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NEW APPROACHES ON NOTIONS FROM ADVANCED MECHANICS

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Abstract: *The dynamical study of the current and sudden motions of the rigid body, and in accordance with differential principles typical to analytical dynamics of systems, is based, between others, on advanced notions, such as: angular momentum, kinetic energy, acceleration energies of different orders and their absolute time derivatives of higher order. Advanced notions are developed in connection with the generalized variables, also named independent parameters of position and orientation corresponding to holonomic rigid body. But, mechanically, the expressions of definition of the advanced notions contain on the one hand kinematical parameters and their differential transformations, corresponding to absolute motions, on the other hand the mass properties, highlighted by mass and inertial tensors and their generalized laws. By means of the researches of the author, in this paper a few essential reformulations and new formulations concerning input expressions and parameters from advanced kinematics and dynamics will be presented. They become input expressions compulsory included in dynamics equations of higher order, corresponding to the current and sudden motions in the case of the rigid body. These are extended on the multibody systems. Within of the paper a few reformulations on the fundamental theorems from dynamics, differential generalized principle in analytical dynamics, as well as generalization of Gibbs – Appell's equations will be also defined.*

Key words: *advanced mechanics, dynamics, advanced notions, dynamics equations, matrix exponentials.*

1. INTRODUCTION

The solid body consists one of physical form of existence of the matter in material universe. As a result, the solid body is considered material continuum. According to this property, to obtain an exact solution, geometrically, the solid body is decomposed in the infinity of the elementary particles, having elementary mass infinitesimal with continuous distribution entire geometrical shape of the solid body. If the distances between elementary particles are kept constant entire solid body, then it has the rigid character (rigid solid body (S)). When the density property is kept constant inside rigid structure, it obtains homogeneous rigid solid. In this case, when the integration limits on the geometrical outline are well-defined, it obtains homogeneous body with simple or regular geometrical shape. In this last case, geometrical and mass integrals are applied.

Before mechanical study (statics, kinematics and dynamics), it must compulsory established the geometrical state to any rigid solid to each moment in Cartesian space. In the view of this, at beginning, geometrical state of the simplest model, named material point, is studied (Fig.1).

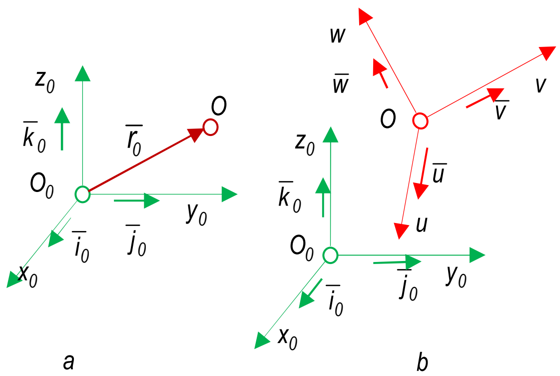

Fig.1 Position and Orientation

Using [1] and [2], a few notations are applied:

$$\left\{\begin{matrix} \chi=\{u;v;w\};\ \chi_0=\{u_0;v_0;w_0\}\ \ where \\ u=\{x;y;z\};v=\{y;z;x\}\neq u;w=\{z;x;y\}\neq v \end{matrix}\right\} \quad (1)$$

$$\left\{\begin{matrix} \bar{\chi}=\{\bar{u};\ \bar{v};\ \bar{w}\};\ \bar{\chi}_0=\{\bar{u}_0;\ \bar{v}_0;\ \bar{w}_0\}\ \ where \\ \bar{u}=\{\bar{i};\bar{j};\bar{k}\};\bar{v}=\{\bar{j};\bar{k};\bar{i}\}\neq\bar{u};\bar{w}=\{\bar{k};\bar{i};\bar{j}\}\neq\bar{v} \end{matrix}\right\} \quad (2)$$

$$\delta_\chi=\{\alpha_\chi;\ \beta_\chi;\ \gamma_\chi\};\ \cos\delta_\chi=c\delta_\chi;\ \sin\delta_\chi=s\delta_\chi \quad (3)$$

$$O_0x_0y_0z_0\equiv\{0\};\ O_0'x_0'y_0'z_0'\equiv\{0'\};\ Oxyz\equiv\{S\}. \quad (4)$$

Notations (1) refer to the Cartesian coordinates or axes, the symbols (2) highlight unit vectors, while (3) express angles and direction cosines.

According to Fig.1a, the geometrical state to any material point (as example O) is named *position*. This is defined by means of *the position vector*:

$$\overline{r_0} = \begin{bmatrix} x_0 & y_0 & z_0 \end{bmatrix}^T, \text{ relative to } \{0\} \text{ frame}. \quad (5)$$

If material point is free in Cartesian frame, the three linear coordinates of (5) are independent. They are also named degrees of freedom (*d.o.f.*).

The study is extended on vector or Cartesian axis (see Fig.1b). In this case, the geometrical state is named *orientation*. This is highlighted by means of *the unit vector*. Using (3) – (13), any unit vector $\overline{\chi} \in \{S\}$ in relation to $\{0\}/\{0'\}$ frame is characterized by the direction cosines, that is:

$$\overline{\chi} = \overline{\chi}^T \cdot \begin{pmatrix} \overline{i_0} \\ \overline{j_0} \\ \overline{k_0} \end{pmatrix} = \begin{pmatrix} c\alpha_\chi \\ c\beta_\chi \\ c\gamma_\chi \end{pmatrix} \equiv \begin{pmatrix} c\alpha \\ c\beta \\ c\gamma \end{pmatrix}_\chi \quad (6)$$

where $\overline{\chi}^T \cdot \overline{\chi} = c^2\alpha_\chi + c^2\beta_\chi + c^2\gamma_\chi = 1$. (7)

Due to (7), *the orientation* to any vector or axis is defined by means of *two independent angles*. The above geometrical aspects are extended on a reference system orthogonal and right oriented (see Fig.1b, $Ouvw \equiv Oxyz \equiv \{S\}$) relative to $\{0\}$. Its geometrical state is *position* and *orientation*. Unlike position defined by (5), for orientation, according to [1] – [8], first of all is established:

$$_S^0[R] = \begin{bmatrix} \overline{i} & \overline{j} & \overline{k} \end{bmatrix} = \begin{bmatrix} \begin{pmatrix} c\alpha \\ c\beta \\ c\gamma \end{pmatrix}_x & \begin{pmatrix} c\alpha \\ c\beta \\ c\gamma \end{pmatrix}_y & \begin{pmatrix} c\alpha \\ c\beta \\ c\gamma \end{pmatrix}_z \end{bmatrix} \quad (8)$$

This is named the resultant rotation (orientation) matrix. It contains the unit vectors belonging to $\{S\}$ in relation to $\{0\}/\{0'\}$. Every unit vector has *two independent angles*. Beside (7), between the unit vectors there are other three relationships:

$$\overline{u}^T \cdot \overline{v} = c\alpha_u \cdot c\alpha_v + c\beta_u \cdot c\beta_v + c\gamma_u \cdot c\gamma_v = 0, \; v \neq u \quad (9)$$

So, in the general case, resultant orientation to any reference frame $\{S\}$ relative to another for example $\{0\}/\{0'\}$ is defined by means of *three orientation angles and independent (three d.o.f)*. According to [3] – [13], they are symbolized as:

$$\overline{\overline{\psi}}(t) = \begin{bmatrix} \alpha_u(t) & \beta_v(t) & \gamma_w(t) \end{bmatrix}^T; \quad (10)$$

Every angle from (10) is, geometrically, dihedral angle between two geometrical plans, that is:

$$\chi_0 = \{u_0; v_0; w_0\} = cst - \text{fixed plan} \in \{0'\}/\{0\},$$

and $\chi = \{u; v; w\} = 0 - \text{mobile plan} \in \{S\}$ (11)

Physically, every orientation angle expresses a simple rotation around of the axes: $\chi = \{u; v; w\}$. Considering the researches from [2] – [13], by combining the three simple rotations, twelve sets of the orientation angles (10) are obtained. The symbol from (10) is named the column matrix of orientation. According to same researches are developed expressions of definition for the three simple rotation matrices, below symbolized as:

$$R(\overline{\chi}; \delta_\chi) = \{R(\overline{x}; \alpha_x); R(\overline{y}; \beta_y); R(\overline{z}; \gamma_z)\} \quad (12)$$

