TECHNICAL UNIVERSITY OF CLUJ-NAPOCA

ACTA TECHNICA NAPOCENSIS

Series: Applied Mathematics and Mechanics
Vol. 56, Issue IV, November, 2013

STABILISATION OF THE INVERSE PENDULUM ON A CART BASED ON A SEARCH METHOD

Adina BĂLAN, Alin PLEŞA, Radu DONCA, Mihai DRĂGHICI, Vistrian MĂTIEŞ

Abstract: *Inverse pendulum on a cart is usually used to illustrate control methods for unstable nonlinear processes. The pendulum is attached to a cart that can move in a horizontal plane by a pivot. As a result of the cart movement, the pendulum will begin to rotate around the fulcrum. Using an adequate motion control system for the cart, the pendulum can be raised in superior vertical unstable position. There are two essential stages in the control of this nonlinear process: initial pendulum raising phase and final stabilization in vertical superior instable position. A common approach for the final phase is state space control. This paper presents a method for calculating the control law for the final phase, law whose parameters are obtained by a search process in the space of possible solutions.*

Key words: *Inverse pendulum on a cart, modeling, simulations, state space control, search algorithms.*

1. INTRODUCTION

Well known problem of the inverse pendulum on a cart is used typically for testing new types of control algorithms [1], [2]. The problem is important because it allows highlighting the components of a mechatronic system and in particular the problems of interfacing and control.

Let us consider the inverted pendulum system presented in fig. 1 where a cart can move on a finite rail. The pendulum is represented as an *m* mass point attached to a rod of *l* length. It is considered that the rod has no mass and that the pendulum can swing 360°, unconstrained.

The main purpose of the problem is to get the pendulum in an inverted vertical position (the *m* mass point is up) and to maintain it, starting from the initial position of stable equilibrium (pendulum of mass *m* is below). This is achieved by applying variable horizontal forces, obtained through an advanced control algorithm, forces that move the pendulum-cart system along the rail in one way or the other.

The applicability of the problem is broad: robotics (biped robots), aviation, astronautics, military field, etc.

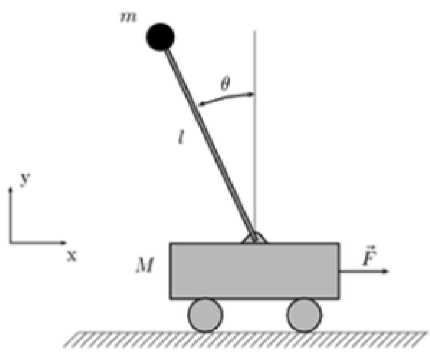

Fig. 1. Inverted pendulum on a cart [3]

Two stages can be emphasized in this nonlinear process control: initial phase, the swing of the pendulum to allow it's raise and the final phase of stabilization in superior vertical unstable position.

2. THE MODEL

State space equations can be obtained based on Lagrange equation [4]:

$$\frac{d}{dt}\left[\frac{\partial}{\partial \dot{q}}L\right]-\frac{\partial}{\partial q}L=\tau \tag{1}$$

where L is the Lagrange function $L=K-V$, K respectively V being the kinetic and potential energy.

Following notations were made: $x_1=\theta$, $x_2=\dot{\theta}$ representing the angular position respectively the angular speed of the pendulum, $x_3=x$ representing the cart position, $x_4=\dot{x}$ representing the cart speed and u represents the force applied to the cart.

For an inverted pendulum the kinetic energy is given by:

$$K=\frac{1}{2}M\dot{x}^2+\frac{1}{2}m\left(\left(\dot{x}+l\dot{\theta}\cos\theta\right)^2+\left(l\dot{\theta}\sin\theta\right)^2\right) \tag{2}$$

And the potential energy is given by:

$$V=mgl\cos\theta \tag{3}$$

The following equations are obtained:

$$\begin{aligned}&(M+m)\ddot{x}+ml\ddot{\theta}\cos\theta-ml\dot{\theta}^2\sin\theta=u-b\dot{x}\\&l\ddot{\theta}+\ddot{x}\cos\theta-g\sin\theta=-f\dot{\theta}\end{aligned} \tag{4}$$

where the $b\dot{x}$ term on the right side of first equation represents the cart friction and $f\dot{\theta}$ term on the right side of the second equation represents the friction in the rotary joint of the pendulum.

