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HIGHER-ORDER ACCELERATIONS ON RIGID BODIES MOTIONS. A TENSORS AND DUAL LIE ALGEBRA APPROACH

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Abstract: *This paper extends the results from velocities and accelerations fields of rigid bodies motion to higher-order accelerations. Using the tensor calculus and the dual numbers algebra, a computing method for studying the higher order acceleration field properties is proposed in the case of the general motion. The vector and tensor invariants in the distribution of the n-th order acceleration field are highlighted. For the case of the spatial kinematics chains, an equation that allows the determination of the n-th order field accelerations are given, using a Brockett-like formulas specific to the dual algebra. The results are free of coordinate and in a closed form. In particular cases the properties for velocities, accelerations, jerks, and hyper-jerks fields are given. This approach uses the isomorphism between the Lie group of the rigid displacements* $S\mathbb{E}3$ *and the Lie group of the orthogonal dual tensors.*

Key words: *higher-order acceleration, velocity tensor, acceleration tensor, Lie group, dual-number algebra*

1. INTRODUCTION

The problem of determining the tensor and the vector invariants that describe the vector field of the n^{th} order accelerations is generally avoided in rigid body kinematics.

This paper extends the discussion from velocities and accelerations to n^{th} order accelerations. It is proved that the tensors that describe the field of n^{th} order accelerations may be determined from direct computations, as well as the vector invariants. It is sufficient to know some cinematic characteristics of three or four points of the rigid. Two computation methods are highlighted:

1. Determining the n^{th} order accelerations tensor and vector invariant when the n^{th} order absolute accelerations of four non-coplanar points of the rigid are known.

2. Determining the n^{th} order accelerations tensor and vector invariant when the k^{th} order absolute accelerations and the absolute positions of three non-collinear points of the rigid body are known, $1 \leq k \leq n$.

Using the dual numbers algebra, a computing method for studying the n-th order acceleration field properties is proposed for the case of the general motion of the rigid body and of the systems of rigid bodies.

This approach uses the isomorphism between the Lie group of the rigid displacements $S\mathbb{E}_3$ and the Lie group of the orthogonal dual tensors $\underline{S\mathbb{O}}_3$.

2. THEORETICAL CONSIDERATION ON RIGID BODY MOTION

The general framework of this paper is a rigid body that moves with respect to a fixed reference frame $\{\mathfrak{R}^0\}$. Consider another reference frame $\{\mathfrak{R}\}$ originated in a point Q that moves together with the rigid body. Let $\boldsymbol{\rho}_Q$ denote the position vector of point Q with respect to frame $\{\mathfrak{R}^0\}$, $\mathbf{v}_Q$

its absolute velocity and $\mathbf{a}_Q$ its absolute acceleration.

The vector parametric equation of motion is:

$$\boldsymbol{\rho} = \boldsymbol{\rho}_Q + \mathbf{Rr} \quad (1)$$

where $\boldsymbol{\rho}$ represents the absolute position of a generic point P of the rigid body with respect to $\{\mathfrak{R}^0\}$ and $\mathbf{R} = \mathbf{R}(t)$ is an orthogonal proper tensor function in $\mathbf{SO}_3^{\mathbb{R}}$. Vector $\mathbf{r}$ is constant and it represents the relative position vector of the arbitrary point P with respect to $\{\mathfrak{R}\}$.

The results of this section succinctly present the velocity and acceleration vector fields in rigid body motion. These results lead to the generalization presented in the next section.

With the denotations that were introduced, the vector fields of velocities and accelerations are described, respectively, by:

$$\begin{cases} \mathbf{v} - \mathbf{v}_Q = \dot{\mathbf{R}}\mathbf{R}^T(\boldsymbol{\rho} - \boldsymbol{\rho}_Q) \\ \mathbf{a} - \mathbf{a}_Q = \ddot{\mathbf{R}}\mathbf{R}^T(\boldsymbol{\rho} - \boldsymbol{\rho}_Q) \end{cases} \quad (2)$$

Tensors:

$$\begin{cases} \boldsymbol{\Phi}_1 = \dot{\mathbf{R}}\mathbf{R}^T \\ \boldsymbol{\Phi}_2 = \ddot{\mathbf{R}}\mathbf{R}^T \end{cases} \quad (3)$$

represent the **velocity tensor** respectively the **acceleration tensor** [1]. Tensor $\boldsymbol{\Phi}_1 = \tilde{\boldsymbol{\omega}} \in \mathbf{so}_3^{\mathbb{R}}$ is the skew-symmetric tensor associated to the instantaneous angular velocity $\boldsymbol{\omega} \in \mathbf{V}_3^{\mathbb{R}}$. Tensor $\boldsymbol{\Phi}_2 = \tilde{\boldsymbol{\omega}}^2 + \tilde{\boldsymbol{\varepsilon}}$, where $\boldsymbol{\varepsilon} = \dot{\boldsymbol{\omega}}$ is the instantaneous angular acceleration of the rigid body. One may remark that vectors:

$$\begin{cases} \mathbf{a}_1 = \mathbf{v} - \boldsymbol{\Phi}_1\boldsymbol{\rho} = \mathbf{v}_Q - \boldsymbol{\Phi}_1\boldsymbol{\rho}_Q \\ \mathbf{a}_2 = \mathbf{a} - \boldsymbol{\Phi}_2\boldsymbol{\rho} = \mathbf{a}_Q - \boldsymbol{\Phi}_2\boldsymbol{\rho}_Q \end{cases} \quad (4)$$

do not depend on the choice of point P of the rigid body [1], [8], [9]. They are called the **velocity invariant** respectively the **acceleration invariant** (at a given moment of time).

2.1. The velocity field in rigid body motion

It is described by:

$$\mathbf{v} - \mathbf{v}_Q = \boldsymbol{\Phi}_1(\boldsymbol{\rho} - \boldsymbol{\rho}_Q) \quad (5)$$

The instantaneous angular velocity $\boldsymbol{\omega}$ of the rigid body may be determined as $\boldsymbol{\omega} = \mathrm{vect}\boldsymbol{\Phi}_1$. The major property that may be highlighted from eq (4) is that the velocity of a given point of the rigid may be computed when knowing the velocity tensor $\boldsymbol{\Phi}_1$ and the velocity invariant $\mathbf{a}_1$:

$$\mathbf{v} = \mathbf{a}_1 + \boldsymbol{\Phi}_1\boldsymbol{\rho} \quad (6)$$

2.2. The acceleration field in rigid body motion

It is described by:

$$\mathbf{a} - \mathbf{a}_Q = \boldsymbol{\Phi}_2(\boldsymbol{\rho} - \boldsymbol{\rho}_Q) \quad (7)$$

The absolute acceleration of a given point of the rigid body may be computed when knowing the acceleration tensor $\boldsymbol{\Phi}_2$ and the acceleration invariant $\mathbf{a}_2$:

$$\mathbf{a} = \mathbf{a}_2 + \boldsymbol{\Phi}_2\boldsymbol{\rho} \quad (8)$$

The instantaneous angular acceleration of the rigid body may be determined as:

$$\boldsymbol{\varepsilon} = \mathrm{vect}\boldsymbol{\Phi}_2 \quad (9)$$

The determinant of tensor $\boldsymbol{\Phi}_2$ is (see [9]): $\det\boldsymbol{\Phi}_2 = -(\boldsymbol{\omega} \times \boldsymbol{\varepsilon})^2$. It follows that if $\boldsymbol{\omega} \times \boldsymbol{\varepsilon} \neq \mathbf{0}$, then tensor $\boldsymbol{\Phi}_2$ is invertible and its inverse is (see [9]):

$$\boldsymbol{\Phi}_2^{-1} = \frac{1}{(\boldsymbol{\omega} \times \boldsymbol{\varepsilon})^2}\left[\boldsymbol{\varepsilon} \otimes \boldsymbol{\varepsilon} + (\boldsymbol{\omega} \otimes \boldsymbol{\omega})^2 - \widetilde{\boldsymbol{\omega}^2\boldsymbol{\varepsilon}}\right] \quad (10)$$

It follows that, if tensor $\boldsymbol{\Phi}_2$ is non-singular, then for an arbitrary given acceleration $\mathbf{a}$**,** we may find a point of the rigid that has this acceleration. Its absolute position is given by (see also eq. (8)):

$$\boldsymbol{\rho} = \boldsymbol{\Phi}_2^{-1}(\mathbf{a} - \mathbf{a}_2) \quad (11)$$