In this paper it proposes the generalized matrix:

$$\left\{ \begin{array}{l} R(\overline{\chi}; \delta_\chi) = \{R(\overline{x}; \alpha_x); R(\overline{y}; \beta_y); R(\overline{z}; \gamma_z)\} \\ = \begin{bmatrix} c(\delta_\chi \cdot \Delta_{yz}) & -s(\delta_\chi \cdot \Delta_z) & s(\delta_\chi \cdot \Delta_y) \\ s(\delta_\chi \cdot \Delta_z) & c(\delta_\chi \cdot \Delta_{zx}) & -s(\delta_\chi \cdot \Delta_x) \\ -s(\delta_\chi \cdot \Delta_y) & s(\delta_\chi \cdot \Delta_x) & c(\delta_\chi \cdot \Delta_{xy}) \end{bmatrix} \end{array} \right\} \quad (13)$$

$$\text{where} \quad \underset{\{\chi=\{u;v\}\}}{\Delta_{uv}} = \{\Delta_{yz}; \Delta_{zx}; \Delta_{xy}\} = \quad (14)$$

$$= \begin{Bmatrix} 1 \\ 0 \end{Bmatrix} \text{ if } \delta_\chi = \left\{ \begin{Bmatrix} (\beta_y; \gamma_z) \\ \alpha_x \end{Bmatrix}; \begin{Bmatrix} (\gamma_z; \alpha_x) \\ \beta_y \end{Bmatrix}; \begin{Bmatrix} (\alpha_x; \beta_y) \\ \gamma_z \end{Bmatrix} \right\};$$

$$\text{and} \quad \underset{\{\chi=u\}}{\Delta_u} = \{\Delta_x; \Delta_y; \Delta_z\} = 1 - \underset{\{\chi=\{u;v\}\}}{\Delta_{uv}}. \quad (15)$$

Successively, substituting (14) and (15) in the generalized form (13) simple rotation matrices (12) are obtained as expressions of definition. In consonance with (13) – (15), new notations are:

$$\underset{(3\times1)}{\overline{\overline{\Delta_{uv}}}} = \begin{bmatrix} c(\delta_\chi \cdot \Delta_{yz}) & c(\delta_\chi \cdot \Delta_{zx}) & c(\delta_\chi \cdot \Delta_{xy}) \end{bmatrix}^T; \quad (16)$$

$$\underset{(3\times1)}{\overline{\overline{\Delta_u}}} = \begin{bmatrix} s(\delta_\chi \cdot \Delta_x) & s(\delta_\chi \cdot \Delta_y) & s(\delta_\chi \cdot \Delta_z) \end{bmatrix}^T. \quad (17)$$

The generalized matrix becomes new expression:

$$R(\overline{\chi}; \delta_\chi) = I_3 \cdot \overline{\overline{\Delta_{uv}}} + \left[\overline{\overline{\Delta_u}} \times \right]; \quad (18)$$

where symbol I_3 is unit matrix, and $\left[\overline{\overline{\Delta_u}} \times \right]$ is skew-symmetric matrix associated to (17). The matrix (18) is obtained with classical formula:

$$R(\overline{\chi}; \delta_\chi) = \overline{\chi} \cdot \overline{\chi}^T \cdot (1 - c\delta_\chi) + I_3 \cdot c\delta_\chi + (\overline{\chi} \times s\delta_\chi) \quad (19)$$

According to researchers from [1] – [6], the three simple rotations from (10) are performed either around of the moving axes or (relatively) fixed axes belonging to $\{S\}$ or $\{0'\}/\{0\}$. Thus, the resultant rotation matrix is determined with:

$$_S^0[R] = R(\overline{u}; \alpha_u) \cdot R(\overline{v}; \beta_v) \cdot R(\overline{w}; \gamma_w). \quad (20)$$

Using results of the author regarding to matrix exponentials [2] – [13], resultant rotation matrix (20) is below written by means of exponentials:

$$\left\{\begin{array}{l} \prod\limits_{\{\bar{\chi};\,\delta_\chi\}} R(\bar{\chi};\,\delta_\chi) = exp\left[\sum\limits_{\{\chi=\{u;v;w\}\}}\left[\bar{\chi}\times\delta_\chi\right]\right] = \\ \prod\limits_{\{\chi=\{u;v;w\}\}} exp\left[\bar{\chi}\times\delta_\chi\right] = \prod\limits_{\{u;\,v\}}\left\{I_3\cdot\overline{\overline{\Delta_{uv}}} + \left[\overline{\overline{\Delta_u}}\times\right]\right\} \end{array}\right\} \quad (21)$$

$$\left\{\begin{array}{l} {}^0_S[R] = \prod\limits_{\{\bar{\chi};\,\delta_\chi\}} R(\bar{\chi};\,\delta_\chi) = e^{\bar{u}\times\alpha_u}\cdot e^{\bar{v}\times\beta_v}\cdot e^{\bar{w}\times\gamma_w} = \\ = exp[\bar{u}\times\alpha_u]\cdot exp[\bar{v}\times\beta_v]\cdot exp[\bar{w}\times\gamma_w] = \\ = exp[\bar{u}\times\alpha_u + \bar{v}\times\beta_v + \bar{w}\times\gamma_w] \end{array}\right\} \quad (22)$$

Considering the geometrical state of position (5) and the column matrix of orientation (10), the geometrical state corresponding to reference system orthogonal and right oriented is named position and orientation. This is characterized by six independent parameters (see (5) and (10)).

The above mathematical conclusions about position and orientation are generalized in the case of the rigid solid. Considering definitions form the first aligned of this section, any rigid body is composed on the one hand by infinity of material points, one the other hand by infinity of geometrical axes parallel and perpendicular one to another. They have a continuous distributed inside of the geometrical shape of the rigid (S).

The same body is also composed by infinity of assemblies of three geometrical plans orthogonal and continuous distributed in entire rigid solid.

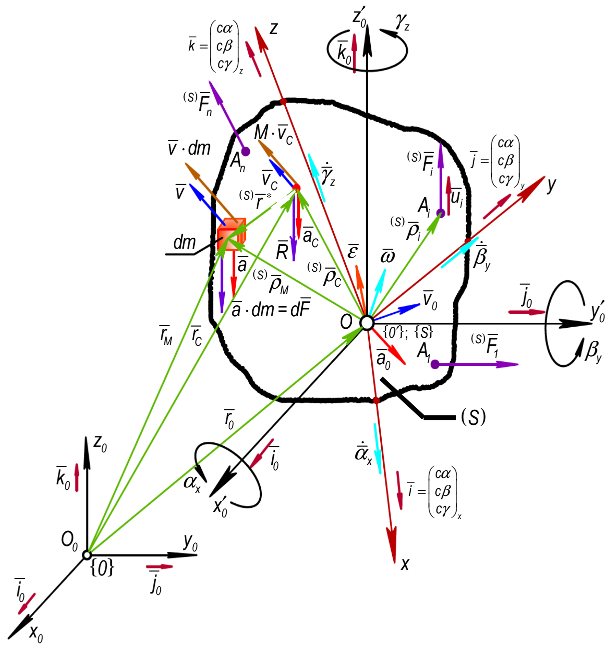

Fig.2 Rigid Body Free in Cartesian Frame

Geometrically, only one ensemble composed of three geometrical plans and orthogonal is enough to choose. It determines a reference system right oriented with the origin in arbitrary point O of the rigid body. According to (4) and Fig. 2, this is symbolized $Oxyz \equiv \{S\}$. Due to rigid character, the system $\{S\}$ is linked of inside body structure. But, the position and orientation of this frame is defined by the (5) and (10) expressions. Taking two material points from internal structure of the body, as example $M \neq O$ and $C \neq \{O;M\}$ it can write the following position expressions [4]:

$$\bar{r}_M(t) = \bar{r}_0(t) + \bar{\rho}_M(t) = \bar{r}_0(t) + {}^0_S[R](t)\cdot{}^S\bar{\rho}_M; \quad (23)$$

$$\bar{r}_C(t) = \bar{r}_0(t) + \bar{\rho}_C(t) = \bar{r}_0(t) + {}^0_S[R](t)\cdot{}^S\bar{\rho}_C; \quad (24)$$

where $\bar{\rho}_M(t) \neq \bar{\rho}_C(t)$ and $\bar{r}_M(t) \neq \bar{r}_C(t)$. (25)

When position $\bar{r}_0(t)$ and orientation ${}^0_S[R](t)$ of the moving frame $Oxyz \equiv \{S\}$ are known, then the position equation for any material point of the body can be determined. At the same time it observes that the orientation is invariant for all points of the rigid solid. So, geometrically and mechanically the body is substituted by means of its moving frame $Oxyz \equiv \{S\}$. These aspects demonstrate the authenticity that the geometrical state of the any rigid solid, free in the Cartesian space, is named *position and orientation*. This is geometrically characterized by means of the six independent parameters (six d.o.f.), as follows:

$$\overline{\overline{X}}_{(6\times1)}(t) = \begin{bmatrix} \bar{r}_0(t) \\ \cdots\cdots \\ \overline{\overline{\psi}}(t) \end{bmatrix} = \begin{bmatrix} \left[x_0(t) \quad y_0(t) \quad z_0(t)\right]^T \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \left[\alpha_u(t) \quad \beta_v(t) \quad \gamma_w(t)\right]^T \end{bmatrix} \quad (26)$$

