher order, corresponding to the current and sudden motions in the case of multibody systems (MBS), for example mechanical robot structure.

So, unlike the classical models the author has presented in first section of paper reformulations and new formulations regarding the parameters of advanced kinematics. So, the time derivatives of higher orders were applied, concerning linear and angular accelerations of higher order. In this section, important differential properties have been developed concerning position of the mass center and orientation vector. They are also used for determine the same linear and angular accelerations of higher order above mentioned. According to author researches, the parameters of advanced kinematics have been developed as time functions with the polynomial interpolating functions of higher order, defined in this section. The main objective of second section was a few reformulations of the fundamental theorems, in consonance with the general motion of MBS: motion theorem of the mass center (momentum theorem), theorem of the angular momentum and theorem of the kinetic energy. After a few transformations was obtained the differential generalized principle in analytical dynamics.

Unlike the classical approaches, in the third sections of this paper, the author presents formulations concerning generalized active and inertia forces. It observed that they have unique character namely they are mathematically identical as form of expression. These aspects have important advantage, in the establishment of the dynamics equations, corresponding to every kinetic ensemble from mechanical system.

In the fourth sections of this paper, the author presented the expressions for the generalized inertia forces of higher order. They are included in dynamics equations, whose central functions are the acceleration energies of various orders. But, the fundamental aspect of the fourth section consist in the fact that author has proposed the generalized differential equations of higher order for the sudden and transitory motions of MBS.

7. REFERENCES

[1] Appell, P., *Sur une forme générale des equations de la dynamique*, Paris, 1899.

[2] Appell, P., *Traité de mécanique rationnelle*, Paris, 1903.

[3] Fu, K., et al, *Robotics: Control, Sensing, Vision and Intelligence*, McGraw-Hill Book Co., International Edition, 1987

[4] Negrean, I., *Robotics - Kinematic and Dynamic Modelling*, EDP-R.A., Bucureşti, ISBN 973-30-5958-7, 1998.

[5] Negrean, I., Negrean, D. C., *Matrix Exponentials to Robot Kinematics,* 17th International Conference on CAD/CAM, Robotics and Factories of the Future, CARS&FOF 2001, Durban South Africa, 2001, Vol.2, pp. 1250-1257, 32 rel., 4 ref.

[6] Negrean, I., Negrean, D. C., *The Acceleration Energy to Robot Dynamics,* International Conference on Automation, Quality and Testing, Robotics, AQTR 2002, May 23-25, Cluj-Napoca, Tome II, pp. 59-64

[7] Negrean I., *Mecanică avansată în Robotică*, ISBN 978-973-662-420-9, UT Press, Cluj-Napoca, 2008.

[8] Negrean I., *Mecanică, Teorie şi Aplicaţii*, Editura UT PRESS, Cluj – Napoca, ISBN 978-973-662-523-7, 2012.

[9] Negrean, I., *Formulations in Advanced Dynamics of Mechanical Systems*, The 11 th IFTOMM International Symposium on Science of Mechanisms and Machines, Mechanisms and Machine Science 18, DOI:10.1007/978-3-319-01845-4-19 Springer International Publishing Switzerland, pp. 185-195.

[10] Negrean, I., *Energies of Acceleration in Advanced Robotics Dynamics,* Applied Mechanics and Materials, ISSN: 1662-7482, vol 762, pp 67-73 Submitted: 2014-08-05 ©(2015)TransTech Publications Switzerland DOI:10.4028/www.scientific.net/AMM.762.67, Revised:2014-11-16.
[11] Negrean I., *New Formulations on Acceleration Energy in Analytical Dynamics,* Applied Mechanics and Materials, vol. 823 pp 43-48 © (2016) TransTech Publications Switzerland Revised: 2015-09, Doi: 10.4028 /www.scientific.net/AMM.823.43.
[12] Negrean I., *New Formulations on Motion Equations in Analytical Dynamics,* Applied Mechanics and Materials, vol. 823 (2016), pp 49-54 © (2016) TransTech Publications Switzerland Revised: 2015-09-09, DOI: 10.4028/www.scientific.net/AMM.823.49.
[13] Negrean I., *Formulations on the Advanced Notions in analytical Dynamics of Mechanical Systems,* published in IJERM, ISSN: 2349-2058, Volume-03, Issue-04, pp 123-130, 2016.
[14] Negrean I., *Advanced Notions and Differential Principles of Motion in Analytical Dynamics,* Journal of Engineering Sciences and Innovation, Technical Sciences Academy of Romania, Vol. 1/2016, Issue 1, pp. 49-72, ISSN 2537-320X.
[15] Negrean I., *New Formulations in Analytical Dynamics of Systems*, Acta Technica Napocensis, Series: Applied Mathematics, Mechanics, and Engineering, Vol.60, Issue I, March 2017, pp. 49 – 56.
[16] Negrean I., *Mass Distribution in Analytical Dynamics of Systems*, Acta Technica Napocensis, Series: Applied Mathematics, Mechanics, and Engineering, Vol.60, Issue II, June 2017, pp. 175 – 184.
[17] Negrean I., *Generalized Forces in Analytical Dynamics of Systems*, Acta Technica Napocensis, Series: Applied Mathematics, Mechanics, and Engineering, Vol.60, Issue III, September 2017, pp.
[18] Negrean I., *Advanced Notions in Analytical Dynamics of Systems*, Acta Technica Napocensis, Series: Applied Mathematics, Mechanics, and Engineering, Vol.60, Issue IV, October 2017, pp.
[19] Pars L.A., *A Treatise on Analytical Dynamics*, Heinemann, London, (2007), Volume I, pp. 1-122.
[20] Ursu – Fischer, N., *Elemente de Mecanică Analitică*, Casa Cărții de Știință, 2015.

ECUAȚII AVANSATE ÎN DINAMICA ANALITICĂ A SISTEMELOR

Studiul dinamic al mișcărilor curente și rapide ale sistemelor multicorp (MBS), spre exemplu structurile mecanice de roboți seriali și în conformitate cu principiile diferențiale specifice dinamicii analitice a sistemelor, se bazează pe noțiunile avansate, cum sunt: forțele generalizate, energia cinetică, energiile de accelerații de diferite ordine și derivatele absolute în raport cu timpul a acestora de ordin superior. Noțiunile avansate sunt dezvoltate în conexiune directă cu variabilele generalizate, de asemenea, denumite parametrii independenți corespunzători sistemelor mecanice olonome. Dar, sub aspect mecanic, ecuațiile avansate ale dinamicii conțin pe de o parte noțiuni avansate din cinematică și transformările diferențiale ale acestora, iar pe de altă parte proprietățile maselor și forțele generalizate. În special pe baza cercetărilor autorului, în această lucrare se vor prezenta reformulări și formulări noi cu privire la parametrii cinematicii avansate, precum și funcțiile polinomiale de interpolare de ordin superior. În continuarea lucrării se vor prezenta reformulări asupra teoremelor fundamentale ale dinamicii și principiul diferențial generalizat al dinamicii analitice. Dar, aspectul fundamental al acestei lucrări va consta în faptul că autorul lucrării va propune ecuațiile diferențiale generalizate de ordin superior pentru mișcările rapide și tranzitorii. Aceste ecuații conțin energiile de accelerații de ordin superior în forma matematică generalizată.

Iuliu NEGREAN Professor Ph.D., Member of the Academy of Technical Sciences of Romania, Director of Department of Mechanical Systems Engineering, Technical University of Cluj-Napoca, iuliu.negrean@mep.utcluj.ro, http://users.utcluj.ro/~inegrean, Office Phone 0264/401616.