Particularly, if $\boldsymbol{\Phi}_2$ is non-singular, then there exists a point G of zero acceleration, named the **acceleration center**. Its absolute position vector is given by:

$$\boldsymbol{\rho}_G = -\boldsymbol{\Phi}_2^{-1}\mathbf{a}_2 \quad (12)$$

3. THE VECTOR FIELD OF THE n^{th} ORDER ACCELERATIONS

This section extends some of the previous considerations to the case of the n^{th} order accelerations. We define the n^{th} order acceleration of a point as:

$$\mathbf{a}_{\boldsymbol{\rho}}^{[n]} \stackrel{\text{def}}{=} \frac{d^n}{dt^n}\boldsymbol{\rho}, n \geq 1 \quad (13)$$

For $n = 1$ it represents the velocity and for $n = 2$ the acceleration. By derivation with respect to time successively in eqs (2), it follows that:

$$\mathbf{a}_{\boldsymbol{\rho}}^{[n]} - \mathbf{a}_Q^{[n]} = \mathbf{R}^{(n)}\mathbf{R}^T(\boldsymbol{\rho} - \boldsymbol{\rho}_Q), \text{ where: } \mathbf{R}^{(n)} \stackrel{\text{def}}{=} \frac{d^n}{dt^n}\mathbf{R}, n \geq 1 \quad (14)$$

We define:

$$\boldsymbol{\Phi}_n \stackrel{\text{def}}{=} \mathbf{R}^{(n)}\mathbf{R}^T \quad (15)$$

the $\boldsymbol{n^{th}}$ **order acceleration tensor** in rigid body motion. A vector invariant is highlighted from eq. (14) with the denotation (15). Vector:

$$\mathbf{a}_n = \mathbf{a}_{\boldsymbol{\rho}}^{[n]} - \boldsymbol{\Phi}_n\boldsymbol{\rho} = \mathbf{a}_Q^{[n]} - \boldsymbol{\Phi}_n\boldsymbol{\rho}_Q \qquad (16)$$

does not depend on the choice of the point of the rigid body for which the acceleration $\mathbf{a}^{[n]}$ is computed. Vector $\mathbf{a}_n$ is named the **invariant vector** of the $\boldsymbol{n^{th}}$ **order accelerations**. Then eq. (7) may be generalized as it follows:

$$\mathbf{a}_{\boldsymbol{\rho}}^{[n]} - \mathbf{a}_Q^{[n]} = \boldsymbol{\Phi}_n(\boldsymbol{\rho} - \boldsymbol{\rho}_Q) \qquad (17)$$

The next Theorem gives the fundamental properties of the vector fields of the n^{th} order accelerations.

Theorem 1. *In the rigid body motion, at a moment of time* t*, there exist tensor* $\boldsymbol{\Phi}_n$ *defined by eq. (15) and vector* $\boldsymbol{a}_n$ *such as*:

$$\mathbf{a}_{\boldsymbol{\rho}}^{[n]} - \mathbf{a}_Q^{[n]} = \boldsymbol{\Phi}_n(\boldsymbol{\rho} - \boldsymbol{\rho}_Q)$$

$$\mathbf{a}_n = \mathbf{a}_{\boldsymbol{\rho}}^{[n]} - \boldsymbol{\Phi}_n\boldsymbol{\rho} = \mathbf{a}_Q^{[n]} - \boldsymbol{\Phi}_n\boldsymbol{\rho}_Q$$

for any point P *of the rigid body with the absolute position defined by vector* $\boldsymbol{\rho}$*.*

Remark 1. *Given the absolute position of a point of the rigid body and knowing* $\boldsymbol{\Phi}_n$ *and* $\boldsymbol{a}_n$*, its acceleration is computed from:*

$$\mathbf{a}_{\boldsymbol{\rho}}^{[n]} = \mathbf{a}_n + \boldsymbol{\Phi}_n\boldsymbol{\rho}$$

Remark 2. *Tensor* $\boldsymbol{\Phi}_n$ *and vector* $\boldsymbol{a}_n$ *generalize the notions of velocity / acceleration tensor, respectively, velocity / acceleration invariant. They are fundamental in the study of the vector field of the* n^{th} *order accelerations. The recursive formulas for computing* $\boldsymbol{\Phi}_n$ *and* $\boldsymbol{a}_n$ *are:*

$$\begin{cases}\boldsymbol{\Phi}_{n+1} = \dot{\boldsymbol{\Phi}}_n + \boldsymbol{\Phi}_n\boldsymbol{\Phi}_1 \\ \mathbf{a}_{n+1} = \dot{\mathbf{a}}_n + \boldsymbol{\Phi}_n\mathbf{a}_1\end{cases}, \quad n \geq 1, \qquad (18)$$

where $\boldsymbol{\Phi}_1 = \tilde{\boldsymbol{\omega}}, \mathbf{a}_1 = \mathbf{v}_Q - \boldsymbol{\Phi}_1\boldsymbol{\rho}_Q \stackrel{\text{def}}{=} \mathbf{v}$

Remark 3. *One may remark that from eq. (18) it follows by direct computation:*

$$\begin{cases}\boldsymbol{\Phi}_n = \boldsymbol{\Phi}_{n+1}\boldsymbol{\Phi}_1 + \left(\dfrac{d^{n-1}}{dt^{n-1}}\boldsymbol{\Phi}_1\right) + \\ \displaystyle\sum_{k=1}^{n-2}\left[\frac{d^k}{dt^{n-1}}(\boldsymbol{\Phi}_{n-k-1}\boldsymbol{\Phi}_1)\right] \\ \mathbf{a}_n = \boldsymbol{\Phi}_{n-1}\mathbf{a}_1 + \left(\dfrac{d^{n-1}}{dt^{n-1}}\mathbf{a}_1\right) \\ \displaystyle +\sum_{k=1}^{n-2}\left[\frac{d^k}{dt^{n-1}}(\boldsymbol{\Phi}_{n-k-1}\mathbf{a}_1)\right]\end{cases}, \qquad (19)$$

$n \geq 3.$

Remark 4. *By defining the* n^{th} *order instantaneous angular acceleration of the rigid body* $\boldsymbol{\varepsilon}^{[n]} \stackrel{\text{def}}{=} \frac{d^{n-1}}{dt^{n-1}}\boldsymbol{\omega}$*, it follows from eqs (19) that its associated skew-symmetric tensor may be expressed as* $\tilde{\boldsymbol{\varepsilon}}^{[n]} = \frac{d^{n-1}}{dt^{n-1}}\boldsymbol{\Phi}_1$*. The expression of the* n^{th}*order instantaneous angular acceleration is:*

$$\boldsymbol{\varepsilon}^{[n]} = = vect\left\{\boldsymbol{\Phi}_n - \boldsymbol{\Phi}_{n-1}\boldsymbol{\Phi}_1 - \sum_{k=1}^{n-2}\left[\frac{d^k}{dt^k}(\boldsymbol{\Phi}_{n-k-1}\boldsymbol{\Phi}_1)\right]\right\}, n \geq 3 \qquad (20)$$

Remark 5. *If tensor* $\boldsymbol{\Phi}_n$ *is invertible, when given a* n^{th} *order acceleration* $\boldsymbol{a}^{[n]}$*, then there exists a unique point of the rigid that has this acceleration. Its absolute position vector is:*

$$\boldsymbol{\rho} = \boldsymbol{\Phi}_n^{-1}(\mathbf{a}^{[n]} - \mathbf{a}_n) \qquad (21)$$

When tensor $\boldsymbol{\Phi}_n$ *is invertible, then there exists a point* G_n *of zero* n^{th} *order acceleration, named the* $\boldsymbol{n^{th}}$ ***order acceleration center****. It has the absolute position vector:*

$$\boldsymbol{\rho}_{G_n} = -\boldsymbol{\Phi}_n^{-1}\mathbf{a}_n$$

3.1. Computing the field of the n^{th} order accelerations by direct measurements

This subsection offers the method for computing numerically tensor $\boldsymbol{\Phi}_n$ and the vector invariant $\mathbf{a}_n$, by direct measurements. Two situations occur.

Method of four points

First, consider four non-coplanar points of the rigid body that have the absolute position vectors $\boldsymbol{\rho}_k, k = \overline{0,3}$ at a given moment of time and the absolute accelerations $\mathbf{a}_k^{[n]}, k = \overline{0,3}$. Consider a Cartesian orthonormate positive oriented system of coordinates associated to the reference frame $\{\Re^0\}$. It follows that:

$$\mathbf{a}_k^{[n]} - \mathbf{a}_0^{[n]} = \boldsymbol{\Phi}_n(\boldsymbol{\rho}_k - \boldsymbol{\rho}_0), k = \overline{1,3} \qquad (22)$$

We denote:

$$\begin{cases}\mathbf{r}_k = \boldsymbol{\rho}_k - \boldsymbol{\rho}_0 \\ \boldsymbol{\alpha}_k^{[n]} = \mathbf{a}_k^{[n]} - \mathbf{a}_0^{[n]},\end{cases} \quad k = \overline{1,3} \qquad (23)$$

the relative positions, respectively the relative n^{th} order accelerations of the last three points with respect to the first one.

