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NEW FORMULATIONS ON KINEMATICS OF MECHANICAL SYSTEMS

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Abstract: *The main objective of this work is the developing of new formulations about the kinematics of the mechanical multibody systems. In order to achieve of this goal will be applied a kinematic study, in matrix form. The approaches in the paper, about new formulations in kinematics of the mechanical multibody systems, are linked to the expressions of the rotation parameters as absolute angular velocity and absolute angular acceleration. Hence, by using the exponential form to express the rotation matrices, it will been developed the generalized expressions for the angular velocity and respectively for the angular acceleration. In view of the meaning of the symbols with respect to the rotation axes, by substituting the expressions of their angular velocity and angular acceleration, there will be obtained the twelve sets of orientation angles, and their corresponding matrices of rotation, respectively the angular velocity and angular acceleration, which are expressing the absolute rotation motion, from kinematic point of view.*

Key words: *matrix exponentials, kinematics, multibody systems, angular velocity , angular acceleration*

1. KINEMATIC ANALYSIS

In this section, will be presented in matrix form, a geometrical study concerning the structure of multibody systems [1]. As shown in Fig.1, there will be considered a linked system of bodies, denoted (S_i), where $i = 1,2,...n = 1 \rightarrow n$.

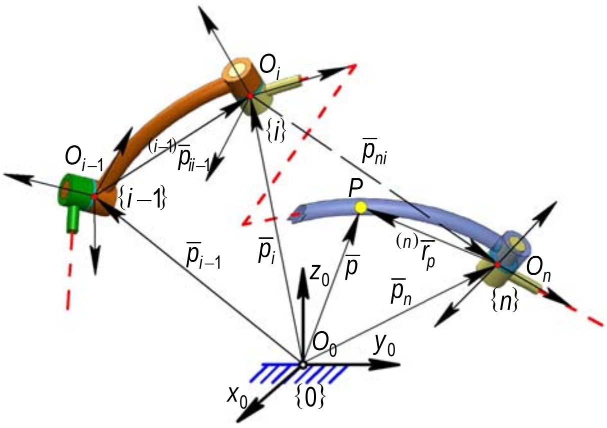

Fig. 1. Sequence form a mechanical robot structure (MRS)

Assuming, that the (n) bodies would be free, the whole system would have $6 \cdot n$ degrees of freedom. The, geometric and kinematic study of each (S_i) body, is made using a reference system, invariably linked to the body with its origin, in an arbitrary point of it. There is entered the hypothesis that mechanical system consisting of (n) rigid bodies is covered in the direct sense from $(0) \rightarrow (S_n)$. For the geometrical study, also is considered, the sequence of two bodies $(S_{i-1}) \rightarrow (S_i)$ connected, having attached the reference frames $\{i-1\}$ and $\{i\}$. Since the connections between the bodies that composing the mechanical system are allowing translations and/or rotations to the body (S_i), besides (S_{i-1}) is characterized by the number of degrees of freedom called, in analytical mechanics, generalized coordinates. They can be linear and/or angular coordinates, symbolized q_j, and are included in the symbol $\bar{\theta}_{ii-1}$ having, according with [1], [2], the signification of a column vector:

$$\bar{\theta}_{ii-1} = \left(q_j,\ j = \left(N_{i-1}+1\right) \rightarrow N_i\right)^T,\ \ i = 1 \rightarrow n \quad (1)$$

$$\bar{\theta}_i = \left[\bar{\theta}_{jj-1}^T,\ \ j = 1 \rightarrow i\right]^T \quad (2)$$

where, according to matrix algebra, T means the transposed of a matrix, and $\bar{\theta}_i$ is a column vector which characterizing the degrees of freedom the same (S_i) body, with respect to the fixed reference system. There is introduced the operator Δ_i, having two possible values:

$$\Delta_i = 1,\ for\ rotation;\ \Delta_i = 0,\ for\ translation \quad (3)$$

1.1 Orientation of the system bodies

Taking into account the issues covered in works such as [1], [2], [3] the orientation of the body (S_{i-1}), with respect to the fixed reference system is defined by the orientation matrix:

$$^{0}_{i-1}(R) = R\left[\bar{\theta}_{i-1\Delta}(t)\right] \quad (4)$$

which is a matrix function of the column vector $\bar{\theta}_{i-1\Delta}(t)$ components, represented by independent angular coordinates:

$$\bar{\theta}_{i-1\Delta}(t) = \left[\bar{\theta}^{T}_{jj-1\Delta},\ j = 1 \rightarrow i-1\right]^{T} \quad (5)$$

where $\bar{\theta}_{jj-1\Delta} = \left[q_m \cdot \Delta_m,\ m = \left(N_{j-1}+1\right) \rightarrow N_j\right]$ (6)

Therefore, the orientation of (S_i) body, with respect to $\{i-1\}$ and $\{0\}$ frames is:

$$^{i-1}_{i}(R) = R\left[q_j \cdot \Delta_j,\ j = \left(N_{i-1}+1\right) \rightarrow N_i\right] \quad (7)$$

$$^{0}_{i}(R) = {}^{0}_{i-1}(R) \cdot {}^{i-1}_{i}(R) \quad (8)$$

The previous expression is rewritten as:

$$^{0}_{i}(R) = {}^{0}_{1}(R) \cdot \ldots \cdot {}^{j-1}_{j}(R) \cdot \ldots \cdot {}^{i-1}_{i}(R) = \prod_{j=1}^{i} {}^{j-1}_{j}(R) \quad (9)$$

For $i = n$, the orientation of each axis of $\{n\}$ frame, attached to (S_n) body, with respect to $\{0\}$ fixed reference frame, is expressed with the rotation matrix:

$$\begin{aligned} ^{0}_{n}(R) &= \prod_{i=1}^{n} {}^{i-1}_{i}(R) = {}^{0}_{n-1}(R) \cdot {}^{n-1}_{n}(R) = \\ &= {}^{0}_{n}(R)\left(q_j \cdot \Delta_j,\ j = 1 \rightarrow k\right) \end{aligned} \quad (10)$$

