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SIMULTANEOUS MULTI – COLLISION OF A RIGID BODY WITHOUT FRICTION

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Abstract: In our paper we discuss the collision without friction of a rigid body with many obstacles at the same time. The problem is solved using the notion of inertance. We deduced the expressions of the impulse at any contact point, distribution of velocities after the collision, energy of lost velocities and variation of kinetic energy. The obtained formulae are general and written in matrix form. An example highlights the theory.

Key words: multi-collision, simultaneous, inertance, coefficients of restitution, impulse, velocities, energy of lost velocities, variation of kinetic energy.

1. INTRODUCTION

The collision of two rigid bodies or a rigid body by an impulse is studied in many papers and monographs [1-18, 22-32].

The study is considered only for one point of contact and one impulse, no matter the authors study the phenomenon with or without friction.

The authors of this paper also studied the collision without [19] and with [20] friction using the notion of inertance.

In the present paper we generalize the results deduced in [19] for the simultaneous multi-collision of a rigid body.

2. THEORETICAL ASPECTS

Referring to Fig. 1, in which the rigid body is collided by a single impulse \mathbf{P} at the point A , and using the notations:

– $Cxyz$ – the system of the principal central axes of inertia;

– \mathbf{n} – the unit vector of the impulse \mathbf{P} ;

– a, b, c , and d, e, f , respectively – the projections of the vectors \mathbf{n} , and $\mathbf{CA} \times \mathbf{n}$, respectively, onto the axes of the system $Cxyz$;

– m – the mass of the rigid body;

– J_{Cx}, J_{Cy}, J_{Cz} – the principal central moments of inertia;

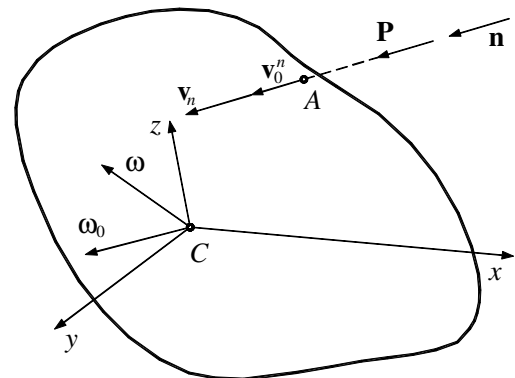


Fig. 1. Collision of a rigid body by an impulse

– $[\mathbf{M}_C]$ – the matrix of inertia relative to the principal central axes of inertia

$$[\mathbf{M}_C] = \begin{bmatrix} 0 & 0 & 0 & m & 0 & 0 \\ 0 & 0 & 0 & 0 & m & 0 \\ 0 & 0 & 0 & 0 & 0 & m \\ J_{Cx} & 0 & 0 & 0 & 0 & 0 \\ 0 & J_{Cy} & 0 & 0 & 0 & 0 \\ 0 & 0 & J_{Cz} & 0 & 0 & 0 \end{bmatrix}, \quad (1)$$

$$[\mathbf{M}_C]^{-1} = \begin{bmatrix} 0 & 0 & 0 & J_{Cx}^{-1} & 0 & 0 \\ 0 & 0 & 0 & 0 & J_{Cy}^{-1} & 0 \\ 0 & 0 & 0 & 0 & 0 & J_{Cz}^{-1} \\ m^{-1} & 0 & 0 & 0 & 0 & 0 \\ 0 & m^{-1} & 0 & 0 & 0 & 0 \\ 0 & 0 & m^{-1} & 0 & 0 & 0 \end{bmatrix}; \quad (2)$$

– \mathbf{v}^0 , \mathbf{v} – the velocities of the point C before and after the application of the impulse \mathbf{P} ;

– $v_x^0, v_y^0, v_z^0, v_x, v_y, v_z, \omega_x^0, \omega_y^0, \omega_z^0$, and $\omega_x, \omega_y, \omega_z$, respectively – the projections onto the coordinate axes of the vectors $\mathbf{v}^0, \mathbf{v}, \boldsymbol{\omega}^0$, and $\boldsymbol{\omega}$, respectively;

– $\{\mathbf{N}\}$ – the column matrix

$$\{\mathbf{N}\} = [a \ b \ c \ d \ e \ f]^T; \quad (3)$$

– $\{\boldsymbol{\eta}\}$ – the square matrix

$$\{\boldsymbol{\eta}\} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}; \quad (4)$$