Since vectors $\boldsymbol{\rho}_k, k = \overline{0,3}$ are non-coplanar. Eq. (22) may be put in the form:

$$\boldsymbol{\alpha}_k^{[n]} = \boldsymbol{\Phi}_n \mathbf{r}_k, k = \overline{1,3} \quad (24)$$

Since vectors $\mathbf{r}_k, k = \overline{1,3}$ are non-coplanar, then they form a basis $\{B\}$ in $\mathbf{V}_3$. Let $\mathbf{r}^k, k = \overline{1,3}$ be the vectors of the reciprocal basis $\{B^*\}$ associated to $\{B\}$. They are defined by:

$$\mathbf{r}^k = \varepsilon^{kpj} \frac{\mathbf{r}_p \times \mathbf{r}_j}{2(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3)} \quad (25)$$

where ε^{kpj} represent Ricci's permutation symbols, $(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3)$ the triple scalar product of vectors $\mathbf{r}_k, k = \overline{1,3}$. Einstein summation convention is used in eq. (25). From eq. (24) it follows that the n^{th} order acceleration tensor $\boldsymbol{\Phi}_n$ may be expressed as:

$$\boldsymbol{\Phi}_n = \boldsymbol{\alpha}_k^{[n]} \otimes \mathbf{r}^k \quad (26)$$

where $\otimes$ denotes the tensor product of two vectors.

The n^{th} order acceleration invariant may now be computed from:

$$\mathbf{a}_n = \mathbf{a}_0^{[n]} - \boldsymbol{\Phi}_n \boldsymbol{\rho}_0 \quad (27)$$

We introduce the matrices:

$$\begin{cases} \mathbf{A}^{[n]} = \left[\boldsymbol{\alpha}_1^{[n]}, \quad \boldsymbol{\alpha}_2^{[n]}, \quad \boldsymbol{\alpha}_3^{[n]}\right] \\ \quad \mathbf{P} = \left[\boldsymbol{r}_1, \quad \boldsymbol{r}_2, \quad \boldsymbol{r}_3\right] \end{cases} \quad (28)$$

where $\boldsymbol{u}$ represents the 3×1 matrix associated to vector $\mathbf{u}$ with respect to the Cartesian orthonormate positive oriented coordinate system associated to the reference frame.

The matrix form of eq. (26) is:

$$\boldsymbol{\Phi}_n = \mathbf{A}^{[n]} \mathbf{P}^{-1} \quad (29)$$

This procedure allows the computation of $\boldsymbol{\Phi}_n$ and $\mathbf{a}_n$ when knowing the absolute n^{th} order accelerations of four non-coplanar points of the rigid body. An open question is how arbitrary may be these accelerations in order to remain compatible with a rigid motion. This method offers only $\boldsymbol{\Phi}_n$ and $\mathbf{a}_n$ and does not determine the instantaneous angular n^{th} order acceleration.

Method of three points

The second computation method assumes that the absolute positions $\boldsymbol{\rho}_k, k = \overline{0,2}$ of three non-collinear points and the j^{th} order accelerations, $= \overline{1,n}$, are known, $\mathbf{a}_k^{[j]}, k = \overline{0,2}$.

Here the compatibility with a rigid motion may be verified. It also allows computing not only $\boldsymbol{\Phi}_n$ and $\mathbf{a}_n$, but all $\boldsymbol{\Phi}_j$ and $\mathbf{a}_j$, $j = \overline{1,n}$, together with the j^{th} order instantaneous angular accelerations, $j = \overline{1,n}$.

Consider now the relative positions $\mathbf{r}_k, k = \overline{1,2}$ and the relative accelerations $\boldsymbol{\alpha}_k, k = \overline{1,2}$ defined by:

$$\begin{cases} \mathbf{r}_k = \boldsymbol{\rho}_k - \boldsymbol{\rho}_0 \\ \boldsymbol{\alpha}_k^{[n]} = \mathbf{a}_k^{[n]} - \mathbf{a}_0^{[n]} \end{cases}, k = \overline{1,2} \quad (30)$$

We consider a fourth point of the rigid body described by the relative position vector $\mathbf{r}_3$ such as:

$$\mathbf{r}_3 = \mathbf{r}_1 \times \mathbf{r}_2 \quad (31)$$

Since vectors $\mathbf{r}_1, \mathbf{r}_2$ are not collinear, from eq (31) it follows that vectors $\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3$ are not coplanar.

The j^{th} order relative accelerations of the point described by $\mathbf{r}_3$ may be determined from eq (31) like it follows:

$$\boldsymbol{\alpha}_3^{[j]} = \sum_{i=0}^{j} \mathbf{C}_j^i \boldsymbol{\alpha}_1^{[j-i]} \times \boldsymbol{\alpha}_2^{[i]} \quad (32)$$

where we denoted $\boldsymbol{\alpha}^{[0]} = \mathbf{r}$ (the zero-order acceleration is the position vector) and $\mathbf{C}_j^i$ the binomial coefficient.

Now we have the relative j^{th} order accelerations of three non-collinear points of the rigid body, so the j^{th} order acceleration tensors may be expressed exactly like in eq (26):

$$\boldsymbol{\Phi}_j = \boldsymbol{\alpha}_k^{[j]} \otimes \mathbf{r}^k, j = \overline{1,n} \quad (33)$$

Consider the matrices:

$$\begin{cases} \mathbf{A}^{[j]} = \left[\boldsymbol{\alpha}_1^{[j]}, \quad \boldsymbol{\alpha}_2^{[j]}, \quad \boldsymbol{\alpha}_3^{[j]}\right] \\ \quad \mathbf{P} = \left[\boldsymbol{r}_1, \quad \boldsymbol{r}_2, \quad \boldsymbol{r}_3\right] \end{cases}, j = \overline{1,n}, \quad (34)$$

It follows that eq. (33) may be put in the matrix form:

$$\boldsymbol{\Phi}_j = \mathbf{A}^{[j]} \mathbf{P}^{-1} \quad (35)$$

The vector invariants $\mathbf{a}_j, j = \overline{1,n}$ may be computed from:

$$\begin{aligned} \mathbf{a}_j &= \mathbf{a}_0^{[j]} - \boldsymbol{\Phi}_j \boldsymbol{\rho}_0 \\ &= \mathbf{a}_0^{[j]} - \mathbf{A}^{[j]} \mathbf{P}^{-1} \boldsymbol{\rho}_0 \end{aligned} \quad (36)$$

This procedure offers the method for computing the main elements that describe the n^{th} order accelerations field from a set of three non-collinear points of the rigid body. For that, it is necessary to known all the j^{th} order accelerations of the considered points, $1 \leq j \leq n$. The amount of computations is bigger than in the previous case, but this method gives the answer to the question "how arbitrary may be the j^{th} order accelerations, $j = \overline{1,n}$, of four non-collinear points in order to remain compatible with a rigid motion?" The compatibility conditions are written as:

$$\sum_{k=0}^{j} C_j^k \left[\mathbf{A}^{[j-k]}\right]^T \mathbf{A}^{[k]} = \mathbf{0}_3 \;\; \mathbf{A}^{[0]} = \mathbf{P} \qquad (37)$$

This method also allows computing the j^{th} order instantaneous angular acceleration $\boldsymbol{\varepsilon}^{[j]}$ by using eq. (20) and the recursive formulas from eq. (18). For j small enough, the computations are easy, but for greater values of j, the amount of computations is huge.