According to [1], [2] the orientation of the (S_n) rigid body is expressed by a set of three independent orientation angles as:

$$\begin{gathered} ^{0}_{n}(R) = R\left(\alpha_A - \beta_B - \gamma_C\right), \\ A = \{x;y;z\};\ B = \{y;z;x\};\ C = \{z;x;y\} \end{gathered} \quad (11)$$

where A, B, C is the type of axis around which is performed the rotation. For orientation, there are known 12 sets of angles, one of these, being the set of Euler angles, included in the orientation vector: $\underset{(3\times1)}{\Omega} = [\psi \quad \theta \quad \varphi]^{T}$. (12)

1.2 Position of the system bodies

Knowing the position vector, between the $\{i\}-\{i-1\}$ frames, projected on $\{i-1\}$ reference frame, symbolized with $^{i-1}\bar{p}_{ii-1}$, and the rotation matrix (8), the position of $\{i\}$ frame with respect to the fixed frame $\{0\}$ is:

$$^{i-1}\bar{p}_{i\,i-1} = {}^{i-1}\bar{p}^{(0)}_{i\,i-1} + \sigma_i \cdot {}^{i-1}\bar{p}_{i\,i-1}(t) \quad (13)$$

$$\bar{p}_{ii-1} = {}^{0}_{i}(R) \cdot {}^{i-1}\bar{p}_{ii-1} \quad (14)$$

$$\bar{p}_i = \sum_{j=1}^{i} p_{jj-1} = \sum_{j=1}^{i} {}^{0}_{j-1}(R) \cdot {}^{j-1}\bar{p}_{jj-1} \quad (15)$$

where the operator $\sigma_i = \{1; 0\}$, has the signification: $\left(1, if\ {}^{i-1}\bar{p}_{ii-1} = {}^{i-1}\bar{p}_{i\,i-1}(t)\right)$, respectively $\left(0, if\ {}^{i-1}\bar{p}_{i\,i-1} = cst.\right)$.

Going through the system of bodies: $(S_1),\ldots,(S_i),\ldots,(S_n)$ in direct sense, and replacing $i = n$ in (15) results:

$$\bar{p}_n = \sum_{i=1}^{n} \bar{p}_{ii-1} = \sum_{i=1}^{n} {}^{0}_{i-1}(R) \cdot {}^{i-1}\bar{p}_{ii-1} = \bar{p}_n\left[q_j(t);\ j = 1 \rightarrow k\right] \quad (16)$$

the position of $\{n\}$ reference frame, affixed to (S_n), regard the fixed reference frame.

1.3 Angular velocities and accelerations

According to [2],[4] the starting expression in determining of the angular velocity and acceleration is:

$$^{0}_{i}(R) = {}^{0}_{i-1}(R)(t) \cdot {}^{i-1}_{i}(R) \quad (17)$$

There is applied the property $^{0}_{i}(\dot{R}) \cdot {}^{0}_{i}(R)^{T} = (\bar{\omega}_i \times)$, representing the skew symmetric matrix associated to angular velocity vector. On the expression (17) there are applied the absolute first order and second order time derivatives. By successive transformations, there are obtained the definition expressions for absolute angular velocities and accelerations of the (S_i) rigid body as follows:

$$\bar{\omega}_i = \bar{\omega}_0 + \sum_{j=1}^{i} \Delta_j \cdot \bar{\omega}_{jj-1} \quad (18)$$

$$\bar{\varepsilon}_i = \bar{\varepsilon}_0 + \sum_{j=1}^{i} \Delta_j \cdot \left(\bar{\omega}_{j-1} \times \bar{\omega}_{jj-1} + \bar{\varepsilon}_{jj-1}\right) \quad (19)$$

where $\bar{\omega}_0$ and $\bar{\varepsilon}_0$ are expressing the absolute rotation of the fixed frame, having the values:

$\bar{\omega}_0 = 0$, $\bar{\varepsilon}_0 = 0$, while the vectors $\bar{\omega}_{jj-1}$, $\bar{\varepsilon}_{jj-1}$ are characterizing the relative rotation of (S_i) body with respect to (S_{i-1}).

The projection of $\bar{\omega}_i$ and $\bar{\varepsilon}_i$ defined with (18) and (19), on its reference frame axes $\{i\}$, are established by the following expression:

$$^{i}\bar{\omega}_i = {}^{0}_{i}(R)^T \cdot \bar{\omega}_i; \qquad ^{i}\bar{\varepsilon}_i = {}^{0}_{i}(R)^T \cdot \bar{\varepsilon}_i \qquad (20)$$

Going through the cinematic chain of the system bodies: $(S_1),\ldots,(S_i),\ldots,(S_n)$, namely for $i = 1 \to n$, the expressions (18) and (19), turns into:

$$\bar{\omega}_n = \bar{\omega}_0 + \sum_{i=1}^{n} \Delta_i \cdot \bar{\omega}_{ii-1} \qquad (21)$$

$$\bar{\varepsilon}_n = \bar{\varepsilon}_0 + \sum_{i=1}^{n} \Delta_i \cdot \left(\bar{\omega}_{i-1} \times \bar{\omega}_{ii-1} + \bar{\varepsilon}_{ii-1} \right) \qquad (22)$$

Therefore, the absolute rotation of the frame $\{n\}$, attached to the (S_n) body is completely defined by the rotation matrix (10), respectively by the absolute angular velocity (21) and absolute angular acceleration (22).