Considering [2] – [9], $(six\ d.o.f.)$ are symbolized:

$$\bar{\theta} \neq \bar{\theta}^{(0)}; \quad \bar{\theta}(t) = \left[q_j(t); \quad j = 1 \to 6\right]^T, \quad (27)$$

$$\Delta_j = \left\{(0\ for\ q_j - linear);\,(1\ for\ q_j - angular)\right\} \quad (28)$$

where $q_j(t)$ is named the generalized coordinate; and (28) is operator that highlights type $(d.o.f.)$;

$$\left\{\begin{array}{l} \left\{\bar{\theta}(t);\, \dot{\bar{\theta}}(t);\, \ddot{\bar{\theta}}(t);\cdots;\, \overset{(m)}{\bar{\theta}}(t)\right\} = \\ = \left\{\begin{array}{c} q_i(t);\, \dot{q}_i(t);\, \ddot{q}_i(t);\cdots;\, \overset{(m)}{q_i}(t) \\ i = 1 \to n,\ m \geq 1 \end{array}\right\} \end{array}\right\}. \quad (29)$$

The symbols, from expressions (29), highlight the generalized variables of higher order in the case of the current and sudden movements. The character (m) represents the time deriving order.

In the advanced mechanics, instead of (10), named column matrix of orientation, the angular vector of orientation is used, according to [4]. Its expression of definition is below written thus:

$$\left\{\begin{matrix} {}^{0}J_{\psi}\left[\alpha_u(t)-\beta_v(t)-\gamma_w(t)\right]= \\ \left[\bar{u} \quad R(\bar{u};\alpha_u)\cdot\bar{v} \quad R(\bar{u};\alpha_u)\cdot R(\bar{v};\beta_v)\cdot\bar{w}\right] \end{matrix}\right\}; \quad (30)$$

$$\left\{\begin{matrix} \bar{\psi}(t)={}^{0}J_{\psi}\left[\alpha_u(t)-\beta_v(t)-\gamma_w(t)\right]\cdot\bar{\bar{\psi}}(t)= \\ =\bar{\psi}\left[q_j(t)\cdot\Delta_j; \quad j=1\rightarrow k^{*}=6, \; t\right] \end{matrix}\right\}; \quad (31)$$

The conclusions and expressions of definition, synthetically disseminated in this introductory section, are compulsory applied in the advanced kinematics and dynamics of mechanical system.

2. INPUT EXPRESSIONS IN DYNAMICS

Based on especially of the author researches, in this paper, a few reformulations and new formulations regarding the advanced notions of dynamics will be presented. In the view of this beside the equations (1) – (31) from first section, the other equations regarding general motion, mass properties and the distribution of the active forces must be synthetically disseminated. For this analysis, it considers the rigid solid (S) in accordance Fig. 2, found in the general motion.

● The parametric equations of motion are (26). The first three (5) express resultant translation motion, while the last three (10) define resultant rotation movement. This is also characterized by (12) – (22), (30) and (31). Taking to study (23), it expresses the absolute position equation. It shows the variable distribution from to another material point of the body. Applying the first time and absolute derivative on (23), it obtains:

$$\left\{\begin{matrix} \dot{\bar{r}}_M(t)=\dot{\bar{r}}_0(t)+\dot{\bar{\rho}}_M(t)=\dot{\bar{r}}_0(t)+{}_{S}^{0}\left[\dot{R}\right](t)\cdot{}^{S}\bar{\rho}_M \\ =\dot{\bar{r}}_0(t)+{}_{S}^{0}\left[\dot{R}\right](t)\cdot{}_{S}^{0}[R]^{T}(t)\cdot{}_{S}^{0}[R](t)\cdot{}^{S}\bar{\rho}_M \end{matrix}\right\}. \quad (32)$$

In accordance to [1] – [7], the skew symmetric matrix associated to angular velocity vector is:

$${}_{S}^{0}\left[\dot{R}\right](t)\cdot{}_{S}^{0}[R]^{T}(t)=(\bar{\omega}\times). \quad (33)$$

As a result, linear velocity and acceleration are:

$$\bar{v}_M=\bar{v}_0+\bar{\omega}\times\bar{\rho}_M; \quad (34)$$

$$\bar{a}_M=\bar{a}_0+\bar{\varepsilon}\times\bar{\rho}_M+\bar{\omega}\times\bar{\omega}\times\bar{\rho}_M. \quad (35)$$

Using (21) and (22), the position equation is written by matrix exponentials, according to:

$$\left\{\begin{matrix} \bar{r}_M(t)=\bar{r}_0(t)+exp\left[\sum_{\{\chi=\{u;v;w\}\}}\left[\bar{\chi}\times\delta_{\chi}\right]\right]\cdot{}^{S}\bar{\rho}_M= \\ \bar{r}_0(t)+\left\{exp\left[\bar{u}\times\alpha_u+\bar{v}\times\beta_v+\bar{w}\times\gamma_w\right]\right\}\cdot{}^{S}\bar{\rho}_M \end{matrix}\right\} \quad (36)$$

In advanced kinematics and dynamics, the time derivatives of higher order for position vectors and rotation matrices must be used as follows:

$$\left\{\begin{matrix} \dfrac{d^k\bar{r}_M(t)}{dt^k}=\overset{(k-1)}{\bar{v}_M}=\sum_{j=1}^{6}\left[\dfrac{\partial\overset{(m)}{\bar{r}_M}}{\partial\overset{(m)}{q_j}}\cdot\overset{(k)}{q_j}\right]+ \\ \sum_{j=1}^{6}\sum_{r=1}^{k-1}\left\{\dfrac{\prod_{p=1}^{r}(k-p)}{p!}\cdot\left[\dfrac{p!\cdot m!}{(m+p)!}\cdot\dfrac{\partial\overset{(m+p)}{\bar{r}_M}}{\partial\overset{(m)}{q_j}}\cdot\overset{(k-p)}{q_j}\right]\right\} \end{matrix}\right\} \quad (37)$$

$$\frac{d^k}{dt^k}\left\{{}_{S}^{0}[R](t)\right\}={}_{S}^{0}[R]\left[\overset{(k)}{q_j}(t)\cdot\Delta_j\right]= \quad (38)$$

$$\left\{\begin{matrix} \sum_{j=1}^{i}\left\{\dfrac{\partial}{\partial\overset{(m)}{q_j}}\left\{{}_{S}^{0}\overset{(m)}{[R]}\right\}\cdot\Delta_j\cdot\overset{(k)}{q_j}\right\}+ \\ +\sum_{j=1}^{6}\sum_{r=1}^{k-1}\left\{\dfrac{\prod_{p=1}^{r}(k-p)}{p!}\cdot\left\{\dfrac{d^p}{dt^p}\left\{\dfrac{\partial}{\partial\overset{(m)}{q_j}}\left\{{}_{S}^{0}\overset{(m)}{[R]}\right\}\right\}\cdot\Delta_j\cdot\overset{(k-p)}{q_j}\right\}\right\}= \\ =\sum_{j=1}^{6}\left\{\dfrac{\partial}{\partial\overset{(m)}{q_j}}\left\{{}_{S}^{0}\overset{(m)}{[R]}\right\}\cdot\Delta_j\cdot\overset{(k)}{q_j}\right\}+ \\ \sum_{j=1}^{6}\sum_{r=1}^{k-1}\left\{\dfrac{\prod_{p=1}^{r}(k-p)}{p!}\cdot\left\{\dfrac{p!\cdot m!}{(m+p)!}\cdot\dfrac{\partial}{\partial\overset{(m)}{q_j}}\left\{{}_{S}^{0}\overset{(m+p)}{[R]}\right\}\cdot\Delta_j\cdot\overset{(k-p)}{q_j}\right\}\right\} \end{matrix}\right\}$$

and $\left\{\begin{matrix} k\geq 1; \quad k=\{1;2;3;4;5;.....\} \\ m\geq(k+1); \quad m=\{2;3;4;5;.....\} \end{matrix}\right\};$

where the symbols: (k) and (m) are the orders of the time derivatives concerning (37) and (38). According to researches of author [2] – [13], expressions of definition for angular velocities, and then angular accelerations of higher order are established on the basis of matrix exponentials:

$$\bar{\omega}\left[\alpha_u(t)-\beta_v(t)-\gamma_w(t)\right]= \quad (39)$$

$$\left\{\begin{matrix} =\dot{\alpha}_u(t)\cdot\{exp[0]\}\cdot\bar{u}^{(0)}+ \\ +\dot{\beta}_v(t)\cdot\left\{exp\left[\bar{u}(t)\times\alpha_u(t)\right]\right\}\cdot\bar{v}^{(0)}+ \\ +\dot{\gamma}_w(t)\cdot\left\{exp\left[\bar{u}(t)\times\alpha_u(t)\right]\cdot exp\left[\bar{v}(t)\times\beta_v(t)\right]\right\}\cdot\bar{w}^{(0)} \end{matrix}\right\};$$

$$\overset{(k)}{\bar{\omega}}\left[\alpha_u(t)-\beta_v(t)-\gamma_w(t)\right]= \quad (40)$$

$$\left\{\begin{array}{c} =\frac{d^k}{dt^k}\left\{\dot{\alpha}_u(t)\cdot\{exp[0]\}\cdot\bar{u}^{(0)}+\right. \\ +\dot{\beta}_v(t)\cdot\left\{exp\left[\bar{u}(t)\times\alpha_u(t)\right]\right\}\cdot\bar{v}^{(0)}+ \\ \left.+\dot{\gamma}_w(t)\cdot\left\{exp\left[\bar{u}(t)\times\alpha_u(t)\right]\cdot exp\left[\bar{v}(t)\times\beta_v(t)\right]\right\}\cdot\bar{w}^{(0)}\right\} \end{array}\right\}$$

Remark: The using of matrix exponentials apparently seems to be complicatedly, but these have great advantages of not using reference systems. This is visible in above equations by $\bar{\chi}^{(0)}=\left\{\bar{u}^{(0)};\ \bar{v}^{(0)};\ \bar{w}^{(0)}\right\}$. They are corresponding to initial state of the moving frame $Oxyz\equiv\{S\}$.

● An essential aspect in advanced dynamics is reflected by *the mass properties*. First of all, the position of the mass center is determined in the relation with $O_0'x_0'y_0'z_0'\equiv\{0'\}$, as follows below:

$$\bar{\rho}_C(t)=\frac{\int\bar{\rho}_M(t)\cdot dm}{\int dm}=\frac{\int\bar{\rho}_M(t)\cdot dm}{M}; \quad (41)$$

$$\int\bar{\rho}_M(t)\cdot dm=M\cdot\bar{\rho}_C(t); \quad (42)$$

where (42) is the static moment relative to $\{0'\}$. The position of the mass center, in the relation to $O_0x_0y_0z_0\equiv\{0\}$, is expressed in classical form, and then on the basis of matrix exponentials as:

$$\left\{\begin{array}{c} \bar{r}_C(t)=\frac{\int\bar{r}_M(t)\cdot dm}{\int dm}=\frac{\int\bar{r}_M(t)\cdot dm}{M}= \\ =\bar{r}_0(t)+\bar{\rho}_C(t)=\bar{r}_0(t)+{}_S^0[R](t)\cdot{}^S\bar{\rho}_C \end{array}\right\}; \quad (43)$$

$$\left\{\begin{array}{l} \bar{r}_C(t)=\bar{r}_0(t)+\exp\left[\sum\limits_{\{\chi=\{u;v;w\}\}}\left[\bar{\chi}\times\delta_\chi\right]\right]\cdot{}^S\bar{\rho}_C= \\ \bar{r}_0(t)+\left\{\exp[\bar{u}\times\alpha_u+\bar{v}\times\beta_v+\bar{w}\times\gamma_w]\right\}\cdot{}^S\bar{\rho}_C \end{array}\right\} \quad (44)$$

Applying the time derivative on (43), the linear velocity and acceleration of the mass center are:

$$\left\{\begin{array}{c} \dot{\bar{r}}_C(t)=\dot{\bar{r}}_0(t)+\dot{\bar{\rho}}_C(t)= \\ \dot{\bar{r}}_0(t)+{}_S^0\left[\dot{R}\right](t)\cdot{}_S^0[R]^T(t)\cdot{}_S^0[R](t)\cdot{}^S\bar{\rho}_C \end{array}\right\}; \quad (45)$$

$$\bar{v}_C(t)=\bar{v}_0(t)+\bar{\omega}(t)\times\bar{\rho}_C(t); \quad (46)$$

$$\bar{a}_C=\bar{a}_0+\bar{\varepsilon}\times\bar{\rho}_C+\bar{\omega}\times\bar{\omega}\times\bar{\rho}_C \quad (47)$$

Linear and absolute accelerations of higher order corresponding to mass center are below defined:

$$\overset{(k)}{\bar{v}}_C(t)=\frac{d^{k+1}}{dt^{k+1}}\left\{\bar{r}_0(t)+{}_S^0[R](t)\cdot{}^S\bar{\rho}_C\right\}; \quad (48)$$

$$\left\{\begin{array}{l} \overset{(k)}{\bar{v}}_C(t)=\frac{d^{k+1}}{dt^{k+1}}\left\{\bar{r}_0(t)+exp\left[\sum\limits_{\{\chi=\{u;v;w\}\}}\left[\bar{\chi}\times\delta_\chi\right]\right]\cdot{}^S\bar{\rho}_C\right\}= \\ \frac{d^{k+1}}{dt^{k+1}}\left\{\bar{r}_0(t)+\left\{exp[\bar{u}\times\alpha_u+\bar{v}\times\beta_v+\bar{w}\times\gamma_w]\right\}\cdot{}^S\bar{\rho}_C\right\} \end{array}\right\}$$

Using classical expression (46)/(48) it obtains:

$$\overset{(k)}{\bar{v}}_{C_i}(t)=\overset{(k)}{\bar{v}}_0(t)+\frac{d^k}{dt^k}\left[\bar{\omega}(t)\times\bar{\rho}_C(t)\right], \quad (49)$$

$where\ u=\{(\bar{\omega}\times);\bar{\rho}_C\};\ v=\{\bar{\rho}_C;\bar{\omega}\};\ v\neq u;\ \chi=\{u;v\}:$

$$\left\{\begin{array}{l} \frac{d^k}{dt^k}\left[\bar{\omega}(t)\times\bar{\rho}_C(t)\right]=\sum\limits_{\{u;v\}}\left\{\overset{(k)}{u}\cdot v\right\}+\frac{k}{0!}\cdot\sum\limits_{\{u;v\}}\left\{\overset{(k-1)}{u}\cdot\dot{v}\right\} \\ +\left(k-\Delta_k\right)\cdot\left[k-\left(j+1-\Delta_k\right)\right]\cdot\sum\limits_{\{u;v\}}\left\{\overset{(k-2)}{u}\cdot\ddot{v}\right\}\cdot\delta_k \\ +k\cdot\left[k-\left(2-\Delta_k\right)\right]\cdot\sum\limits_{\{u;v\}}\left\{\overset{(k-2)}{u}\cdot\overset{(3)}{v}\right\}\cdot\delta_{kk} \\ +(k-1)\cdot\left[k-2\cdot(1-\delta_{kk})\right]\cdot\sum\limits_{\{u;v\}}\left\{\overset{(k-j)}{u}\cdot\overset{(k-j)}{v}\right\}\cdot\Delta_k \end{array}\right\}.$$

In the cases of the time derivatives of order (k) position of terms from (49) must be respected:

$$\left\{\begin{array}{c} and\ 1\leq k\leq 8;\ \ \delta_{k(k)}=\{(0;\ k\leq 4(6));\ (1;\ k\geq 4(6))\} \\ \Delta_k=\{(1;\ k=2\cdot j);\ (0;\ k\neq 2\cdot j),\ and\ j\geq 1\} \end{array}\right\}.$$

In dynamics next expression is necessary:

$$\bar{\rho}_M(t)=\bar{\rho}_C(t)+\bar{r}^*(t)={}_S^0[R](t)\cdot\left({}^S\bar{\rho}_C+{}^S\bar{r}^*\right);$$

$$\bar{\omega}(t)\times\bar{\rho}_M(t)=\bar{\omega}(t)\times\bar{\rho}_C(t)+\bar{\omega}(t)\times\bar{r}^*(t). \quad (50)$$