3.2. Homogenous Matrix Approach to the Field of n^{th} Order Accelerations

The set of affine maps, $g: \mathbf{V}_3 \to \mathbf{V}_3, g(\mathbf{u}) = \mathbf{R}\mathbf{u} + \mathbf{w}$, where $\mathbf{R}$ is an orthogonal proper tensor and $\mathbf{w}$ a vector in $\mathbf{V}_3$ is a group under composition and it is called *the group of direct affine isometric of rigid motions* and it is denoted $S\mathbb{E}_3$. Any rigid finite displacement may be described by such a map: Tensor $\mathbf{R}$ models the rotation of the considered rigid body and vector $\mathbf{w}$ its translation. An affine map from $S\mathbb{E}_3$ may be represented with a 4×4 square matrix:

$$g = \begin{bmatrix} \mathbf{R} & \mathbf{w} \\ \mathbf{0} & 1 \end{bmatrix} \qquad (38)$$

One may remark that the following relations hold true:

$$\begin{cases} \begin{bmatrix} \mathbf{R}_1 & \mathbf{w}_1 \\ \mathbf{0} & 1 \end{bmatrix} \begin{bmatrix} \mathbf{R}_2 & \mathbf{w}_2 \\ \mathbf{0} & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{R}_1\mathbf{R}_2 & \mathbf{R}_1\mathbf{w}_2 + \mathbf{w}_1 \\ \mathbf{0} & 1 \end{bmatrix} \\ \begin{bmatrix} \mathbf{R} & \mathbf{w} \\ \mathbf{0} & 1 \end{bmatrix}^{-1} = \begin{bmatrix} \mathbf{R}^T & -\mathbf{R}^T\mathbf{w} \\ \mathbf{0} & 1 \end{bmatrix} \end{cases} \qquad (39)$$

We may extend now $S\mathbb{E}_3$ to $S\mathbb{E}_3^{\mathbb{R}}$, the set of the functions with the domain $\mathbb{R}$ and the range $S\mathbb{E}_3$. The parametric vector equation of the rigid body motion (1) may be rewritten with the help of a homogenous matrix function in $S\mathbb{E}_3^{\mathbb{R}}$ like it follows:

$$\begin{bmatrix} \boldsymbol{\rho} \\ 1 \end{bmatrix} = \begin{bmatrix} \mathbf{R} & \boldsymbol{\rho}_Q \\ \mathbf{0} & 1 \end{bmatrix} \begin{bmatrix} \mathbf{r} \\ 1 \end{bmatrix} \qquad (40)$$

From eq. (40) it follows that:

$$\begin{bmatrix} \dot{\boldsymbol{\rho}} \\ \mathbf{0} \end{bmatrix} = \begin{bmatrix} \dot{\mathbf{R}} & \dot{\boldsymbol{\rho}}_Q \\ \mathbf{0} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{r} \\ 1 \end{bmatrix} = \begin{bmatrix} \dot{\mathbf{R}} & \dot{\boldsymbol{\rho}}_Q \\ \mathbf{0} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{R}^T & -\mathbf{R}^T\boldsymbol{\rho}_Q \\ \mathbf{0} & 1 \end{bmatrix} \begin{bmatrix} \boldsymbol{\rho} \\ 1 \end{bmatrix} \qquad (41)$$

and by making the computations and taking into account eq. (3), (4) it follows that:

$$\begin{bmatrix} \dot{\boldsymbol{\rho}} \\ 0 \end{bmatrix} = \begin{bmatrix} \boldsymbol{\Phi}_1 & \mathbf{a}_1 \\ \mathbf{0} & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{\rho} \\ 1 \end{bmatrix} \qquad (42)$$

By using the previous considerations, it follows that eq. (40) may be extended like:

$$\begin{bmatrix} \mathbf{a}^{[n]} \\ \mathbf{0} \end{bmatrix} = \begin{bmatrix} \boldsymbol{\Phi}_n & \mathbf{a}_n \\ \mathbf{0} & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{\rho} \\ 1 \end{bmatrix} \qquad (43)$$

Eq. (43) represents a unified form of describing the vector field of the n^{th} order accelerations in rigid body motion. The matrix:

$$\boldsymbol{\Psi}_n = \begin{bmatrix} \boldsymbol{\Phi}_n & \mathbf{a}_n \\ \mathbf{0} & 0 \end{bmatrix} \qquad (44)$$

contains both the n^{th} order acceleration tensor $\boldsymbol{\Phi}_n$ and the vector invariant $\mathbf{a}_n$. Eq. (18) may be put in a compact form:

$$\boldsymbol{\Psi}_{n+1} = \dot{\boldsymbol{\Psi}}_n + \boldsymbol{\Psi}_n\boldsymbol{\Psi}_1, n \geq 1 \qquad (45)$$

If follows that $\boldsymbol{\Psi}_n$ may be written as:

$$\boldsymbol{\Psi}_n = \boldsymbol{\Psi}_{n-1}\boldsymbol{\Psi}_1 + \left(\frac{d^{n-1}}{dt^{n-1}}\boldsymbol{\Psi}_1\right) + \sum_{k=1}^{n-2}\left[\frac{d^k}{dt^k}(\boldsymbol{\Psi}_n\boldsymbol{\Psi}_1)\right], \quad n \geq 3. \qquad (46)$$

4. SYMBOLIC CALCULUS OF HIGHER-ORDER KINEMATICS INVARIANTS

We will present a method for the symbolic calculation of higher-order kinematics invariants for rigid motion.

Let be $\mathbf{a}_n$ and $\boldsymbol{\Phi}_n$, $n \in \mathbb{N}$ vector invariants, respectively, tensor invariants for the n^{th} order accelerations fields. We denote by:

$$\boldsymbol{\Psi}_n = \begin{bmatrix} \boldsymbol{\Phi}_n & \mathbf{a}_n \\ \mathbf{0} & 0 \end{bmatrix} \qquad (47)$$

and we have the following relationship of recurrence:

$$\boldsymbol{\Psi}_{n+1} = \dot{\boldsymbol{\Psi}}_n + \boldsymbol{\Psi}_n\boldsymbol{\Psi}_1, n \in \mathbb{N} \qquad \boldsymbol{\Psi}_1 = \begin{bmatrix} \tilde{\boldsymbol{\omega}} & \mathbf{v} \\ \mathbf{0} & 0 \end{bmatrix} \qquad (48)$$

The pair of vectors $(\boldsymbol{\omega}, \mathbf{v})$ is also known as *the spatial twist of rigid body* [1], [8].

Let be $\boldsymbol{\mathcal{A}}$ the ring of matrixes

$$\boldsymbol{\mathcal{A}} = \left\{\mathbf{A} \in \mathcal{M}_{4\times4}(\mathbb{R}) \,\middle|\, \mathbf{A} = \begin{bmatrix} \boldsymbol{\Phi} & \mathbf{a} \\ \mathbf{0} & 0 \end{bmatrix};\ \boldsymbol{\Phi} \in \mathbf{L}(\mathbf{V}_3, \mathbf{V}_3), \mathbf{a} \in \mathbf{V}_3\right\} \qquad (49)$$

and $\boldsymbol{\mathcal{A}}[X]$ the set of polynomials with coefficients in the non-commutative ring $\boldsymbol{\mathcal{A}}$. A generic element of $\boldsymbol{\mathcal{A}}[X]$ has the form:

$$\mathbf{P}(\mathrm{X}) = \mathbf{A}_0\mathrm{X}^m + \mathbf{A}_1\mathrm{X}^{m-1} + \cdots + \mathbf{A}_{m-1}\mathrm{X} + \mathbf{A}_m, \quad \mathbf{A}_k \in \boldsymbol{\mathcal{A}}, k = \overline{0,m} \qquad (50)$$

Theorem 2. There *is a unique polynomial* $\mathbf{P}_\mathrm{n} \in \boldsymbol{\mathcal{A}}[X]$ *such that* $\boldsymbol{\Psi}_\mathrm{n}$ *will be written as:*

$$\boldsymbol{\Psi}_n = \mathbf{P}_n(\mathbf{D})\boldsymbol{\Psi}_1, n \in \mathbb{N}, \qquad (51)$$

where $\mathbf{D} = \frac{d}{dt}$ *is the operator of time derivative.*

Proof:

Taking into account the eq. (51) and eq. (48) we will have the following relationship of recurrence for $\mathbf{P}_n(\mathbf{D})$:

$$\begin{cases} \mathbf{P}_{n+1} = \mathbf{D}\mathbf{P}_n + \mathbf{P}_n(\boldsymbol{\Psi}_1) \\ \mathbf{P}_0 = \mathbf{I} \end{cases} \qquad (52)$$

Since $\boldsymbol{\Psi}_1 = \begin{bmatrix} \widetilde{\boldsymbol{\omega}} & \mathbf{v} \\ 0 & 0 \end{bmatrix}$ it follows the next outcome.