1.4 Linear velocities and accelerations

In order to establish the definition expressions for absolute linear velocities and accelerations, according to [5]-[8], the starting equation is:

$$\bar{p}_i(t) = \bar{p}_{i-1}(t) + \bar{p}_{ii-1}(t) \qquad (23)$$

On the expression (23) there are applied the absolute first and second order derivatives with respect to time. By successive transformations there are obtained the absolute linear velocities and accelerations of (S_i) body, as:

$$\bar{v}_i = \bar{v}_0 + \sum_{j=1}^{i} \left[\bar{\omega}_{j-1} \times \bar{p}_{jj-1} + \sigma_j \cdot \bar{v}_{jj-1} \right] \qquad (24)$$

$$\begin{aligned} \bar{a}_i = \bar{a}_0^* &+ \sum_{j=1}^{i} \left(\bar{\varepsilon}_{j-1} \times \bar{p}_{jj-1} + \bar{\omega}_{j-1} \times \bar{\omega}_{j-1} \times \bar{p}_{jj-1} \right) + \\ &+ \sum_{j=1}^{i} \sigma_j \cdot \left(2 \cdot \bar{\omega}_j \times \bar{v}_{jj-1} + \bar{a}_{jj-1} \right) \end{aligned} \qquad (25)$$

where $\bar{v}_{jj-1}$ and $\bar{a}_{jj-1}$ are representing the relative linear velocity and acceleration of the origin $O_i \in \{i\}$ with respect to $\{i-1\}$ reference frame. The Kinematical parameters $\bar{v}_0$ and $\bar{a}_0^*$ from (24) and (25), are characterizing, alongside $\bar{\omega}_0$ and $\bar{\varepsilon}_0$, the absolute motion of the fixed frame, having the values: $\bar{v}_0 = 0$, $\bar{a}_0^* = 0$, and in the studies concerning the dynamics of the material systems $\bar{a}_0^* = \tau \cdot g \cdot \bar{k}_0$, representing a vector, equal in modulus and in opposite sense of the gravitational acceleration $(\tau = \pm 1)$. [1]

In order to project the vectors $\bar{v}_i$ and $\bar{a}_i$ defined with (24) and (25), on its $\{i\}$ reference frame axes, there are established by the following transfer matrix expression:

$$^{i}\bar{v}_i = {}^{0}_{i}(R)^T \cdot \bar{v}_i; \qquad ^{i}\bar{a}_i = {}^{0}_{i}(R)^T \cdot \bar{a}_i \qquad (26)$$

Going through the cinematic chain of the system bodies: $(S_1),\ldots,(S_i),\ldots,(S_n)$, namely for $i = 1 \to n$, the expressions (24) and (25), turns into:

$$\bar{v}_n = \bar{v}_0 + \sum_{i=1}^{n} \left(\bar{\omega}_{i-1} \times \bar{p}_{ii-1} + \sigma_i \cdot \bar{v}_{ii-1} \right) \qquad (27)$$

$$\begin{aligned} \bar{a}_n = \bar{a}_0^* &+ \sum_{i=1}^{n} \left(\bar{\varepsilon}_{i-1} \times \bar{p}_{ii-1} + \bar{\omega}_{i-1} \times \bar{\omega}_{i-1} \times \bar{p}_{ii-1} \right) + \\ &+ \sum_{i=1}^{n} \sigma_i \cdot \left[2 \left(\bar{\omega}_0 + \sum_{j=1}^{i} \Delta_j \cdot \bar{\omega}_{jj-1} \right) \times \bar{v}_{ii-1} + \bar{a}_{ii-1} \right] \end{aligned} \qquad (28)$$

Therefore, the absolute rotation of the $\{n\}$ frame, attached to the (S_n) body is completely defined by the position and orientation parameters, by the angular velocity and acceleration (21) - (22), respectively by the linear velocity and acceleration, defined by (27)-(28).

2 MATRIX EXPONENTIALS

The kinematical parameters, developed in previous section, can be expressed, according to [3], by using of matrix exponential functions. First, there is expressed the rotation matrix and the position vector between $\{j\}$ and $\{0\}$ reference frames as:

$$\begin{aligned} R_{j0} &= \left\{ exp \left\{ \sum_{i=1}^{j} \left\{ \bar{u}_i^{(0)} \times \right\} q_i \Delta_i \right\} \right\} \cdot R_{j0}^{(0)} = \\ &= \prod_{i=1}^{j} exp \left\{ \left\{ \bar{u}_i^{(0)} \times \right\} q_i \Delta_i \right\} \cdot R_{j0}^{(0)} \end{aligned} \qquad (29)$$

$$\bar{p}_j = \sum_{i=1}^{j} \left\{ exp \left\{ \sum_{k=0}^{i-1} \left\{ \bar{u}_k^{(0)} \times \right\} q_k \cdot \Delta_k \right\} \right\} \cdot \bar{b}_i \qquad (30)$$

where $R_{j0}^{(0)}$ corresponds to initial configuration of the multibody system. The column vector $\overline{b}_i$ from the expression (30), according to [1], is:

$$\overline{b}_j = \left\{ I_3 \cdot q_j + \left\{ \overline{u}_j^{(0)} \times \right\} \left[1 - \cos\left(q_j \cdot \Delta_j \right) \right] + \right. \left. + \overline{u}_j^{(0)} \cdot \overline{u}_j^{(0)T} \cdot \left[q_j - \sin\left(q_j \cdot \Delta_j \right) \right] \right\} \cdot \overline{v}_j^{(0)}. \quad (31)$$