● Beside mass and position of the mass center an essential aspect they have inertia properties in the cases of the rotation motions. These are named mechanical moments of inertia [4] – [5]. According to Fig. 2, the position of elementary mass (dm) relative to mass center is defined by means of the position vector as: $\bar{r}^*=\bar{r}^*(t)$. But, considering (43), the next property is obtained:

$$\frac{\int\bar{r}^*(t)\cdot dm}{\int dm}=\frac{\int\bar{r}^*(t)\cdot dm}{M}=0; \quad (51)$$

$$wence\ \int\bar{r}^*(t)\cdot dm={}_S^0[R](t)\cdot\int{}^S\bar{r}^*\cdot dm=0. \quad (52)$$

Using the researches of the author [1] – [4], the inertial tensor and its variation law relative to concurrent frames is established, as follows:

$$\left\{\begin{array}{c} I_S^*=\int\left(\bar{r}^*\times\right)\cdot\left(\bar{r}^*\times\right)^T\cdot dm= \\ ={}_S^0[R]\cdot\left[\int\left({}^S\bar{r}^*\times\right)\cdot\left({}^S\bar{r}^*\times\right)^T\cdot dm\right]\cdot{}_S^0[R]^T \end{array}\right\}; \quad (53)$$

where the mass integral is squared matrix, thus:

$$\int\left({}^{S}\bar{r}^{*}\times\right)\cdot\left({}^{S}\bar{r}^{*}\times\right)^{T}\cdot dm={}^{S}I_{S}^{*}. \quad (54)$$

This is inertial tensor axial and centrifugal of the body (S) in with relation $\{S^{*}\}$ applied in the mass center (C), having property: $\{S^{*}\}_{OR}\equiv\{S\}_{OR}$.

From (53) the variation law of the inertial tensor relative to concurrent frames in the mass center: $\{S^{*}\}$ *and* $\{0^{*}\}_{OR}\equiv\{0'\}_{OR}$ is established, as follows

$$I_{S}^{*}=\int\left(\bar{r}^{*}\times\right)\cdot\left(\bar{r}^{*}\times\right)^{T}\cdot dm={}_{S}^{0}[R]\cdot{}^{S}I_{S}^{*}\cdot{}_{S}^{0}[R]^{T}. \quad (55)$$

In the following steps the inertial tensor axial and centrifugal in relation with $\{0'\}$ is defined:

$$\left\{\begin{array}{c}I_{S}'=\int(\bar{\rho}_{M}\times)\cdot(\bar{\rho}_{M}\times)^{T}\cdot dm=\\=\,{}_{S}^{0}[R]\cdot\left[\int\left({}^{S}\bar{\rho}_{M}\times\right)\cdot\left({}^{S}\bar{\rho}_{M}\times\right)^{T}\cdot dm\right]\cdot{}_{S}^{0}[R]^{T}\end{array}\right\}; \quad (56)$$

$$where\ \int\left({}^{S}\bar{\rho}_{M}\times\right)\cdot\left({}^{S}\bar{\rho}_{M}\times\right)^{T}\cdot dm={}^{S}I_{S}; \quad (57)$$

$$and\ I_{S}'=\int(\bar{\rho}_{M}\times)\cdot(\bar{\rho}_{M}\times)^{T}\cdot dm={}_{S}^{0}[R]\cdot{}^{S}I_{S}\cdot{}_{S}^{0}[R]^{T}.$$

The position equation (50) is changed in a skew symmetric matrix, and this is substituted in (56):

$$(\bar{\rho}_{M}\times)=(\bar{\rho}_{C}\times)+\left(\bar{r}^{*}\times\right);$$

$$I_{S}'=(\bar{\rho}_{C}\times)\cdot(\bar{\rho}_{C}\times)^{T}\cdot\int dm+\int\left(\bar{r}^{*}\times\right)\cdot\left(\bar{r}^{*}\times\right)\cdot dm; \quad (58)$$

$$(\bar{\rho}_{C}\times)\cdot(\bar{\rho}_{C}\times)^{T}\cdot\int dm=M\cdot(\bar{\rho}_{C}\times)\cdot(\bar{\rho}_{C}\times)^{T}=I_{SC}' \quad (59)$$

$$I_{S}'={}_{S}^{0}[R]\cdot{}^{S}I_{S}\cdot{}_{S}^{0}[R]^{T}=I_{SC}'+I_{S}^{*}. \quad (60)$$

According to [4] – [5], the matrix expression (60) characterizes the generalized variation law of *the inertial tensor axial and centrifugal* in relation with frame $\{0'\}$. The expression (59) is named the inertia matrix axial and centrifugal of the mass center relative to $\{0'\}$. Sometimes, the inertial tensor axial and centrifugal is defined in relation with absolute frame $\{0\}_{OR}=\{0'\}_{OR}$, thus:

$$\left\{\begin{array}{c}I_{S}=\int(\bar{r}_{M}\times)\cdot(\bar{r}_{M}\times)^{T}\cdot dm=\\M\cdot(\bar{r}_{0}\times)\cdot(\bar{r}_{0}\times)^{T}+I_{S}'=I_{SO}+I_{S}'=I_{SO}+I_{SC}'+I_{S}^{*}\end{array}\right\}. \quad (61)$$

● Another essential aspect for any dynamical study it consists in the distribution of the active forces, that determine the general motion of the rigid solid. Its distribution is shown below as:

$$\left\{{}^{(S)}\bar{F}_{i}={}^{(S)}\bar{u}_{i}\cdot F_{i};\ A_{i};\ {}^{(S)}\bar{\rho}_{i};\ i=1\rightarrow n\right\}; \quad (62)$$

$$where\ \bar{F}_{i}={}_{S}^{0}[R]\cdot{}^{S}\bar{F}_{i}={}_{S}^{0}[R]\cdot{}^{S}\bar{u}_{i}\cdot F_{i}; \quad (63)$$

$$and\ \bar{r}_{i}(t)=\bar{r}_{0}(t)+\bar{\rho}_{i}(t)=\bar{r}_{0}(t)+{}_{S}^{0}[R](t)\cdot{}^{S}\bar{\rho}_{i}. \quad (64)$$

As a result, the reduction torsor relative $\{0'\}$ is:

$$\left\{\begin{array}{c}\bar{R}^{*}=\sum_{i=1}^{n}\bar{F}_{i}={}_{S}^{0}[R]\cdot\sum_{i=1}^{n}{}^{S}\bar{F}_{i}=\\{}_{S}^{0}[R]\cdot\sum_{i=1}^{n}{}^{S}\bar{u}_{i}\cdot F_{i}=\int d\bar{F}=\int\bar{a}_{M}\cdot dm\end{array}\right\}; \quad (65)$$

$$\left\{\begin{array}{c}\bar{M}_{O}'=\sum_{i=1}^{n}\bar{\rho}_{i}\times\bar{F}_{i}={}_{S}^{0}[R]\cdot\sum_{i=1}^{n}\left({}^{S}\bar{\rho}_{i}\times{}^{S}\bar{F}_{i}\right)=\\{}_{S}^{0}[R]\cdot\sum_{i=1}^{n}F_{i}\cdot\left({}^{S}\bar{\rho}_{i}\times{}^{S}\bar{u}_{i}\right)=\int\bar{\rho}_{M}\times d\bar{F}\end{array}\right\}. \quad (66)$$

Since the resultant vector (65) is invariant with any reduction pole, this means that changing the pole from (O) in $(O_{0})\in\{0\}$ resultant moment is highlighted by the variation law, as follows:

$$\bar{M}_{O}=\bar{r}_{0}\times\bar{R}+\bar{M}_{O}'=\int\bar{r}_{M}\times d\bar{F}=\int\bar{r}_{M}\times\bar{a}_{M}\cdot dm. \quad (67)$$

In the above equations $\bar{a}_{M}$ is substituted by (35).

3. PARAMETERS OF HIGHER ORDER

Velocities, as well as the accelerations of higher order (39), (40), (46) – (48) can be also established by means of the following vectors:

$$\left\{\begin{array}{c}\bar{r}_{0}(t)+exp\left[\sum_{\{\chi=\{u;v;w\}\}}\left[\bar{\chi}(t)\times\delta_{\chi}\right]\right]\cdot{}^{S}\bar{\rho}_{C}=\\=\bar{r}_{C}\left[q_{j}(t);\quad j=1\rightarrow k^{*}=6,\ t\right]=\bar{r}_{C}(t)\end{array}\right\}; \quad (68)$$

$$\bar{\psi}(t)={}^{0}J_{\psi}\left[\alpha_{u}(t)-\beta_{v}(t)-\gamma_{Cw}(t)\right]\cdot\bar{\bar{\psi}}(t); \quad (69)$$

$$\left\{\begin{array}{c}\bar{\psi}(t)=\alpha_{u}(t)\cdot\{exp[0]\}\cdot\bar{u}^{(0)}+\\+\beta_{v}(t)\cdot\left\{exp\left[\bar{u}(t)\times\alpha_{u}(t)\right]\right\}\cdot\bar{v}^{(0)}+\\+\gamma_{w}(t)\cdot\left\{exp\left[\bar{u}(t)\times\alpha_{u}(t)\right]\cdot exp\left[\bar{v}(t)\times\beta_{v}(t)\right]\right\}\cdot\bar{w}^{(0)}\\=\bar{\psi}_{i}\left[q_{j}(t)\cdot\Delta_{j};\quad j=1\rightarrow k^{*}=6,\ t\right]\end{array}\right\}$$

where (68) is identical with (43) / (44), and (69) named the orientation vector is written by means of expressions: (10), (22), (28), (30) and (31).