Theorem 3. *There is an unique polynomial with the coefficients in the non-commutative ring* $\mathbf{L}(\boldsymbol{V}_3, \boldsymbol{V}_3)$ *such that the vector, respectively, the tensor invariants of the* n^{th} *order accelerations will be written as:*

$$\begin{matrix} \mathbf{a}_n = \mathbf{P}_n\mathbf{v} \\ \boldsymbol{\Phi}_n = \mathbf{P}_n\widetilde{\boldsymbol{\omega}} \end{matrix}, n \in \mathbb{N}^* \qquad (53)$$

where $\boldsymbol{P}_n$ *fulfills the relationship of recurrence*

$$\begin{cases} \mathbf{P}_{n+1} = \mathbf{D}\mathbf{P}_n + \mathbf{P}_n(\widetilde{\boldsymbol{\omega}}), n \in \mathbb{N}^* \\ \mathbf{P}_1 = \mathbf{I} \end{cases}. \qquad (54)$$

It follows

$$\begin{gathered} \mathbf{P}_2 = \mathbf{D} + \widetilde{\boldsymbol{\omega}} \\ \mathbf{P}_3 = \mathbf{D}^2 + \widetilde{\boldsymbol{\omega}}\mathbf{D} + 2\dot{\widetilde{\boldsymbol{\omega}}} + \widetilde{\boldsymbol{\omega}}^2 \\ \mathbf{P}_4 = \mathbf{D}^3 + \widetilde{\boldsymbol{\omega}}\mathbf{D}^2 + \left(3\dot{\widetilde{\boldsymbol{\omega}}} + \widetilde{\boldsymbol{\omega}}^2\right)\mathbf{D} + \ddot{\widetilde{\boldsymbol{\omega}}} + 3\dot{\widetilde{\boldsymbol{\omega}}}\widetilde{\boldsymbol{\omega}} + 2\widetilde{\boldsymbol{\omega}}\dot{\widetilde{\boldsymbol{\omega}}} + \widetilde{\boldsymbol{\omega}}^3. \end{gathered} \qquad (55)$$

Thus it follows:

- The velocity field invariants:

$$\begin{cases} \mathbf{a}_1 = \mathbf{v} \\ \boldsymbol{\Phi}_1 = \widetilde{\boldsymbol{\omega}} \end{cases} \qquad (56)$$

- The acceleration field invariants:

$$\begin{cases} \mathbf{a}_2 = \dot{\mathbf{v}} + \widetilde{\boldsymbol{\omega}}\mathbf{v} \\ \boldsymbol{\Phi}_2 = \dot{\widetilde{\boldsymbol{\omega}}} + \widetilde{\boldsymbol{\omega}}^2 \end{cases} \qquad (57)$$

- The jerk field invariants:

$$\begin{cases} \mathbf{a}_3 = \ddot{\mathbf{v}} + \widetilde{\boldsymbol{\omega}}\mathbf{v} + 2\dot{\widetilde{\boldsymbol{\omega}}}\mathbf{v} + \widetilde{\boldsymbol{\omega}}^2\mathbf{v} \\ \boldsymbol{\Phi}_3 = \ddot{\widetilde{\boldsymbol{\omega}}} + \widetilde{\boldsymbol{\omega}}\dot{\widetilde{\boldsymbol{\omega}}} + 2\dot{\widetilde{\boldsymbol{\omega}}}\widetilde{\boldsymbol{\omega}} + \widetilde{\boldsymbol{\omega}}^3 \end{cases} \qquad (58)$$

- The hyper-jerk field invariants:

$$\begin{cases} \mathbf{a}_4 = \dddot{\mathbf{v}} + \widetilde{\boldsymbol{\omega}}\ddot{\mathbf{v}} + \left(3\dot{\widetilde{\boldsymbol{\omega}}} + \widetilde{\boldsymbol{\omega}}^2\right)\dot{\mathbf{v}} + \\ +\ddot{\widetilde{\boldsymbol{\omega}}}\mathbf{v} + 3\dot{\widetilde{\boldsymbol{\omega}}}\widetilde{\boldsymbol{\omega}}\mathbf{v} + 2\widetilde{\boldsymbol{\omega}}\dot{\widetilde{\boldsymbol{\omega}}}\mathbf{v} + \widetilde{\boldsymbol{\omega}}^3\mathbf{v} \\ \boldsymbol{\Phi}_4 = \dddot{\widetilde{\boldsymbol{\omega}}} + \widetilde{\boldsymbol{\omega}}\ddot{\widetilde{\boldsymbol{\omega}}} + \left(3\dot{\widetilde{\boldsymbol{\omega}}} + \widetilde{\boldsymbol{\omega}}^2\right)\dot{\widetilde{\boldsymbol{\omega}}} + \\ +\ddot{\widetilde{\boldsymbol{\omega}}}\widetilde{\boldsymbol{\omega}} + 3\dot{\widetilde{\boldsymbol{\omega}}}\widetilde{\boldsymbol{\omega}}^2 + 2\widetilde{\boldsymbol{\omega}}\dot{\widetilde{\boldsymbol{\omega}}}\widetilde{\boldsymbol{\omega}} + \widetilde{\boldsymbol{\omega}}^4 \end{cases} \qquad (59)$$

Remark 6. *The higher order time derivative of spatial twist solve completely the problem of determining the filed of the* n^{th} *order acceleration of rigid motion.*

5. HIGHER – ORDER KINEMATICS IN DUAL LIE ALGEBRA

Being the rigid body motion given by the following parametric equation in a given reference frame:

$$\begin{cases} \boldsymbol{\rho} = \boldsymbol{\rho}(t) \in \boldsymbol{V}_3 \\ \boldsymbol{R} = \boldsymbol{R}(t) \in \boldsymbol{S}\mathbb{O}_3 \end{cases} \qquad (60)$$

with $t \in \mathbf{I} \subseteq \mathbb{R}$ is time variable.

The dual orthogonal tensor that describes the rigid body motion is [3], [4], [6]:

$$\underline{\boldsymbol{R}} = (\mathbf{I} + \varepsilon\widetilde{\boldsymbol{\rho}})\boldsymbol{R} \qquad (61)$$

In the relation (61), ε is a dual quantity so that $\varepsilon^2 = 0, \varepsilon \neq 0$ ([3], [4], [5], [6], [7], [10], [11], [12]). The skew symmetric tensor associated to the vector $\boldsymbol{\rho}$ is denoted by $\widetilde{\boldsymbol{\rho}}$.

It can be easily demonstrated [4] that:

$$\begin{gathered} \underline{\boldsymbol{R}}\,\underline{\boldsymbol{R}}^T = \underline{\boldsymbol{I}} \\ det\underline{\boldsymbol{R}} = 1 \\ \underline{\boldsymbol{R}}\left(\underline{\boldsymbol{a}} \cdot \underline{\boldsymbol{b}}\right) = \left(\underline{\boldsymbol{R}}\,\underline{\boldsymbol{a}}\right) \cdot \left(\underline{\boldsymbol{R}}\,\underline{\boldsymbol{b}}\right), \forall \underline{\boldsymbol{a}}, \underline{\boldsymbol{b}} \in \underline{\boldsymbol{V}}_3 \\ \underline{\boldsymbol{R}}\left(\underline{\boldsymbol{a}} \times \underline{\boldsymbol{b}}\right) = \underline{\boldsymbol{R}}\left(\underline{\boldsymbol{a}}\right) \times \underline{\boldsymbol{R}}\left(\underline{\boldsymbol{b}}\right), \forall \underline{\boldsymbol{a}}, \underline{\boldsymbol{b}} \in \underline{\boldsymbol{V}}_3 \end{gathered} \qquad (62)$$

The tensor $\underline{\boldsymbol{R}}$ transports the dual vectors from the body frame in the space frame with the conservation of the dual angles and the relative orientation of lines that corresponds to the dual vectors $\underline{\boldsymbol{a}}$ *and* $\underline{\boldsymbol{b}}$.

The dual angular velocity for the rigid body motion (60) is given by (61):

$$\underline{\boldsymbol{\omega}} = vect\dot{\underline{\boldsymbol{R}}}\,\underline{\boldsymbol{R}}^T \qquad (63)$$

It can be demonstrated that [6], [7]:

$$\underline{\boldsymbol{\omega}} = \boldsymbol{\omega} + \varepsilon\boldsymbol{v} \qquad (64)$$

where

$$\boldsymbol{\omega} = vect\dot{\boldsymbol{R}}\boldsymbol{R}^T \qquad (65)$$

is the instantaneous angular velocity of the rigid body and

$$\boldsymbol{v} = \dot{\boldsymbol{\rho}} - \boldsymbol{\omega} \times \boldsymbol{\rho} \qquad (66)$$

is the linear velocity of a point of the rigid body that coincides with the origin of the reference frame at that given moment of time.

The dual angular velocity $\underline{\boldsymbol{\omega}}$ *completely characterizes the distribution of the velocity field of the rigid body. The pair* $(\boldsymbol{\omega}, \boldsymbol{v})$ *is called "the twist of the rigid body motion"* [1].