In the previous equations, are used the symbols $\overline{u}_j = \left\{ \overline{x}_j ; \overline{y}_j ; \overline{z}_j \right\}$ and $\overline{v}_j = \left\{ \overline{p}_j \times \right\} \overline{u}_j \cdot \Delta_j + \left(1 - \Delta_j \right) \cdot \overline{u}_j$, together expressing the screw parameters or the homogeneous parameters of the oriented axis $\{j\}$ around or along which are achieved the generalized coordination. In accordance with [2], the defining expressions for angular velocities and accelerations (18), (19) can be established on the basis of matrix exponential thus:

$${}^{0}\overline{\omega}_j = \left\{ \sum_{i=1}^{j} \left\{ exp \left\{ \sum_{k=1}^{i-1} \left\{ \overline{u}_k^{(0)} \times \right\} q_k \Delta_k \right\} \right\} \overline{u}_i^{(0)} \dot{q}_i \Delta_i \right\}; \quad (32)$$

$${}^{0}\dot{\overline{\omega}}_j = \left\{ \sum_{i=1}^{j} \left\{ Mexp\left\{ V_{i1} \right\} \ddot{q}_i + Mexp\left\{ \dot{V}_{i1} \right\} \dot{q}_i \right\} \overline{u}_i^{(0)} \Delta_i \right\} \quad (33)$$

$$\underset{(3x3)}{ME}\left(V_{j1} \right) = \exp\left\{ \sum_{i=0}^{j-1} \left\{ \overline{u}_i^{(0)} \times \right\} q_i \cdot \Delta_i \right\} \quad (34)$$

$$ME\left\{ \dot{V}_{j1} \right\} = \sum_{i=1}^{j-1} \left\{ e^{V_1^*} \right\} \left\{ \overline{u}_i^{(0)} \times \right\} \dot{q}_i \cdot \Delta_i \left\{ e^{V_2^*} \right\}$$
$$where \;\; V_1^* = \sum_{k=0}^{i-1} \left\{ \overline{u}_k^{(0)} \times \right\} \cdot q_k \cdot \delta_{ik\cdot} \cdot \Delta_k ; \quad (35)$$
$$V_2^* = \sum_{m=i}^{j-1} \left\{ \overline{u}_m^{(0)} x \right\} \cdot q_m \cdot \delta_m \cdot \Delta_m$$

The velocity $\overline{v}_j$ and the acceleration $\overline{a}_j$ are expressed as:

$$\overline{v}_j = \sum_{i=1}^{j} \left\{ \sum_{m=1}^{i-1} \left\{ \exp\left\{ V_1^* \right\} \right\} \left\{ \overline{u}_m^{(0)} \times \right\} \dot{q}_m \Delta_m \left\{ \exp\left\{ V_2^* \right\} \right\} \right\} + $$
$$+ \sum_{i=1}^{j} \left\{ \exp\left\{ V_3^* \right\} \right\} \cdot \dot{\overline{b}}_i$$
$$where \;\; V_1^* = \sum_{k=0}^{m-1} \left\{ \overline{u}_k^{(0)} \times \right\} q_k \delta_{mk\cdot} \Delta_k ;$$
$$V_2^* = \sum_{l=m}^{j} \left\{ \overline{u}_l^{(0)} \times \right\} q_l \delta_l \Delta_l ;$$
$$V_3^* = \sum_{k=0}^{i-1} \left\{ \overline{u}_k^{(0)} \times \right\} q_k \cdot \Delta_k \quad (36)$$

$$\overline{a}_j = \frac{d}{dt} \left\{ \sum_{i=1}^{j} \left\{ \sum_{m=1}^{i-1} \left\{ \exp\left\{ V_1^* \right\} \right\} \left\{ \overline{k}_m^{(0)} \times \right\} \dot{q}_m \Delta_m \left\{ \exp\left\{ V_2^* \right\} \right\} \right\} + \right.$$
$$\left. + \sum_{i=1}^{j} \left\{ \exp\left\{ V_3^* \right\} \right\} \cdot \dot{\overline{b}}_i \right\}, where$$
$$V_1^* = \sum_{k=0}^{m-1} \left\{ \overline{k}_k^{(0)} \times \right\} q_k \delta_{mk\cdot} \Delta_k ; V_2^* = \sum_{l=m}^{j} \left\{ \overline{k}_l^{(0)} \times \right\} q_l \delta_l \Delta_l ; \quad (37)$$
$$V_3^* = \sum_{k=0}^{i-1} \left\{ \overline{u}_k^{(0)} \times \right\} q_k \cdot \Delta_k$$

$$\delta_{ik} = \left\{ (0; i > j-1), (1; i \le j-1) \right\}, \;\; \delta_{mk} = \left\{ (0; m > j); (1; m \le j) \right\}$$

$$where \;\; \dot{\overline{b}}_j = \left\{ I_3 + \left\{ \overline{u}_j^{(0)} \times \right\} \sin\left(q_j \cdot \Delta_j \right) + \right.$$
$$\left. + \left\{ \overline{u}_j^{(0)} \times \right\}^2 \left[1 - \cos\left(q_j \cdot \Delta_j \right) \right] \right\} \cdot \overline{v}_j^{(0)} \cdot \dot{q}_j .$$

The using of matrix exponentials apparently seems to be complicatedly, has the advantage of not using reference systems. This observation is visible in the above equations, by the occurrence of homogeneous coordinates, specific to initial configuration.

3 THE ROTATION MOTION

The analysis from previous sections revealing the fact that the absolute rotation of the frame $\{n\}$ attached to the rigid body (S_n) is defined from kinematic point of view by the rotation matrix (10), the orientation vector (12), respectively by the angular velocity (21), and angular acceleration (22). But, according to [1], [2], orientation vector $\overline{\Omega}(t)$ is substituted in expression (1), which becomes:

$${}^{0}\overline{X}(t) = \begin{bmatrix} \overline{p}_n(t) \\ \cdots\cdots \\ \overline{\Omega}(t) \end{bmatrix} = \begin{bmatrix} \left[p_{xn}(t) \;\; p_{yn}(t) \;\; p_{zn}(t) \right]^T \\ \cdots\cdots\cdots \\ \left[\alpha_A(t) \;\; \beta_B(t) \;\; \gamma_C(t) \right]^T \end{bmatrix} \quad (38)$$