– $\{\mathbf{v}\}, \{\mathbf{v}^0\}$ – the column matrices

$$\{\mathbf{v}\} = [\omega_x \ \omega_y \ \omega_z \ v_x \ v_y \ v_z]^T, \quad (5)$$

$$\{\mathbf{v}^0\} = [\omega_x^0 \ \omega_y^0 \ \omega_z^0 \ v_x^0 \ v_y^0 \ v_z^0]^T; \quad (6)$$

– g – the inertance of the rigid body

$$g = \{\mathbf{N}\}^T \{\boldsymbol{\eta}\} [\mathbf{M}_C]^{-1} \{\mathbf{N}\}, \quad (7)$$

$$g = \frac{1}{m} + \frac{d^2}{J_{Cx}} + \frac{e^2}{J_{Cy}} + \frac{f^2}{J_{Cz}}, \quad (8)$$

one gets the following expressions [19]

$$\{\mathbf{v}\} = \{\mathbf{v}^0\} - P[\mathbf{M}_C]^{-1} \{\mathbf{N}\}, \quad (9)$$

$$E_p = \frac{1}{2} P^2 g, \quad (10)$$

$$\Delta E_c = -\frac{1}{2} P^2 g - P v_n^0, \quad (11)$$

where E_p is the energy of the lost velocities, while ΔE_c is the variation of the kinetic energy.

3. SIMULTANEOUS MULTI-COLLISION OF A RIGID BODY

We consider now the Figure 2.

A generic collision with an obstacle takes place at the point A_i , situated on the frontier of the rigid body.

Let us assume that the frontier is given by the implicit equation

$$f(x, y, z) = 0, \quad (12)$$

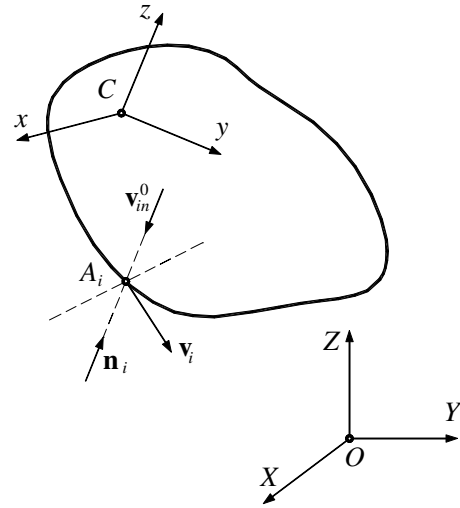


Fig. 2. Simultaneous multi-collision

which implies that the unit vector \mathbf{n}_i has the expression

$$\mathbf{n}_i = -\frac{\nabla f|_{A_i}}{\|\nabla f|_{A_i}\|}. \quad (13)$$

For each point A_i we obtain a different column matrix $\{\mathbf{N}_i\}$,

$$\{\mathbf{N}_i\} = [a_i \ b_i \ c_i \ d_i \ e_i \ f_i]^T, \quad (14)$$

where a_i, b_i, c_i are the projections of the unit vector \mathbf{n}_i onto the principal central axes of inertia, while d_i, e_i, f_i are the projections of the vector product $\mathbf{C}A_i \times \mathbf{n}_i$ onto the same axes,

$$\mathbf{n}_i = a_i \mathbf{i} + b_i \mathbf{j} + c_i \mathbf{k}, \quad (15)$$

$$\mathbf{C}A_i \times \mathbf{n}_i = d_i \mathbf{i} + e_i \mathbf{j} + f_i \mathbf{k}, \quad (16)$$

in which \mathbf{i}, \mathbf{j} , and \mathbf{k} are the unit vectors of the axes Cx, Cy and Cz , respectively.

Let us assume that the coefficient of restitution at the point A_i has the value .

Denoting now the matrices [21]

$$\{\mathbf{R}_i\} = [X_{A_i} \ Y_{A_i} \ Z_{A_i}]^T, \quad (17)$$

$$\{\mathbf{R}_C\} = [X_C \ Y_C \ Z_C]^T, \quad (18)$$

$$[\boldsymbol{\omega}] = \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix}, \quad (19)$$

$$\{\mathbf{r}_i\} = \begin{bmatrix} 0 & -z_{A_i} & y_{A_i} \\ z_{A_i} & 0 & -x_{A_i} \\ -y_{A_i} & x_{A_i} & 0 \end{bmatrix}, \quad (20)$$

$$[\boldsymbol{\psi}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \psi & -\sin \psi \\ 0 & \sin \psi & \cos \psi \end{bmatrix}, \quad (21)$$

$$[\boldsymbol{\theta}] = \begin{bmatrix} \cos \theta & 0 & \sin \theta \\ 0 & 1 & 0 \\ -\sin \theta & 0 & \cos \theta \end{bmatrix}, \quad (22)$$