Being*:*

$$\underline{\boldsymbol{V}}_{\boldsymbol{\rho}} = \boldsymbol{\omega} + \varepsilon \boldsymbol{v}_{\boldsymbol{\rho}} \tag{67}$$

the dual velocity for a point localized in the reference frame by the position vector $\boldsymbol{\rho}$.

In (67), $\boldsymbol{\omega}$ is the instantaneous angular velocity of the rigid body and $\boldsymbol{v}_{\boldsymbol{\rho}}$ is the linear velocity of the point. Using the next equation:

$$e^{\varepsilon\widetilde{\boldsymbol{\rho}}} = \mathbf{I} + \varepsilon\widetilde{\boldsymbol{\rho}} \tag{68}$$

from (64), (66), (67) and (68), results:

$$\boxed{e^{\varepsilon\widetilde{\boldsymbol{\rho}}}\underline{\boldsymbol{V}}_{\boldsymbol{\rho}} = \underline{\boldsymbol{\omega}}} \tag{69}$$

Consequently, $e^{\varepsilon\widetilde{\boldsymbol{\rho}}}\underline{\boldsymbol{V}}_{\boldsymbol{\rho}}$ *is an invariant having the same value for any* $\boldsymbol{\rho}$*.*

Writing this invariant in two different points of the rigid body (noted with P and Q), results that*:*

$$e^{\varepsilon\widetilde{\boldsymbol{\rho}}_P}\underline{\boldsymbol{V}}_P = e^{\varepsilon\widetilde{\boldsymbol{\rho}}_Q}\underline{\boldsymbol{V}}_Q \tag{70}$$

From (70), results:

$$\boxed{\underline{\boldsymbol{V}}_P = e^{\varepsilon\widetilde{PQ}}\underline{\boldsymbol{V}}_Q} \tag{71}$$

with $\mathrm{PQ} = \boldsymbol{\rho}_{\mathrm{Q}} - \boldsymbol{\rho}_{\mathrm{P}}$.

The relation (71) is true for any P and Q.

Analogue with the equation (69), the following invariants take place:

$$\begin{aligned} e^{\varepsilon\widetilde{\boldsymbol{\rho}}}\underline{\mathbf{A}}_{\boldsymbol{\rho}} &= \underline{\dot{\boldsymbol{\omega}}}, \forall\boldsymbol{\rho} \in \mathbf{V}_3 \\ e^{\varepsilon\widetilde{\boldsymbol{\rho}}}\underline{\mathbf{J}}_{\boldsymbol{\rho}} &= \underline{\ddot{\boldsymbol{\omega}}}, \forall\boldsymbol{\rho} \in \mathbf{V}_3 \\ e^{\varepsilon\widetilde{\boldsymbol{\rho}}}\underline{\mathbf{H}}_{\boldsymbol{\rho}} &= \underline{\dddot{\boldsymbol{\omega}}}, \forall\boldsymbol{\rho} \in \mathbf{V}_3 \end{aligned} \tag{72}$$

where we noted:

$$\begin{aligned} \underline{\mathbf{A}}_{\boldsymbol{\rho}} &= \dot{\boldsymbol{\omega}} + \varepsilon\mathbf{A}_{\boldsymbol{\rho}} \\ \underline{\mathbf{J}}_{\boldsymbol{\rho}} &= \ddot{\boldsymbol{\omega}} + \varepsilon\mathbf{J}_{\boldsymbol{\rho}} \\ \underline{\mathbf{H}}_{\boldsymbol{\rho}} &= \dddot{\boldsymbol{\omega}} + \varepsilon\mathbf{H}_{\boldsymbol{\rho}} \end{aligned} \tag{73}$$

with $\mathbf{A}_{\boldsymbol{\rho}}, \mathbf{J}_{\boldsymbol{\rho}}, \mathbf{H}_{\boldsymbol{\rho}}$ the reduced acceleration, reduced jerk, respectively the reduced hyper-jerk, in a point given by the position vector $\boldsymbol{\rho}$:

$$\begin{aligned} \mathbf{A}_{\boldsymbol{\rho}} &= \mathbf{a}_{\boldsymbol{\rho}} - \boldsymbol{\omega} \times \mathbf{v}_{\boldsymbol{\rho}} \\ \mathbf{J}_{\boldsymbol{\rho}} &= \mathbf{j}_{\boldsymbol{\rho}} - \boldsymbol{\omega} \times \mathbf{a}_{\boldsymbol{\rho}} - 2\dot{\boldsymbol{\omega}} \times \mathbf{v}_{\boldsymbol{\rho}} \\ \mathbf{H}_{\boldsymbol{\rho}} &= \mathbf{h}_{\boldsymbol{\rho}} - \boldsymbol{\omega} \times \mathbf{j}_{\boldsymbol{\rho}} - 3\dot{\boldsymbol{\omega}} \times \mathbf{a}_{\boldsymbol{\rho}} - 3\ddot{\boldsymbol{\omega}} \\ &\quad \times \mathbf{v}_{\boldsymbol{\rho}} \end{aligned} \tag{74}$$

In (74), $\mathbf{a}_{\boldsymbol{\rho}}$, $\mathbf{j}_{\boldsymbol{\rho}}$ and $\mathbf{h}_{\boldsymbol{\rho}}$ are, respectively, the acceleration, the jerk, and the hyper-jerk, in a point given by the position vector $\boldsymbol{\rho}$.

Analogue with the equation (71) the following equations take place:

$$\begin{aligned} \underline{\mathbf{A}}_{\mathrm{P}} &= e^{\varepsilon\widetilde{\mathrm{PQ}}}\underline{\mathbf{A}}_{\mathrm{Q}} \\ \underline{\mathbf{J}}_{\mathrm{P}} &= e^{\varepsilon\widetilde{\mathrm{PQ}}}\underline{\mathbf{J}}_{\mathrm{Q}}\ , \\ \underline{\mathbf{H}}_{\mathrm{P}} &= e^{\varepsilon\widetilde{\mathrm{PQ}}}\underline{\mathbf{H}}_{\mathrm{Q}} \end{aligned} \tag{75}$$

The lines corresponding to the dual vectors $\underline{\dot{\boldsymbol{\omega}}}, \underline{\ddot{\boldsymbol{\omega}}}, \underline{\dddot{\boldsymbol{\omega}}}$ represent the geometric place where the vectors $\mathbf{A}_{\boldsymbol{\rho}}, \mathbf{J}_{\boldsymbol{\rho}}, \mathbf{H}_{\boldsymbol{\rho}}$ have the minimum module value. Supplementary,

$$\begin{aligned} \min_{\boldsymbol{\rho}\in\mathbf{V}_3}\|\mathbf{A}_{\boldsymbol{\rho}}\| &= \|\mathrm{Du}|\underline{\dot{\boldsymbol{\omega}}}|\| \\ \min_{\boldsymbol{\rho}\in\mathbf{V}_3}\|\mathbf{J}_{\boldsymbol{\rho}}\| &= \|\mathrm{Du}|\underline{\ddot{\boldsymbol{\omega}}}|\| \\ \min_{\boldsymbol{\rho}\in\mathbf{V}_3}\|\mathbf{H}_{\boldsymbol{\rho}}\| &= \|\mathrm{Du}|\underline{\dddot{\boldsymbol{\omega}}}|\| \end{aligned} \tag{76}$$

Interesting is the fact that for the plane motion $\min\|\mathbf{A}_{\boldsymbol{\rho}}\| = \min\|\mathbf{J}_{\boldsymbol{\rho}}\| = \min\|\mathbf{H}_{\boldsymbol{\rho}}\| = 0$ because $\mathrm{Du}|\underline{\dot{\boldsymbol{\omega}}}| = \mathrm{Du}|\underline{\ddot{\boldsymbol{\omega}}}| = \mathrm{Du}|\underline{\dddot{\boldsymbol{\omega}}}| = 0$.

All properties are extended for higher-order accelerations. The vector $\underline{\boldsymbol{\omega}}^{(n-1)} = \frac{d^{n-1}\underline{\boldsymbol{\omega}}}{dt^{n-1}}, n \in \mathbb{N}^*$ describes completely the helicoidally field of the n^{th} order reduced accelerations, for $n \in \mathbb{N}^*$:

$$\boxed{e^{\varepsilon\widetilde{\boldsymbol{\rho}}}\underline{\mathbf{A}}_{\boldsymbol{\rho}}^{[n]} = \underline{\boldsymbol{\omega}}^{(n-1)}} \tag{77}$$

In eq. (77), $\underline{\mathbf{A}}_{\boldsymbol{\rho}}^{[n]}$ denote the n^{th} order of the dual reduced acceleration in a given point by the position vector $\boldsymbol{\rho}$.