$${}^{0}\overline{X}(t) = \begin{bmatrix} \overline{p}_n(t) \\ \cdots\cdots \\ \overline{\Omega}(t) \end{bmatrix} = \left\{ \begin{matrix} f_j\left[q_i(t) \cdot \delta_i ; i = 1 \to n \right] \\ j = 1 \to 6 \end{matrix} \right\}^T \quad (39)$$

$$\overline{\Omega}(t) = \begin{pmatrix} \alpha_A(t) \\ \beta_B(t) \\ \gamma_C(t) \end{pmatrix} = \begin{pmatrix} f_4\left(q_i(t) \cdot \Delta_i ; \; i = 1 \to k \right) \\ f_5\left(q_i(t) \cdot \Delta_i ; \; i = 1 \to k \right) \\ f_6\left(q_i(t) \cdot \Delta_i ; \; i = 1 \to k \right) \end{pmatrix} \quad (40)$$

In keeping with [1], [2], it is observed that the resultant rotation can be defined from kinematic point of view by any of the twelve sets of orientation angles, leading to the twelve sets of matrices. As a result, the expression (11) can be rewritten in the following form:

$$\left\{\begin{array}{l} {}^{0}_{n}[R] = R\left[\alpha_A(t) - \beta_B(t) - \gamma_C(t)\right] = \\ = R\left[\bar{A};\alpha_A(t)\right]\cdot R\left[\bar{B};\beta_B(t)\right]\cdot R\left[\bar{C};\gamma_C(t)\right] \end{array}\right\} \quad (41)$$

In the matrix expression (41), occurs the unit vectors of the axes around which the simple rotations are performed. The significance of the unit vectors is in concordance with:

$$\left\{\begin{array}{l} {}^{(n)0}\bar{A} = \left\{{}^{(n)0}\bar{x};{}^{(n)0}\bar{y};{}^{(n)0}\bar{z}\right\}; \\ \bar{A} = \left\{\bar{x} = \begin{pmatrix}1\\0\\0\end{pmatrix};\ \bar{y} = \begin{pmatrix}0\\1\\0\end{pmatrix};\ \bar{z} = \begin{pmatrix}0\\0\\1\end{pmatrix}\right\} \end{array}\right\}$$

$$\left\{\begin{array}{l} {}^{(n)0}\bar{B} = \left\{{}^{(n)0}\bar{y};{}^{(n)0}\bar{z};{}^{(n)0}\bar{x}\right\}; \\ \bar{B} = \left\{\bar{y} = \begin{pmatrix}0\\1\\0\end{pmatrix};\ \bar{z} = \begin{pmatrix}0\\0\\1\end{pmatrix};\ \bar{x} = \begin{pmatrix}1\\0\\0\end{pmatrix}\right\} \neq \bar{A} \end{array}\right\} \quad (42)$$

$$\left\{\begin{array}{l} {}^{(n)0}\bar{C} = \left\{{}^{(n)0}\bar{z};{}^{(n)0}\bar{x};{}^{(n)0}\bar{y}\right\}; \\ \bar{C} = \left\{\bar{z} = \begin{pmatrix}0\\0\\1\end{pmatrix};\ \bar{x} = \begin{pmatrix}1\\0\\0\end{pmatrix};\ \bar{y} = \begin{pmatrix}0\\1\\0\end{pmatrix}\right\} \neq \bar{B} \end{array}\right\}$$

Applying some properties, according to [2], the simple rotation matrix, components of the expression (42), are determined as follows:

$$\left\{\begin{array}{l} R\left[\bar{A};\alpha_A(t)\right] = \exp\left[(\bar{A}\times)\cdot\alpha_A(t)\right] = \\ = I_3\cdot c\left[\alpha_A(t)\right] + \\ +\bar{A}\times s\left[\alpha_A(t)\right] + \bar{A}\cdot\bar{A}^T\cdot\left[1 - c\left[\alpha_A(t)\right]\right] = \\ I_3 + \bar{A}\times s\left[\alpha_A(t)\right] + (\bar{A}\times)^2\cdot\left[1 - c\left[\alpha_A(t)\right]\right] \end{array}\right\} \quad (43)$$

$$\left\{\begin{array}{l} R\left[\bar{B};\beta_B(t)\right] = \exp\left[(\bar{B}\times)\cdot\beta_B(t)\right] = \\ = I_3\cdot c\left[\beta_B(t)\right] + \\ +\bar{B}\times s\left[\beta_B(t)\right] + \bar{B}\cdot\bar{B}^T\cdot\left[1 - c\left[\beta_B(t)\right]\right] = \\ I_3 + \bar{B}\times s\left[\beta_B(t)\right] + (\bar{B}\times)^2\cdot\left[1 - c\left[\beta_B(t)\right]\right] \end{array}\right\} \quad (44)$$

$$\left\{\begin{array}{l} R\left[\bar{C};\gamma_C(t)\right] = \exp\left[(\bar{C}\times)\cdot\gamma_C(t)\right] = \\ = I_3\cdot c\left[\gamma_C(t)\right] + \\ +\bar{C}\times s\left[\gamma_C(t)\right] + \bar{C}\cdot\bar{C}^T\cdot\left[1 - c\left[\gamma_C(t)\right]\right] = \\ I_3 + \bar{C}\times s\left[\gamma_C(t)\right] + (\bar{C}\times)^2\cdot\left[1 - c\left[\gamma_C(t)\right]\right] \end{array}\right\} \quad (45)$$

The transpose of the resultant rotation matrix (41) is:

$$\left\{\begin{array}{l} {}^{0}_{n}[R]^T = R^T\left[\alpha_A(t) - \beta_B(t) - \gamma_C(t)\right] = \\ R^T\left[\bar{C};\gamma_C(t)\right]\cdot R^T\left[\bar{B};\beta_B(t)\right]\cdot R^T\left[\bar{A};\alpha_A(t)\right] \end{array}\right\} \quad (46)$$