$$[\boldsymbol{\phi}] = \begin{bmatrix} \cos \varphi & -\sin \varphi & 0 \\ \sin \varphi & \cos \varphi & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad (23)$$

$$[\mathbf{A}] = [\boldsymbol{\psi}][\boldsymbol{\theta}][\boldsymbol{\phi}], \quad (24)$$

$$\{\mathbf{u}_\psi\} = [1 \ 0 \ 0]^T, \quad (25)$$

$$\{\mathbf{u}_\theta\} = [0 \ 1 \ 0]^T, \quad (26)$$

$$\{\mathbf{u}_\varphi\} = [0 \ 0 \ 1]^T, \quad (27)$$

$$[\mathbf{Q}] = [\boldsymbol{\phi}]^T [\boldsymbol{\theta}]^T \{\mathbf{u}_\psi\} \{\mathbf{u}_\theta\} \{\mathbf{u}_\varphi\}, \quad (28)$$

where $OXYZ$ is a fixed reference system, one gets

$$\{\dot{\mathbf{R}}\} = \{\dot{\mathbf{R}}_C\} + [\mathbf{A}][\mathbf{r}_i]^T [\mathbf{Q}]\{\boldsymbol{\beta}\}, \quad (29)$$

with

$$\{\boldsymbol{\beta}\} = [\psi \ \theta \ \varphi]^T. \quad (30)$$

Moreover, writing

$$\{\dot{\mathbf{R}}\} = [v_x \ v_y \ v_z]^T \quad (31)$$

we obtain the normal velocity at the point A_i ,

$$v_{in} = \{\dot{\mathbf{R}}\}^T \{\mathbf{n}_i\}, \quad (32)$$

with

$$\{\mathbf{n}_i\} = [\mathbf{A}][a_i \ b_i \ c_i]^T. \quad (33)$$

It results

$$\{\mathbf{v}\} = \{\mathbf{v}_0\} + \sum \frac{(1+k_i)v_{in}^0}{g_i} [\mathbf{M}_C]^{-1} \{\mathbf{N}_i\}, \quad (34)$$

$$P_i = -\frac{(1+k_i)v_{in}^0}{g_i}, \quad (35)$$

$$E_p = \frac{1}{2} m(\mathbf{v} - \mathbf{v}_0)^2 + \frac{1}{2} [J_{Cx}(\omega_x - \omega_x^0)^2 + J_{Cy}(\omega_y - \omega_y^0)^2 + J_{Cz}(\omega_z - \omega_z^0)^2], \quad (36)$$

$$\Delta E_c = \frac{1}{2} m(\mathbf{v}_0)^2 + \frac{1}{2} J_{Cx}(\omega_x^0)^2 + \frac{1}{2} J_{Cy}(\omega_y^0)^2 + \frac{1}{2} J_{Cz}(\omega_z^0)^2 - \frac{1}{2} mv^2 - \frac{1}{2} J_{Cx}m_x^2 - \frac{1}{2} J_{Cy}m_y^2 - \frac{1}{2} J_{Cz}m_z^2. \quad (37)$$

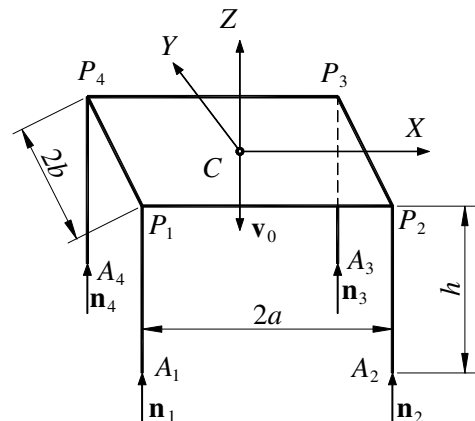


Fig. 3. Example

4. EXAMPLE

The rigid body drawn in Fig. 3 consists in a rectangular shell $P_1P_2P_3P_4$ of dimensions $2a = 2 \text{ m}$, $2b = 4 \text{ m}$ and mass $m = 50 \text{ kg}$ and four equal feet P_iA_i of length h and negligible mass. The rigid body collides four obstacles situated at the points A_i , the coefficients of restitution being $k_i = 0,5 + 0,1i$, $i = \overline{1,4}$. The initial velocity is $v_0 = 1 \text{ m/s}$ and it is orientated vertical descendant.

One asks for the velocity of the shell after collision, the energy of the lost velocities and the variation of the kinetic energy.