It follows that the dual part of $\underline{\boldsymbol{\omega}}^{(n-1)}$:

$$\underline{\boldsymbol{\omega}}^{(n-1)} = \boldsymbol{\omega}^{(n-1)} + \boldsymbol{\varepsilon}\mathbf{v}^{(n-1)} \tag{78}$$

is the n^{th} order reduced acceleration of that point of the rigid body that at the given time passes by the origin of the reference frame.

From equation

$$\mathbf{v} = \dot{\boldsymbol{\rho}} - \boldsymbol{\omega} \times \boldsymbol{\rho} \tag{79}$$

it follows that:

$$\begin{aligned} &\mathbf{v}^{(n-1)} = \\ &= \boldsymbol{\rho}^{(n)} - \sum_{k=0}^{n-1} \mathrm{C}_{n-1}^{k}\boldsymbol{\omega}^{(n-k-1)} \times \boldsymbol{\rho}^{(k)}, \\ &\qquad n \in \mathbb{N}^*. \end{aligned} \tag{80}$$

With the following notations:

$$\mathbf{a}_{\boldsymbol{\rho}}^{[n]} \triangleq \boldsymbol{\rho}^{(n)}, n \in \mathbb{N}, \tag{81}$$

for the n^{th}order acceleration of the point given by the position vector $\boldsymbol{\rho}$ and:

$$\mathbf{A}_{\boldsymbol{\rho}}^{[n]} \triangleq \mathbf{a}_{\boldsymbol{\rho}}^{[n]} - \sum_{k=1}^{n-1} \mathrm{C}_{n-1}^{k}\boldsymbol{\omega}^{(n-k-1)}\mathbf{a}_{\boldsymbol{\rho}}^{[k]} \tag{82}$$

for the n^{th} order reduced acceleration of the same point the equation:

$$\mathbf{A}_{\boldsymbol{\rho}}^{[n]} = \mathbf{v}^{(n-1)} + \boldsymbol{\omega}^{(n-1)} \times \boldsymbol{\rho} \quad (83)$$

which proves which proves the character of the helicoidally field of the n^{th} order reduced accelerations field.

For $\boldsymbol{\rho} = 0$, from eq. (82), the relations between the n^{th} order reduced acceleration and the n^{th} order acceleration, from the point of the rigid body that pass by the origin of the reference frame at the given time, are written:

$$\mathbf{A}^{[n]} = \mathbf{v}^{(n-1)} = \mathbf{a}_n - \sum_{k=1}^{n-1} C_{n-1}^{k} \boldsymbol{\omega}^{(n-k-1)} \mathbf{a}_k, \quad n \in \mathbb{N}^* \quad (84)$$

The invert of previous equation is written:

$$\mathbf{a}_n = \mathbf{P}_n(\mathbf{v}), n \in \mathbb{N}^* \quad (85)$$

where $\mathbf{P}_n$ is the polynomial with the coefficients in the ring of the second order Euclidean tensors. The polynomials $\mathbf{P}_n[\mathbf{D}]$ follow the recurrence relation:

$$\begin{cases} \mathbf{P}_{n+1} = \mathbf{D}\mathbf{P}_n + \mathbf{P}_n(\tilde{\boldsymbol{\omega}}) \\ \mathbf{P}_0 = \mathbf{I} \end{cases} \quad (86)$$

it follows successively

$$\begin{aligned} \mathbf{P}_1 &= \tilde{\boldsymbol{\omega}} \\ \mathbf{P}_2 &= \mathbf{D} + \tilde{\boldsymbol{\omega}} \\ \mathbf{P}_3 &= \mathbf{D}^2 + \tilde{\boldsymbol{\omega}}\mathbf{D} + 2\dot{\tilde{\boldsymbol{\omega}}} + \tilde{\boldsymbol{\omega}}^2 \\ \mathbf{P}_4 &= \mathbf{D}^3 + \tilde{\boldsymbol{\omega}}\mathrm{D} + \left(3\dot{\tilde{\boldsymbol{\omega}}} + \tilde{\boldsymbol{\omega}}^2\right)\mathbf{D} + \\ &+ \ddot{\tilde{\boldsymbol{\omega}}} + 2\tilde{\boldsymbol{\omega}}\dot{\tilde{\boldsymbol{\omega}}} + 3\dot{\tilde{\boldsymbol{\omega}}}\tilde{\boldsymbol{\omega}} + \tilde{\boldsymbol{\omega}}^3 \end{aligned} \quad (87)$$

For the case of the velocities, accelerations, jerk and hyper-jerk, from equations (84) and (85) we will have:

$$\begin{bmatrix} \mathbf{v} \\ \dot{\mathbf{v}} \\ \ddot{\mathbf{v}} \\ \dddot{\mathbf{v}} \end{bmatrix} = \begin{bmatrix} \mathbf{I} & 0 & 0 & 0 \\ -\tilde{\boldsymbol{\omega}} & \mathbf{I} & 0 & 0 \\ -2\dot{\tilde{\boldsymbol{\omega}}} & -\tilde{\boldsymbol{\omega}} & \mathbf{I} & 0 \\ -\ddot{\tilde{\boldsymbol{\omega}}} & -3\dot{\tilde{\boldsymbol{\omega}}} & -\tilde{\boldsymbol{\omega}} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{a}_0 \\ \mathbf{a}_1 \\ \mathbf{a}_2 \\ \mathbf{a}_3 \end{bmatrix} \quad (88)$$

$$\begin{bmatrix} \mathbf{a}_0 \\ \mathbf{a}_1 \\ \mathbf{a}_2 \\ \mathbf{a}_3 \end{bmatrix} = \begin{bmatrix} \mathbf{I} & 0 & 0 & 0 \\ \tilde{\boldsymbol{\omega}} & \mathbf{I} & 0 & 0 \\ 2\dot{\tilde{\boldsymbol{\omega}}} + \tilde{\boldsymbol{\omega}}^2 & \tilde{\boldsymbol{\omega}} & \mathbf{I} & 0 \\ \ddot{\tilde{\boldsymbol{\omega}}} + 2\tilde{\boldsymbol{\omega}}\dot{\tilde{\boldsymbol{\omega}}} + 3\dot{\tilde{\boldsymbol{\omega}}}\tilde{\boldsymbol{\omega}} + \tilde{\boldsymbol{\omega}}^3 & 3\dot{\tilde{\boldsymbol{\omega}}} + \tilde{\boldsymbol{\omega}}^2 & \tilde{\boldsymbol{\omega}} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{v} \\ \dot{\mathbf{v}} \\ \ddot{\mathbf{v}} \\ \dddot{\mathbf{v}} \end{bmatrix} \quad (89)$$

Theorem 4. *The field of the n^{th} order accelerations of a rigid body in a general motion is uniquely determined by the k^{th}order time derivation of a dual twist $\underline{\boldsymbol{\omega}}$, $k = \overline{0, n}$.*

6. HIGHER – ORDER KINEMATICS OF SPATIAL CHAIN USING DUAL LIE ALGEBRA

Let be a space cinematic chain of the bodies $C_k, k = \overline{0, m}$ where the relative motion of the rigid body C_k in relation to C_{k-1} is given by the proper orthogonal dual tensor ${}^{k-1}\underline{\mathbf{R}}_k \in \mathbf{SO}_3^{\mathbb{R}}$. The motion properties of the body C_m in relation to a given reference system are described by the dual orthogonal tensor:

$$\underline{\mathbf{R}} = {}^{0}\underline{\mathbf{R}}_1 \cdot {}^{1}\underline{\mathbf{R}}_2 \dots {}^{m-1}\underline{\mathbf{R}}_m \quad (90)$$

Instantaneous dual angular velocity (dual twist) of the rigid body in relation to the reference frame it will be given by the equation:

$$_0\underline{\boldsymbol{\omega}}_m = \mathrm{vect}\dot{\underline{\mathbf{R}}}\underline{\mathbf{R}}^T \quad (91)$$

It follows from (90)

$$_0\underline{\boldsymbol{\omega}}_m = \underline{\boldsymbol{\Omega}}_1 + {}^{0}\mathbf{R}_1\underline{\boldsymbol{\Omega}}_2 + \cdots + {}^{0}\mathbf{R}_1\,{}^{1}\mathbf{R}_2\,{}^{m}\mathbf{R}_{m-1}\underline{\boldsymbol{\Omega}}_m \quad (92)$$

where

$$\underline{\boldsymbol{\Omega}}_k = \mathrm{vect}\,{}^{k-1}\dot{\underline{\mathbf{R}}}_k\,{}^{k-1}\underline{\mathbf{R}}_k^T \quad (93)$$

Using the notation:

$$\underline{\boldsymbol{\omega}}_k = \underline{\mathbf{R}}_1 \cdot \underline{\mathbf{R}}_2 \dots \underline{\mathbf{R}}_{k-1}\underline{\boldsymbol{\Omega}}_k \quad (94)$$

the equation (91) will be written:

$$_0\underline{\boldsymbol{\omega}}_m = \underline{\boldsymbol{\omega}}_1 + \underline{\boldsymbol{\omega}}_2 + \cdots + \underline{\boldsymbol{\omega}}_m \quad (95)$$

where $\underline{\boldsymbol{\omega}}_k$ is the dual twist of the relative motion of the body C_k in relation to the body C_{k-1} , observed from the body C_0.