An essential component (30) included in (69) is known as angular transfer matrix defined as function of set of orientation angles. Considering (68) and (69) it observes that they are functions of generalized variables (27) – (29). Actually, the six generalized variables are the independent parameters of position and orientation from (26).

● Using researches of author from [9] – [13], on the time vector functions of position (68) and orientation (69), the differentials properties compulsory applied in advanced kinematics and dynamics have been developed as below follows:

$$\frac{\partial \bar{r}_C}{\partial q_j}=\frac{\partial \bar{v}_C}{\partial \dot{q}_j}=\frac{\partial \bar{a}_C}{\partial \ddot{q}_j}=\frac{\partial \dot{\bar{a}}_C}{\partial \dddot{q}_j}=\frac{\partial \ddot{\bar{a}}_C}{\partial \ddddot{q}_j}=\ldots\equiv\frac{\partial \overset{(m)}{\bar{r}_C}}{\partial \overset{(m)}{q_j}}, \quad (70)$$

$$\frac{\partial \bar{\psi}}{\partial q_j}=\frac{\partial \dot{\bar{\psi}}}{\partial \dot{q}_j}=\frac{\partial \bar{\varepsilon}}{\partial \ddot{q}_j}=\ldots=\frac{\partial \ddot{\bar{\varepsilon}}}{\partial \ddddot{q}_j}=\ldots\equiv\frac{\partial \overset{(m)}{\bar{\psi}}}{\partial \overset{(m)}{q_j}}, \quad (71)$$

$$\left\{\begin{array}{l} \dfrac{d}{dt}\left(\dfrac{\partial \bar{r}_C}{\partial q_j}\right)=\dfrac{\partial \bar{v}_C}{\partial q_j}=\dfrac{\partial}{\partial q_j}\left(\sum_{m=1}^{k^*=6}\dfrac{\partial \bar{r}_C}{\partial q_m}\cdot \dot{q}_m\right)= \\ =\dfrac{1}{m+1}\cdot\dfrac{\partial \overset{(m-1)}{\bar{a}_C}}{\partial \overset{(m)}{q_j}}=\dfrac{1}{m+1}\cdot\dfrac{\partial \overset{(m+1)}{\bar{r}_C}}{\partial \overset{(m)}{q_j}},\quad m\geq 0 \end{array}\right\}, \quad (72)$$

$$\left\{\begin{array}{l} \dfrac{d}{dt}\left(\dfrac{\partial \bar{\psi}}{\partial q_j}\right)=\dfrac{\partial \bar{\omega}}{\partial q_j}=\dfrac{\partial}{\partial q_j}\left(\sum_{m=1}^{k^*=6}\dfrac{\partial \bar{\psi}}{\partial q_m}\cdot \dot{q}_m\right)= \\ =\dfrac{1}{m+1}\cdot\dfrac{\partial \overset{(m-1)}{\bar{\varepsilon}}}{\partial \overset{(m)}{q_j}}=\dfrac{1}{m+1}\cdot\dfrac{\partial \overset{(m+1)}{\bar{\psi}}}{\partial \overset{(m)}{q_j}},\quad m\geq 0 \end{array}\right\}, \quad (73)$$

$$\left\{\frac{d^{k-1}}{dt^{k-1}}\left(\frac{\partial \bar{r}_C}{\partial q_j}\right)=\frac{(k-1)!\cdot m!}{(m+k-1)!}\cdot\frac{\partial \overset{(m+k-1)}{\bar{r}_C}}{\partial \overset{(m)}{q_j}}\right\}, \quad (74)$$

$$\left\{\begin{array}{l} \dfrac{d^{k-1}}{dt^{k-1}}\left(\dfrac{\partial \bar{\psi}_i}{\partial q_j}\cdot\Delta_j\right)=\dfrac{(k-1)!\cdot m!}{(m+k-1)!}\cdot\dfrac{\partial \overset{(m+k-3)}{\bar{\varepsilon}}}{\partial \overset{(m)}{q_j}}\cdot\Delta_j= \\ =\dfrac{(k-1)!\cdot m!}{(m+k-1)!}\cdot\dfrac{\partial \overset{(m+k-1)}{\bar{\psi}}}{\partial \overset{(m)}{q_j}}\cdot\Delta_j \end{array}\right\} \quad (75)$$

$$\left\{\begin{array}{l} k\geq 1;\ k=\{1;2;3;4;5;.....\}; \\ m\geq (k+1);\ m=\{2;3;4;5;.....\} \end{array}\right\}. \quad (76)$$

The symbols (76) highlight time deriving orders. Using (69) – (76), the next expressions become:

$$\bar{v}_C(t)=\sum_{j=1}^{k^*=6}\frac{\partial \bar{r}_C(t)}{\partial q_j}\cdot\dot{q}_j(t)=\sum_{j=1}^{k^*=6}\frac{\partial \overset{(m)}{\bar{r}_C}(t)}{\partial \overset{(m)}{q_j}}\cdot\dot{q}_j(t); \quad (77)$$

$$\left\{\begin{array}{c} \bar{a}_C(t)=\dot{\bar{v}}_C(t)=\ddot{\bar{r}}_C(t)= \\ =\sum\limits_{j=1}^{k^*=6}\dfrac{\partial \overset{(m)}{\bar{r}_C}(t)}{\partial \overset{(m)}{q_j}}\cdot\ddot{q}_j(t)+\sum\limits_{j=1}^{k^*=6}\dfrac{1}{m+1}\cdot\dfrac{\partial \overset{(m+1)}{\bar{r}_C}(t)}{\partial \overset{(m)}{q_j}}\cdot\dot{q}_j(t) \end{array}\right\};$$

$$\bar{\omega}_i(t)={}^{0}J_{\psi}\left[\alpha_u(t)-\beta_v(t)-\gamma_{Cw}(t)\right]\cdot\frac{\partial \bar{\bar{\psi}}(t)}{\partial t}= \quad (78)$$

$$\left\{=\sum_{j=1}^{k^*=n}\frac{\partial \bar{\psi}_i(t)}{\partial q_j}\cdot\Delta_j\cdot\dot{q}_j(t)=\sum_{j=1}^{k^*=n}\frac{\partial \overset{(m)}{\bar{\psi}_i}(t)}{\partial \overset{(m)}{q_j}}\cdot\Delta_j\cdot\dot{q}_j(t)\right\};$$

$$\left\{\begin{array}{l} \bar{\varepsilon}_i(t)=\dot{\bar{\omega}}_i(t)=\sum\limits_{j=1}^{k^*=n}\dfrac{\partial \overset{(m)}{\bar{\psi}_i}(t)}{\partial \overset{(m)}{q_j}}\cdot\Delta_j\cdot\ddot{q}_j(t)+ \\ +\dfrac{1}{m+1}\cdot\sum\limits_{j=1}^{k^*=n}\dfrac{\partial \overset{(m+1)}{\bar{\psi}_i}}{\partial \overset{(m)}{q_j}}\cdot\Delta_j\cdot\dot{q}_j(t)=\ddot{\bar{\psi}}_i(t) \end{array}\right\}; \quad (79)$$

$$\left\{\begin{array}{l} \overset{(k-1)}{\bar{a}_{C_i}}(t)=\overset{(k)}{\bar{v}_{C_i}}(t)=\sum\limits_{j=1}^{k^*=n}\dfrac{d^{k-1}}{dt^{k-1}}\left[\dfrac{\partial \overset{(m)}{\bar{r}_{C_i}}(t)}{\partial \overset{(m)}{q_j}}\cdot\ddot{q}_j(t)\right] \\ +\sum\limits_{j=1}^{k^*=n}\dfrac{d^{k-1}}{dt^{k-1}}\left[\dfrac{1}{m+1}\cdot\dfrac{\partial \overset{(m+1)}{\bar{r}_{C_i}}(t)}{\partial \overset{(m)}{q_j}}\cdot\dot{q}_j(t)\right]=\overset{(k+1)}{\bar{r}_{C_i}}(t) \end{array}\right\} \quad (80)$$

$$\left\{\begin{array}{l} \overset{(k-1)}{\bar{\varepsilon}_i}(t)=\overset{(k)}{\bar{\omega}_i}(t)=\sum\limits_{j=1}^{k^*=n}\dfrac{d^{k-1}}{dt^{k-1}}\left[\dfrac{\partial \overset{(m)}{\bar{\psi}_i}(t)}{\partial \overset{(m)}{q_j}}\cdot\Delta_j\cdot\ddot{q}_j(t)\right] \\ +\dfrac{1}{m+1}\cdot\sum\limits_{j=1}^{k^*=n}\dfrac{d^{k-1}}{dt^{k-1}}\left[\dfrac{\partial \overset{(m+1)}{\bar{\psi}_i}}{\partial \overset{(m)}{q_j}}\cdot\Delta_j\cdot\dot{q}_j(t)\right]=\overset{(k+1)}{\bar{\psi}_i}(t) \end{array}\right\} \quad (81)$$