The matrix components of (46) is:

$$\left\{\begin{array}{l} R^T\left[\bar{A};\alpha_A(t)\right] = \exp\left[-(\bar{A}\times)\cdot\alpha_A(t)\right] = \\ = \exp\left[(\bar{A}\times)^T\cdot\alpha_A(t)\right] = I_3\cdot c\left[\alpha_A(t)\right] - \\ -\bar{A}\times s\left[\alpha_A(t)\right] - \bar{A}\cdot\bar{A}^T\cdot\left[1 - c\left[\alpha_A(t)\right]\right] = \\ I_3 - \bar{A}\times s\left[\alpha_A(t)\right] - (\bar{A}\times)^2\cdot\left[1 - c\left[\alpha_A(t)\right]\right] \end{array}\right\} \quad (47)$$

$$\left\{\begin{array}{l} R^T\left[\bar{B};\beta_B(t)\right] = \exp\left[-(\bar{B}\times)\cdot\beta_B(t)\right] = \\ = \exp\left[(\bar{B}\times)^T\cdot\beta_B(t)\right] = I_3\cdot c\left[\beta_B(t)\right] - \\ -\bar{B}\times s\left[\beta_B(t)\right] - \bar{B}\cdot\bar{B}^T\cdot\left[1 - c\left[\beta_B(t)\right]\right] = \\ I_3 - \bar{B}\times s\left[\beta_B(t)\right] - (\bar{B}\times)^2\cdot\left[1 - c\left[\beta_B(t)\right]\right] \end{array}\right\} \quad (48)$$

$$\left\{\begin{array}{l} R^T\left[\bar{C};\gamma_C(t)\right] = \exp\left[-(\bar{C}\times)\cdot\gamma_C(t)\right] = \\ = \exp\left[(\bar{C}\times)^T\cdot\gamma_C(t)\right] = I_3\cdot c\left[\gamma_C(t)\right] - \\ -\bar{C}\times s\left[\gamma_C(t)\right] - \bar{C}\cdot\bar{C}^T\cdot\left[1 - c\left[\gamma_C(t)\right]\right] = \\ I_3 - \bar{C}\times s\left[\gamma_C(t)\right] - (\bar{C}\times)^2\cdot\left[1 - c\left[\gamma_C(t)\right]\right] \end{array}\right\} \quad (49)$$

Substituting (43)-(45) in (41), and (47)-(49) in (46), there are obtained the exponential expressions of the resultant rotation matrix and it`s transpose as:

$$\left\{\begin{array}{l} {}^{0}_{n}[R] = R\left[\alpha_A(t) - \beta_B(t) - \gamma_C(t)\right] = \\ = \exp\left[\sum_{i=1}^{n}\left(\bar{k}_i^{(0)}\times\right)q_i^*\right]\cdot R_{n0}^{(0)} = \\ = \exp\left[\begin{array}{c}(\bar{A}\times)\cdot\alpha_A(t) + (\bar{B}\times)\cdot\beta_B(t) + \\ +(\bar{C}\times)\cdot\gamma_C(t)\end{array}\right] \end{array}\right\} \quad (50)$$

$$\left\{\begin{array}{l}\left\{\exp\begin{bmatrix}(\bar{A}\times)\cdot\alpha_A(t)+(\bar{B}\times)\cdot\beta_B(t)+\\+(\bar{C}\times)\cdot\gamma_C(t)\end{bmatrix}\right\}=\\=\left\{\exp\left[(\bar{A}\times)\cdot\alpha_A(t)\right]\right\}\cdot\left\{\exp\begin{bmatrix}(\bar{B}\times)\cdot\beta_B(t)+\\+(\bar{C}\times)\cdot\gamma_C(t)\end{bmatrix}\right\}\end{array}\right\}$$

$$\left\{\begin{array}{l}\left\{\exp\begin{bmatrix}(\bar{B}\times)\cdot\beta_B(t)+\\+(\bar{C}\times)\cdot\gamma_C(t)\end{bmatrix}\right\}=\\=\left\{\exp\left[(\bar{B}\times)\cdot\beta_B(t)\right]\right\}\cdot\left\{\exp\left[(\bar{C}\times)\cdot\gamma_C(t)\right]\right\}\end{array}\right\}$$

$$\left\{\begin{array}{l}R_{n0}^{T}=R_{n0}^{(0)T}\cdot\exp\left\{-\left\{\sum_{i=1}^{n}\left[\bar{k}_i^{(0)}\times\right]\cdot q_i^{*}\cdot\Delta_i\right\}\right\}=\\=\exp\begin{bmatrix}(\bar{C}\times)^{T}\cdot\gamma_C(t)+(\bar{B}\times)^{T}\cdot\beta_B(t)+\\+(\bar{A}\times)^{T}\cdot\alpha_A(t)\end{bmatrix}\end{array}\right\}\quad(51)$$

$$\left\{\begin{array}{l}\left\{\exp\begin{bmatrix}(\bar{C}\times)^{T}\cdot\gamma_C(t)+(\bar{B}\times)^{T}\cdot\beta_B(t)+\\+(\bar{A}\times)^{T}\cdot\alpha_A(t)\end{bmatrix}\right\}=\\\left\{\exp\left[(\bar{C}\times)^{T}\cdot\gamma_C(t)\right]\right\}\cdot\left\{\exp\begin{bmatrix}(\bar{B}\times)^{T}\cdot\beta_B(t)+\\+(\bar{A}\times)^{T}\cdot\alpha_A(t)\end{bmatrix}\right\}\end{array}\right\}$$

$$\left\{\begin{array}{l}\left\{\exp\begin{bmatrix}(\bar{B}\times)^{T}\cdot\beta_B(t)+\\+(\bar{A}\times)^{T}\cdot\alpha_A(t)\end{bmatrix}\right\}=\\=\left\{\exp\left[(\bar{B}\times)^{T}\cdot\beta_B(t)\right]\right\}\cdot\left\{\exp\left[(\bar{A}\times)^{T}\cdot\alpha_A(t)\right]\right\}\end{array}\right\}$$

where $R_{n0}^{(0)}$ is the resultant rotation matrix in initial configuration, which supposes that $q_i=0,\ i=1\rightarrow n$.