Solution: We successively obtain

$$J_x = \frac{mb^2}{3} = \frac{200}{3} \text{ kgm}^2, \quad (38)$$

$$J_y = \frac{ma^2}{3} = \frac{50}{3} \text{ kgm}^2,$$

$$J_z = \frac{m(a^2 + b^2)}{3} = \frac{250}{3} \text{ kgm}^2,$$

$$[\mathbf{M}_C] = \begin{bmatrix} 0 & 0 & 0 & 50 & 0 & 0 \\ 0 & 0 & 0 & 0 & 50 & 0 \\ 0 & 0 & 0 & 0 & 0 & 50 \\ \frac{200}{3} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{50}{3} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{250}{3} & 0 & 0 & 0 \end{bmatrix},$$

$$[\mathbf{M}_C]^{-1} = \begin{bmatrix} 0 & 0 & 0 & \frac{3}{200} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{3}{200} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{3}{200} \\ \frac{1}{50} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{50} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{50} & 0 & 0 & 0 \end{bmatrix}, \quad (39)$$

the coordinates $A_1(-1, -1, -h)$, $A_2(1, -2, -h)$, $A_3(1, 2, -h)$, $A_4(-1, 2, -h)$,

$$\{\mathbf{N}_1\} = [0 \ 0 \ 1 \ -2 \ 1 \ 0]^T, \quad (40)$$

$$\{\mathbf{N}_2\} = [0 \ 0 \ 1 \ -2 \ -1 \ 0]^T, \quad (41)$$

$$\{\mathbf{N}_3\} = [0 \ 0 \ 1 \ 2 \ -1 \ 0]^T, \quad (42)$$

$$\{\mathbf{N}_4\} = [0 \ 0 \ 1 \ 2 \ 1 \ 0]^T, \quad (43)$$

$$g_i = g$$

$$= \frac{1}{m} + \frac{d_i^2}{J_x} + \frac{e_i^2}{J_y} + \frac{f_i^2}{J_z} = \frac{7}{50} \text{ kg}^{-1}, \quad (44)$$

$$i = \overline{1, 4},$$

$$v_{in}^0 = -1 \text{ m/s}, \quad (45)$$

$$\{\mathbf{v}\} = \left\{ \frac{3}{35} \ 0 \ 0 \ 0 \ 0 \ 0 \right\}^T, \quad (46)$$

$$P_i = -\frac{(1+k_i)v_{in}^0}{g}, \quad P_1 = \frac{80}{7} \text{ Ns}, \quad (47)$$

$$P_2 = \frac{85}{7} \text{ Ns}, \quad P_3 = \frac{90}{7} \text{ Ns}, \quad P_4 = \frac{95}{7} \text{ Ns},$$

$$E_p = \frac{1}{2} \cdot 50 \cdot 1^2 + \frac{1}{2} \cdot \frac{200}{3} \cdot \left(\frac{3}{35}\right)^2,$$

$$= 25 \frac{12}{49} \text{ J} = 25,2449 \text{ J}, \quad (48)$$

$$\Delta E_c = \frac{1}{2} \cdot 50 \cdot 1^2 - \frac{1}{2} \cdot \frac{200}{3} \cdot \left(\frac{3}{35}\right)^2$$

$$= 22 \frac{39}{49} \text{ J} = 22,7959 \text{ J}. \quad (49)$$

The motion of the rigid solid before the multi-collision consists in a simple translation along the z -axis, while the motion of the rigid solid after the multi-collision consists in a rotation around the x -axis.

The example may be also solved starting from the general theorems in the case of collisions: theorem of momentum and theorem of moment of momentum.

5. CONCLUSION

We presented a general procedure for the determination of the parameters of a simultaneous multi-collision of a rigid body. The obtained formulae are given in matrix form and represent the generalization of the formulae deduced in a previous paper. The only exceptions are for the energy of the lost velocities and the variation of the kinetic energy, when the reader has to apply the definition.

The theory is detailed with the aid of a numerical example.

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CIOCNIREA MULTIPLĂ SIMULTANĂ A UNUI RIGID FĂRĂ FRECARÉ

Abstract: În lucrarea de față discutăm ciocnirea fără frecare a unui rigid simultan cu mai multe obstacole. Problema este rezolvată folosind noțiunea de inertanță. Se deduc expresiile percuției în fiecare punct de contact, distribuția de viteze după ciocnire, energia vitezelor pierdute și pierderea de energie cinetică. Formulele obținute sunt generale și scrise sub formă matriceală. Un exemplu evidențiază teoria.

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