The expressions (77) and (80) are identical with (46) – (49), and they are referring to the linear velocity and linear accelerations of higher order, corresponding to mass center. The others: (78), (79) and (81) identical with (39) and (40) are the angular velocities and angular accelerations of higher order for rigid solid in general motion.

● Analyzing all input parameters for advanced kinematics and dynamics, it results that they are functions of generalized variables (27) / (29), as well their time derivatives. So, according to author researches they can be developed using polynomial interpolating functions [4] and [13]. It proposes following functions of higher order:

$$\left\{\begin{array}{l} \overset{(m-p)}{q_{ji}}(\tau)=(-1)^p\cdot\dfrac{(\tau_i-\tau)^{p+1}}{t_i\cdot(p+1)!}\cdot\overset{(m)}{q_{ji-1}}+ \\ +\dfrac{(\tau-\tau_{i-1})^{p+1}}{t_i\cdot(p+1)!}\cdot\overset{(m)}{q_{ji}}+\delta_p\cdot\sum\limits_{k=1}^{p}\dfrac{\tau^{p-k}}{(p-k)!}\cdot a_{jik} \end{array}\right\}; \quad (82)$$

$$\left\{\begin{array}{c} \textit{where}\ \ p=0\rightarrow m \\ m-\textit{deriving order},\ m\geq 2,\ m=2,3,4,5,\ldots \\ \delta_p=\{(0,\ p=0);(1;p\geq 1)\} \\ j=1\rightarrow n\ \ \textit{degrees of freedom}-(\textit{d.o.f.}) \\ i=1\rightarrow s\ \ \textit{intervals of motion trajectories} \\ \tau-\textit{actual time variable} \\ t_i=\tau_i-\tau_{i-1}\ (\textit{time to each trajectory interval}) \end{array}\right\}; \quad (83)$$

For each interval of trajectory $(i=1\to s)$, number of unknowns is $(m+1)$, and their significance is:

$$\left.\begin{cases} \left(a_{jik}\right) \text{ for } k=1\to m; \text{ and } \left(\overset{(m)}{q}_{ji-1}\right) \text{ for } i=2\to s \\ \text{where } \left(a_{jik}\right) - \text{integration constants, and} \\ \left(\overset{(m)}{q}_{ji-1}\right) - \text{generalized accelerations of } (m) \text{ order} \end{cases}\right\} \quad (84)$$

The determination the unknowns (84) requires, in accordance with [4] – [13], the application of the geometrical and kinematical constraints as:

$$\left.\begin{cases} (\tau_0)\Rightarrow \overset{(m-p)}{q}_{j0},\ \ p=0\to m;\ \ (\tau_s)\Rightarrow\left\{\overset{(m)}{q}_{js},\ q_{js}\right\} \\ (\tau_i)\Rightarrow\begin{cases} \overset{(2)}{q}_{ji} - \text{generalized accelerations} \\ \left\{\begin{matrix} \overset{(m-p)}{q}_{ji}\left(\tau^+\right)=\overset{(m-p)}{q}_{ji+1}\left(\tau^-\right),\ \ p=0\to m \\ \text{continuity conditions} \end{matrix}\right\} \\ \text{all conditions are applied to each } (\tau_i) \\ \text{where } \ i=1\to s-1 \end{cases} \end{cases}\right\} \quad (85)$$

Finally, the results (82) will be substituted in the advanced notions of kinematics and dynamics.

Remarks: The input expressions and parameters of higher order form the three sections of this paper are compulsory applied in the definition of the dynamic notions of higher order, such as: momentum, angular momentum, kinetic energy, acceleration energy of higher order. They will be included in the dynamics theorems of the current and sudden mechanical motion of the bodies.

4. ADVANCED DYNAMICS THEOREMS

The fundamental theorems, corresponding to dynamics of the rigid solid are: motion theorem of the mass center (momentum theorem), theorem of the angular momentum and theorem of the kinetic energy in differential form. These are in consonance with scientific literature, for example [4], [5] and [13]. Applying the input expressions and parameters of higher order, see previous sections, the main objective of this section consists in a few reformulations of the fundamental theorems, in consonance with the general motion of the rigid solid, see Fig.2.

So, *the motion theorem of the mass center* is characterized by means of the next equation:

$$M_i\cdot\bar a_C = M_i\cdot\dot{\bar v}_C = M_i\cdot\ddot{\bar r}_C = \bar R^*, \qquad (86)$$

where $\bar R$ is resultant vector of active forces (65).

Substituting the linear acceleration of the mass center with (77), the theorem (86) is changed as:

$$M\cdot\sum_{j=1}^{k^*=6}\left[\frac{\partial\overset{(m)}{\bar r_C}}{\partial\overset{(m)}{q_j}}\cdot\ddot q_j+\frac{1}{m+1}\cdot\frac{\partial\overset{(m+1)}{\bar r_C}}{\partial\overset{(m)}{q_j}}\cdot\dot q_j\right]=\bar R. \qquad (87)$$

The theorem of the angular momentum, relative to mass center (Euler's equation) is defined by:

$$I_S^*\cdot\bar\varepsilon+\frac{d}{dt}\left(I_S^*\right)\cdot\bar\omega=I_S^*\cdot\bar\varepsilon+\bar\omega\times I_S^*\cdot\bar\omega=\bar M_C^*. \qquad (88)$$

Substituting angular velocity and acceleration with (78) and (79), the theorem (88) is changed:

$$\left.\begin{cases} I_S^*\cdot\displaystyle\sum_{j=1}^{k^*=6}\left[\frac{\partial\overset{(m)}{\bar\psi}}{\partial\overset{(m)}{q_j}}\cdot\Delta_j\cdot\ddot q_j+\frac{1}{m+1}\cdot\frac{\partial\overset{(m+1)}{\bar\psi}}{\partial\overset{(m)}{q_j}}\cdot\Delta_j\cdot\dot q_j\right]+ \\ \displaystyle\sum_{j=1}^{k^*=6}\sum_{p=1}^{k^*=6}\left[\frac{\partial\overset{(m)}{\bar\psi}}{\partial\overset{(m)}{q_j}}\right]\times\left[I_S^*\cdot\frac{\partial\overset{(m)}{\bar\psi}}{\partial\overset{(m)}{q_p}}\right]\cdot\Delta_j\cdot\Delta_p\cdot\dot q_j\cdot\dot q_p=\bar M_C^* \end{cases}\right\} \quad (89)$$

$$\text{where } \Delta_{j(p)}=\left\{\left(0,\ q_{j(p)}\in\bar r_C\right);\left(1,\ q_{j(p)}\in\bar\psi\right)\right\}, \quad (90)$$

and $\bar M_C^*\equiv\bar M_O', O\equiv C, \text{and } \bar\rho_C=0$ (see (66)) is the resultant moment of active forces, while I_S^* is inertia tensor axial and centrifugal (55), the two parameters are in relation with the mass center.