We will denote $\underline{\boldsymbol{\omega}}_p^{[n]} = {}^{0}\underline{\mathbf{R}}_1\,{}^{1}\underline{\mathbf{R}}_2 \dots {}^{p-2}\underline{\mathbf{R}}_{p-1}\underline{\boldsymbol{\Omega}}_p^{(n)}$ the n-th order derivative of the relative dual twist $\underline{\boldsymbol{\Omega}}_p$, resolved in the body frame of C_0.

To determine the field of the n^{th} order accelerations of a rigid body C_m we have to determine the $_0\underline{\boldsymbol{\omega}}_m^{(n)}, n \in \mathbb{N}$.

Theorem 5. *The following equation:*

$$\begin{aligned} _0\underline{\boldsymbol{\omega}}_m^{(n)} = \underline{\boldsymbol{\omega}}_1{}^{(n)} &+ \mathbf{p}_n\left(\underline{\boldsymbol{\omega}}_1\right)\underline{\boldsymbol{\omega}}_2 \\ &+ \mathbf{p}_n\left(\underline{\boldsymbol{\omega}}_1 + \underline{\boldsymbol{\omega}}_2\right)\underline{\boldsymbol{\omega}}_3 + \cdots \\ &+ \mathbf{p}_n\left(\underline{\boldsymbol{\omega}}_1 + \underline{\boldsymbol{\omega}}_2 + \cdots + \underline{\boldsymbol{\omega}}_{m-1}\right)\underline{\boldsymbol{\omega}}_m; \\ &\forall n \in \mathbb{N} \end{aligned} \quad (96)$$

where $\boldsymbol{p}_n(\boldsymbol{\omega})$ are polynomials of the derivative operator $\mathbf{D} = \frac{d}{dt}$, with coefficients in the non-commutative ring of Euclidian tensors:

$$\mathbf{p}_n(\underline{\boldsymbol{\omega}}) = \sum_{k=0}^{n} C_n^k \underline{\boldsymbol{\Phi}}_{n-k} \mathbf{D}^{[k]} \quad (97)$$

where $\mathbf{D}^{[k]}\boldsymbol{\omega}_p = \boldsymbol{\omega}_p^{[k]}$ *and* $\underline{\boldsymbol{\Phi}}_p$ *are dual tensors:*

$$\underline{\boldsymbol{\Phi}}_p = \underline{\boldsymbol{R}}^{(p)}\underline{\boldsymbol{R}}^T, \underline{\boldsymbol{R}} \in \underline{\mathbf{SO}}_3^{\mathbb{R}}, p = \overline{0,n} \quad (98)$$

which follow the recurrence relationship

$$\begin{cases} \underline{\boldsymbol{\Phi}}_{p+1} = \dot{\underline{\boldsymbol{\Phi}}}_p + \underline{\boldsymbol{\Phi}}_p \tilde{\underline{\boldsymbol{\omega}}} \\ \underline{\boldsymbol{\Phi}}_0 = \underline{\mathbf{I}} \end{cases}, \ p \in \mathbb{N}. \quad (99)$$

Other equivalent forms of the equations (96) are:

$$\boxed{\begin{aligned} {}_0\underline{\boldsymbol{\omega}}_m^{(n)} = \underline{\boldsymbol{\omega}}_1^{[n]} &+ \mathbf{p}_n({}_0\underline{\boldsymbol{\omega}}_1)\underline{\boldsymbol{\omega}}_2 \\ &+ \mathbf{p}_n({}_0\underline{\boldsymbol{\omega}}_2)\underline{\boldsymbol{\omega}}_3 + \cdots \\ &+ \mathbf{p}_n({}_0\underline{\boldsymbol{\omega}}_{m-1})\underline{\boldsymbol{\omega}}_m, \\ \forall n \in \mathbb{N} \end{aligned}} \quad (100)$$

or, in the explicit form:

$$\begin{aligned} {}_0\underline{\boldsymbol{\omega}}_m^{(n)} &= \underline{\boldsymbol{\omega}}_1^{[n]} + \underline{\boldsymbol{\omega}}_2^{[n]} + \cdots + \underline{\boldsymbol{\omega}}_m^{[n]} + \\ &+ \sum_{p=2}^{m}\sum_{k=1}^{n} C_n^k \underline{\boldsymbol{\Phi}}_k(\underline{\boldsymbol{\omega}}_1 + \underline{\boldsymbol{\omega}}_2 + \cdots \quad (101) \\ &\ldots + \underline{\boldsymbol{\omega}}_{p-1})\underline{\boldsymbol{\omega}}_p^{[n-k]}, \forall n \in \mathbb{N}. \end{aligned}$$

The previous equations are valid in the most general situation where there are no kinematic links between the rigid body $C_0, C_1, \ldots, C_m$.

In the situation where kinematics links are given, the equation (96) can be completed by

$$\begin{aligned} &\underline{\boldsymbol{\Phi}}_k(\underline{\boldsymbol{\omega}}_1 + \underline{\boldsymbol{\omega}}_2 + \cdots + \underline{\boldsymbol{\omega}}_{p-1}) = \\ &= \sum_{k_1+k_2+\cdots+k_{p-1}=k} \boldsymbol{C}_n^{k_1,k_2,\ldots k_{p-1}} \underline{\boldsymbol{\Phi}}_{k_1}(\underline{\boldsymbol{\omega}}_1) \cdot\cdot \quad (102) \\ &\qquad \ldots \underline{\boldsymbol{\Phi}}_{k_{p-1}}(\underline{\boldsymbol{\omega}}_{p-1}) \end{aligned}$$

where $\boldsymbol{C}_n^{k_1,k_2,\ldots k_{p-1}} = \frac{n!}{k!k_1!\ldots k_{p-1}!}$ is the monomial coefficient.

7. CONCLUSIONS

The higher-order kinematics properties of rigid body in general motion had been deeply studied. Using the isomorphism between the Lie group of the rigid displacements $S\mathbb{E}_3$ and the Lie group of the orthogonal dual tensors $\underline{S\mathbb{O}}_3$, a general method for the study of the field of arbitrary high-order accelerations is described. It is proved that all information regarding the properties of the distribution of high-order accelerations are contained in the n-th order derivatives of the dual twist of the rigid body. These derivatives belong to the Lie algebra associated to the Lie group $\underline{S\mathbb{O}}_3$.

For the case of the spatial kinematics chains, an equation that allows the determination of the n-th order field accelerations are given, using a Brockett-like formulas specific to the dual algebra. In particular cases the properties for velocities, accelerations, jerks and hyper-jerks fields are given.

The obtained results interests the theoretical kinematics, jerk with hyper-jerk analysis in the case of parallel manipulators, control theory and multibody kinematics.

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ACCELERAȚII DE ORDIN SUPERIOR ÎN MIȘCAREA CORPURILOR RIGIDE. O ABORDARE TENSORIALĂ ÎN ALGEBRA LIE A VECTORILOR DUALI.

Rezumat: Lucrarea extinde rezultatele specifice câmpurilor vitezelor şi accceleraţiilor în mişcarea corpului rigid la domeniul acceleraţiilor de ordin superior. Folosind calculul tensorial şi algebra numerelor duale, se propune o metodă pentru studiul proprietăților câmpului acceleraţiilor de ordin superior în cazul mișcării generale. Sunt evidenţiaţi invarianții vectoriali şi tensoriali în distribuția câmpului acceleraţiilor de ordin n. În cazul lanțurilor cinematice spațiale, se dă o ecuație care permite determinarea câmpului accelerațiilor de ordin n folosind o formulă de tip Brockett, specifică algebrei duale. Rezultatele sînt independente de sistemul de coordonate și au formă închisă. În particular, sînt date proprietăți ale câmpurilor vitezelor, accelerațiilor, jerks și hyper-jerks. Această abordare folosește izomorfismul dintre grupul Lie al deplasărilor rigide *S*E3, și grupul Lie al tensorilor ortogonali duali, introdus de autor în lucrări anterioare.

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