The expressions (21) and (22) consecrate the absolute angular velocity with projection on $\{0\}$ frame. These are generalized by taking into account the notations (40) and (42), as follows:

$$\left\{\begin{array}{l}{}^{(n)0}\bar{\omega}_n=\dot{\alpha}_A(t)\cdot{}^{(n)0}\bar{A}(t)+\\+\dot{\beta}_B(t)\cdot{}^{(n)0}\bar{B}(t)+\dot{\gamma}_C(t)\cdot{}^{(n)0}\bar{C}(t)\end{array}\right\}\quad(52)$$

The matrix form of the expressions (52) is:

$$\left\{\begin{array}{l}{}^{(n)0}\bar{\omega}_n=\dot{\alpha}_A(t)\cdot{}^{(n)0}\bar{A}(t)+\\+\dot{\beta}_B(t)\cdot{}^{(n)0}\bar{B}(t)+\dot{\gamma}_C(t)\cdot{}^{(n)0}\bar{C}(t)\end{array}\right\}=$$

$$=\left\{\left[{}^{(n)0}\bar{A}(t)\quad{}^{(n)0}\bar{B}(t)\quad{}^{(n)0}\bar{C}(t)\right]\cdot\begin{bmatrix}\dot{\alpha}_A(t)\\\cdots\\\dot{\beta}_B(t)\\\cdots\\\dot{\gamma}_C(t)\end{bmatrix}\right\}\quad(53)$$

$${}^{(n)0}\bar{\omega}_n={}^{(n)0}J_{\Omega}(t)\cdot\left[\dot{\alpha}_A(t)\quad\dot{\beta}_B(t)\quad\dot{\gamma}_C(t)\right]^{T}\quad(54)$$

The angular transfer matrix from (54) is expressed as:

$${}^{0}J_{\Omega}(t)=\left\{\bar{A}\ \vdots\ {}^{0}\bar{B}\ \vdots\ {}^{0}\bar{C}\right\}\quad(55)$$

$$\left\{\begin{array}{l}{}^{0}\bar{B}=R\left[\bar{A};\alpha_A(t)\right]\cdot\bar{B}=\\=\left\{\exp\left[(\bar{A}\times)\cdot\alpha_A(t)\right]\right\}\cdot\bar{B}\end{array}\right\}\quad(56)$$

$$\left\{\begin{array}{l}{}^{0}\bar{C}=R\left[\bar{A};\alpha_A(t)\right]\cdot R\left[\bar{B};\beta_B(t)\right]\cdot\bar{C}=\\=\left\{\exp\left[(\bar{A}\times)\cdot\alpha_A(t)+(\bar{B}\times)\cdot\beta_B(t)\right]\right\}\cdot\bar{C}\end{array}\right\}\quad(57)$$

respectively ${}^{n}J_{\Omega}(t)=\left\{{}^{n}\bar{A}\ \vdots\ {}^{n}\bar{B}\ \vdots\ \bar{C}\right\}$ (58)

$$\left\{\begin{array}{l}{}^{n}\bar{A}=R^{T}\left[\bar{C};\gamma_C(t)\right]\cdot R^{T}\left[\bar{B};\beta_B(t)\right]\cdot\bar{A}=\\=\left\{\exp\left[(\bar{C}\times)^{T}\cdot\gamma_C(t)+(\bar{B}\times)^{T}\cdot\beta_B(t)\right]\right\}\cdot\bar{A}\end{array}\right\}\quad(59)$$

$$\left\{\begin{array}{l}{}^{n}\bar{B}=R^{T}\left[\bar{C};\gamma_C(t)\right]\cdot\bar{B}=\\=\left\{\exp\left[(\bar{B}\times)^{T}\cdot\beta_B(t)\right]\right\}\cdot\bar{B}\end{array}\right\}\quad(60)$$

Taking into account (55) and (58) the absolute angular velocity of (S_n), projected on $\{0\}$ and $\{n\}$ is expressed as:

$${}^{0}\bar{\omega}_n=\left\{\begin{array}{c}\bar{A}\\R\left[\bar{A};\alpha_A(t)\right]\cdot\bar{B}\\R\left[\bar{A};\alpha_A(t)\right]\cdot R\left[\bar{B};\beta_B(t)\right]\cdot\bar{C}\end{array}\right\}^{T}\cdot\begin{bmatrix}\dot{\alpha}_A(t)\\\dot{\beta}_B(t)\\\dot{\gamma}_C(t)\end{bmatrix}\quad(61)$$

$${}^{n}\bar{\omega}_n=\left\{\begin{array}{c}R^{T}\left[\bar{C};\gamma_C(t)\right]\cdot R^{T}\left[\bar{B};\beta_B(t)\right]\cdot\bar{A}\\R^{T}\left[\bar{C};\gamma_C(t)\right]\cdot\bar{B}\\\bar{C}\end{array}\right\}^{T}\cdot\begin{bmatrix}\dot{\alpha}_A(t)\\\dot{\beta}_B(t)\\\dot{\gamma}_C(t)\end{bmatrix}\quad(62)$$

Applying the first order derivative with respect to time on expression (54) is obtained the absolute angular acceleration of the body (S_i) as follows:

$$\left\{\begin{array}{l}{}^{(n)0}\dot{\bar{\omega}}_n={}^{(n)0}J_{\psi}(t)\cdot\left[\ddot{\alpha}_A(t)\quad\ddot{\beta}_B(t)\quad\ddot{\gamma}_C(t)\right]^{T}+\\+{}^{(n)0}\dot{J}_{\psi}(t)\cdot\left[\dot{\alpha}_A(t)\quad\dot{\beta}_B(t)\quad\dot{\gamma}_C(t)\right]^{T}\end{array}\right\}\quad(63)$$