The theorem of the kinetic energy in differential form is considered the most general theorem of dynamics. Its equation of definition is written as:

$$E_C=\frac{1}{2}\cdot M_i\cdot\bar v_0^T\cdot\bar v_0+M_i\cdot\bar v_0^T\cdot\left(\bar\omega\times\bar\rho_C\right)+\frac{1}{2}\cdot\bar\omega\cdot I_S'\cdot\bar\omega \quad (91)$$

Considering $O\equiv C, \text{and } \bar\rho_C=0, \text{while } I_S'\equiv I_S^*$, the kinetic energy is determined with the following:

$$E_C=\frac{1}{2}\cdot M\cdot\bar v_C^T\cdot\bar v_C+\frac{1}{2}\cdot\bar\omega\cdot I_s^*\cdot\bar\omega, \qquad (92)$$

$$\left.\begin{cases} dE_C=\displaystyle\sum_{j=1}^{k^*=n}M_i\cdot\bar a_C^T\cdot\frac{\partial\bar r_C}{\partial q_j}\cdot dq_j+ \\ +\displaystyle\sum_{j=1}^{k^*=6}\left(I_S^*\cdot\bar\varepsilon+\bar\omega\times I_S^*\cdot\bar\omega\right)^T\cdot\frac{\partial\bar\psi}{\partial q_j}\cdot dq_j\cdot\Delta_j\equiv dL \end{cases}\right\} \quad (93)$$

$$\left\{\begin{matrix} dE_C\equiv dL=\bar R^{*T}\cdot d\bar r_C+\bar M_C^{*T}\cdot d\bar\psi \\ \displaystyle\sum_{j=1}^{k^*=6}\bar R^{*T}\cdot\frac{\partial\bar r_C}{\partial q_j}\cdot dq_j+\sum_{j=1}^{k^*=6}\bar M_C^{*T}\cdot\frac{\partial\bar\psi}{\partial q_j}\cdot dq_j\cdot\Delta_j \end{matrix}\right\}; \quad (94)$$

where (92) is named König's theorem, (93) is differential expression of the kinetic energy, and (94) elementary work. Expressions (91) – (94) are corresponding to general motion of body.

Substituting (70) and (71) in (93) and (94), and left member from (89) and (90) in (93), theorem of the kinetic energy, under the differential form (94) finally this is mathematically reformulated.

In the case of the body, free in the Cartesian space, then it becomes holonomic body. A few conditions are applied on (93) and (94):

$$\left\{\begin{array}{c} q_j \neq 0,\ dq_j \neq 0,\ j = 1 \rightarrow 6 \\ q_i = 0,\ dq_i = 0,\ i = 1 \rightarrow 6,\ i \neq j \end{array}\right\}. \qquad (95)$$

They are referring to independent parameters in in the both finite and elementary displacements. After a few transformations on the differential of the theorem of the kinetic energy it obtains:

$$\left\{\begin{array}{c} \sum_{i=1}^{6}\left\{\left[\overline{R}^{*T} - M \cdot \overline{a}_{C_i}^{T}\right] \cdot \frac{\partial \overline{r}_C}{\partial q_j} + \right. \\ \left. \sum_{i=1}^{6}\left[\overline{M}_C^{*T} - \left(I_S^* \cdot \overline{\varepsilon} + \overline{\omega} \times I_S^* \cdot \overline{\omega}\right)^T\right] \cdot \frac{\partial \overline{\psi}}{\partial q_j} \cdot \Delta_j \right\} = 0 \end{array}\right\} (96)$$

According to [3] – [13], expression (96) is considered *differential generalized principle* (generalization of the D'Alembert — Lagrange principle) in analytical dynamics of systems.

Applying important transformations on (96) in consonance with researches of the author and considering acceleration energy of first order [3] – [13], it obtains the following equations:

$$\left\{\begin{array}{l} E_A^{(1)} = \frac{1}{2} \cdot M \cdot \overline{a}_0^T \cdot \overline{a}_0 + M \cdot \overline{a}_0^T \cdot \left(\overline{\varepsilon} \times \overline{\rho}_C\right) + \\ +M \cdot \overline{a}_0^T \cdot \left(\overline{\omega} \times \overline{\omega} \times \overline{\rho}_C\right) + \frac{1}{2} \cdot \overline{\varepsilon}^T \cdot I_S' \cdot \overline{\varepsilon} + \\ +\overline{\varepsilon}^T \cdot \left(\overline{\omega} \times I_S' \cdot \overline{\omega}\right) + \frac{1}{2} \cdot \overline{\omega}^T \cdot \left[\overline{\omega}^T \cdot I_S' \cdot \overline{\omega}\right] \cdot \overline{\omega} \end{array}\right\} (97)$$

When $O \equiv C,\ \overline{\rho}_C = 0,\ and\ I_S' \equiv I_S^*$, (97) becomes:

$$\left\{\begin{array}{l} E_A^{(1)} = \frac{1}{2} \cdot M \cdot \overline{a}_C^T \cdot \overline{a}_C + \frac{1}{2} \cdot \overline{\varepsilon}^T \cdot I_S^* \cdot \overline{\varepsilon} + \\ +\overline{\varepsilon}^T \cdot \left(\overline{\omega} \times I_S^* \cdot \overline{\omega}\right) + \frac{1}{2} \cdot \overline{\omega}^T \cdot \left[\overline{\omega}^T \cdot I_S^* \cdot \overline{\omega}\right] \cdot \overline{\omega} \end{array}\right\};(98)$$

$$\frac{\partial}{\partial \overset{(m)}{q_j}}\left\{\overset{(m-2)}{E_A^{(1)}}\left[\overline{\theta}(t);\cdots;\overset{(m)}{\overline{\theta}}(t)\right]\right\} = Q_{i\ddot{o}}^j\ \left[\overline{\theta}(t)\right];\ (99)$$

$$\left\{\begin{array}{c} where\ \ \overset{(0)}{E_A^{(1)}} = E_A^{(1)} \quad j = 1 \rightarrow 6, \quad k = 1 \\ m \geq \left[(k+1) = 2\right],\ and\,(k)\ are\ time\ deriving\ orders \end{array}\right\}.$$

Therefore, according to [1] – [13], (97)/(98) is named acceleration energy of first order and (99) *generalization of Gibbs – Appell's equations*.

5. CONCLUSIONS

The currently paper was devoted especially to presentation a few essential reformulations and new formulations concerning some expressions and parameters from advanced kinematics and dynamics. They become input expressions compulsory included in dynamics equations of higher order, corresponding to the current and sudden motions in the case of the rigid body. These are extended on the multibody systems.

So, unlike the classical models the author has presented in first section of paper reformulations and new formulations regarding the independent parameters of position and orientation, for any rigid body found free in the Cartesian space. In the same section is proposes a new general expression for the simple rotation matrices. In the second section of the paper, they have been presented input expressions that define the general motion of the body. In the view of this matrix exponentials and the time derivatives of higher order have been applied, concerning the linear and angular accelerations of higher order. In the same section were presented the mass and inertia properties, as well as the distribution of active forces corresponding to general motion.
In the third section, important differential properties have been developed concerning position of the mass center and orientation vector. They are also used for determine the same linear and angular accelerations of higher order above mentioned. According to author researches, the parameters of advanced kinematics have been developed as time functions with the polynomial interpolating functions of higher order, defined in this section.
The fourth section was devoted to reformulation of the fundamental theorems, in consonance with the general motion of the rigid solid. By means of a few transformations applied on the theorem of the kinetic energy, finally it was obtained the *differential generalized principle* in analytical dynamics of systems, as of the D'Alembert — Lagrange principle. Using the researches of the author in last part of this section, the expression of definition for the acceleration energy of first order corresponding to general motion of any rigid solid, and of Gibbs – Appell's equations were presented.

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Noi abordări asupra noțiunilor din mecanica avansată

Studiul dinamic al mișcărilor curente și rapide ale corpului rigid și în conformitate cu principiile diferențiale specifice dinamicii analitice a sistemelor, se bazează, printre altele, pe noțiunile avansate, cum sunt: momentul cinetic, energia cinetică, energiile de accelerații de diferite ordine și derivatele absolute în raport cu timpul a acestora de ordin superior. Noțiunile avansate sunt dezvoltate în conexiune cu variabilele generalizate, de asemenea, denumite parametrii independenți de poziție și orientare corespunzători corpului rigid olonom. Dar, sub aspect mecanic, expresiile de definiție ale noțiunilor avansate conțin pe de o parte parametrii cinematici și transformările lor diferențiale corespunzătoare mișcării absolute, iar pe de altă parte proprietățile maselor, evidențiate prin masa și tensorii inerțiali, și legea de variație generalizată a acestora. Cu ajutorul, cercetărilor autorului în această lucrare se vor prezenta reformulări și formulări noi cu privire la expresiile de intrare și parametrii cinematicii și dinamicii avansate. Aceștia devin expresii de intrare în ecuațiile dinamicii de ordin superior corespunzătoare mișcărilor curente și rapide ale corpului rigid și sistemelor multicorp. În lucrare vor fi de asemenea prezentate reformulări asupra teoremelor fundamentale ale dinamicii, asupra principiului diferențial generalizat, precum și o generalizare a ecuațiilor Gibbs – Appell.

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