The time derivative of the angular transfer matrix is determined as:

$$ {}^{0}J_{\psi}(t)=\left\{\dot{\bar{A}} : {}^{0}\dot{\bar{B}} : {}^{0}\dot{\bar{C}}\right\} \tag{64}$$

where $$\dot{\bar{A}}=\begin{pmatrix}0 & 0 & 0\end{pmatrix}^{T} \tag{65}$$

$$\left\{\begin{aligned} {}^{0}\dot{\bar{B}} &= \frac{d}{dt}\left\{R\left[\bar{A};\alpha_A(t)\right]\cdot\bar{B}\right\} = \\ &= \frac{d}{dt}\left\{\left\{\exp\left[(\bar{A}\times)\cdot\alpha_A(t)\right]\right\}\cdot\bar{B}\right\} = \\ &= \left[(\bar{A}\times)\cdot\dot{\alpha}_A(t)\right]\cdot\left\{\exp\left[(\bar{A}\times)\cdot\alpha_A(t)\right]\right\}\cdot\bar{B} \end{aligned}\right\} \tag{66}$$

$$\dot{R}\left[\bar{A};\alpha_A(t)\right]=\left[(\bar{A}\times)\cdot\dot{\alpha}_A(t)\right]\cdot\left\{\exp\left[(\bar{A}\times)\cdot\alpha_A(t)\right]\right\}$$

$$\dot{R}\left[\bar{B};\beta_B(t)\right]=\left[(\bar{B}\times)\cdot\dot{\beta}_B(t)\right]\left\{\exp\left[(\bar{B}\times)\cdot\beta_B(t)\right]\right\}$$

$$\left\{\begin{aligned} &{}^{0}\dot{\bar{C}} = \frac{d}{dt}\left\{R\left[\bar{A};\alpha_A(t)\right]\cdot R\left[\bar{B};\beta_B(t)\right]\cdot\bar{C}\right\} = \\ &\frac{d}{dt}\left\{\left\{\exp\left[(\bar{A}\times)\cdot\alpha_A(t)+(\bar{B}\times)\cdot\beta_B(t)\right]\right\}\cdot\bar{C}\right\} \end{aligned}\right\} \tag{67}$$

$$\left\{\begin{aligned} &=\left[(\bar{A}\times)\cdot\dot{\alpha}_A(t)\right]\cdot R\left[\bar{A};\alpha_A(t)\right]\cdot R\left[\bar{B};\beta_B(t)\right]\cdot\bar{C} \\ &+R\left[\bar{A};\alpha_A(t)\right]\cdot\left[(\bar{B}\times)\cdot\dot{\beta}_B(t)\right]\cdot R\left[\bar{B};\beta_B(t)\right]\cdot\bar{C} \end{aligned}\right\}$$

Observations: The orientation angles and their time derivatives, including the notations (42) substituted into generalized expressions are (53) and (54) according to the meanings of symbols from [1] and [2]. As a result of these generalized transformations there are obtained twelve matrix expressions for the angular transfer matrix, and the absolute rotational angular velocities. Depending on the set of orientation angles used any of the twelve pairs of matrix are expressing the absolute angular velocity and absolute angular acceleration. To these are added the resultant rotation matrix (11) together characterizing in a generalized form the absolute rotation of the body (S_n) component of the multibody mechanical system.

6. CONCLUSIONS

The main objective of this work was the developing of new formulations about the kinematics of the mechanical systems. The achieving of this goal required initially a laborious kinematic study, in matrix form, applied to multibody mechanical systems. In this study were highlighted relations concerning the position and orientation of multibody system and generalized defining relations for linear and angular velocities and accelerations. An important aspect was the using of matrix exponential functions, whose undeniable advantage to classical transformations, refers to not using of reference systems, which is the effect of applying specific homogeneous coordinates of the initial configurations of mechanical systems. The new formulations in kinematics, developed in the paper are expressing the rotation parameters as absolute angular velocity and absolute angular acceleration. Using the exponential form to express the rotation matrices, it have been developed the generalized expressions for the angular velocity and respectively for the angular acceleration. In view of the meaning of the symbols with respect to the axes of rotation, by substituting the expressions of their angular velocity and angular acceleration, there have been obtained the twelve sets of orientation angles, and their corresponding matrices of rotation, respectively the angular velocity and angular acceleration, which are expressing the absolute rotation motion, from kinematic point of view.

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Noi formulări asupra cinematicii sistemelor mecanice

Rezumat: Obiectivul principal al acestei lucrări este dezvoltarea de noi formulări cu privire la cinematica sistemelor mecanice multicorp. Pentru realizarea acestui obiectiv, se va realiza un studiu cinematic în formă matriceală. Abordările din lucrare, cu privire la noile formulări în cinematica sistemelor mecanice multicorp, sunt legate de expresiile parametrilor ce exprima rotația absolută, precum viteza unghiulară absolută şi accelerația absolută unghiulare. Prin urmare, utilizând forma de exprimare exponențială a matricei de rotație, se vor dezvolta expresiile generalizate pentru viteza unghiulară şi respectiv pentru accelerația unghiulară. Ținând seama de semnificația simbolurilor cu privire la axele de rotație, prin substituirea lor în expresiile vitezei unghiulare şi accelerației unghiulare, se obțin cele douăsprezece seturi de unghiuri de orientare, matricele de rotație şi corespunzătoare lor, vitezele unghiulare şi respectiv accelerațiile unghiulare, ce exprimă mişcarea de rotație absolută sub aspect cinematic.

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