Choosing state variables like so:

$$x_1=\theta,\quad x_2=\dot{\theta},\quad x_3=x,\quad x_4=\dot{x} \tag{5}$$

We obtain the equations:

$$\begin{aligned}&\dot{x}_1=\dot{\theta}=x_2\\&\dot{x}_2=\ddot{\theta}=\frac{(M+m)(g\sin x_1-fx_2)-\left(lmx_2^2\sin x_1+u-bx_4\right)\cos x_1}{l\left(M+m\sin^2 x_1\right)}\\&\dot{x}_3=\dot{x}=x_4\\&\dot{x}_4=\ddot{x}=\frac{-mg\sin x_1\cos x_1+mlx_2^2\sin x_1+fmx_2\cos x_1+u-bx_4}{M+m\sin^2 x_1}\end{aligned} \tag{6}$$

Thus obtained model is nonlinear. To use the linear control methods, the model must be linearized. Because the goal is to stabilize the pendulum in a superior vertical position where $x_1=\theta$ angle is small, the following approximations can be made:

$$\sin x_1=x_1;\ \cos x_1=1 \tag{7}$$

Thus we obtain:

$$\begin{aligned}&\dot{x}_1=\dot{\theta}=x_2\\&\dot{x}_2=\ddot{\theta}=\frac{(M+m)(gx_1-fx_2)-\left(lmx_2^2x_1+u-bx_4\right)}{l\left(M+mx_1^2\right)}\\&\dot{x}_3=\dot{x}=x_4\\&\dot{x}_4=\ddot{x}=\frac{-mgx_1+mlx_2^2x_1+fmx_2+u-bx_4}{M+mx_1^2}\end{aligned} \tag{8}$$

For angular speed values $x_2=\dot{\theta}<<1$ the term $mlx_2^2x_1$ can be neglected and also the term $mlx_1^2<<M$ can be neglected thus obtaining the linearized model:

$$\begin{aligned}&\dot{x}_1=\dot{\theta}=x_2\\&\dot{x}_2=\ddot{\theta}=\frac{(M+m)(gx_1-fx_2)-u+bx_4}{lM}\\&\dot{x}_3=\dot{x}=x_4\\&\dot{x}_4=\ddot{x}=\frac{-mgx_1+fmx_2+u-bx_4}{M}\end{aligned} \tag{9}$$

or:

$$\dot{x}=Ax+Bu \tag{10}$$

where:

$$A=\begin{bmatrix}0&1&0&0\\ g(M+m)/lM&-f(M+m)/lM&0&b/lM\\0&0&0&1\\-mg/M&fm/M&0&-b/M\end{bmatrix} \tag{11}$$

$$B=\begin{bmatrix}0\\-1/lM\\0\\1/M\end{bmatrix} \tag{12}$$

Therefore we obtained a linear model of the pendulum-cart system, which approximates the behavior of the model when the angle θ is small (usually under 5 degrees) and the angular speed $\dot{\theta}$ is sufficiently small as well. The linear model allows the design of a control algorithm based on techniques commonly applied to linear systems. However, in the experimental applications at least three problems may occur: (1) the possible reduced robustness on model parameters variation, (2) switch between the algorithms used to raise the pendulum and to stabilize the pendulum as well as (3) noise when measuring the state signals. These difficulties can be solved by adopting appropriate measures: on-line parameter identification, robust control system design, data filtering and using observers.

3. STATE SPACE CONTROLLER

We will use the full state model written as:

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned} \quad (13)$$

and the following model parameters values: *g*=9.81;*l*=0.11;*f*=1;*M*=0.2;*m*=0.022; *b*=10. For the model, the following code can be introduced in Matlab:

```
A=[0 1 0 0; g*(M+m)/(l*M) -f*(M+m)/(l*M) 0 b/(l*M);
                    0 0 0 1; -m*g/M f*m/M 0 -b/M]
B=[0; -1/(l*M); 0; 1/M]
C = [1 0 0 0; 0 0 1 0];
D = [0; 0];
Pendul=ss(A, B, C, D)
PendulTf=tf(Pendul)
```

The control algorithm design is made in such way for the system to respond correspondingly when a disturbance occurs. It is intended both to maintain the pendulum in the upper unstable equilibrium position and bringing the cart in zero position. The problem can be solved by using the complete reaction after state. Using the command:

```
poles=eig(A)
```

we obtain the system poles: 0, 5.9547, -4.5385, -51.5071. Because one of the poles has the real part positive, as expected, the system is unstable in open loop.

Considering that all the four states can be measured, the next step in the design process is to find the gain matrix K. This can be achieved in various ways. If we know the desired positions of the closed loop system poles, Matlab commands *place* or *acker* can be used. Of course, the closed-loop pole positions results from the performance criteria imposed by the system. Another possibility is to use the MATLAB command *LQR* that returns the optimal gain factor of the controller considering a linear process and a quadratic performance function and the reference equal to zero.

Before designing the controller it is necessary to check that the system is controllable. Satisfying this property implies that the system status can be brought in a finite time anywhere in the state space, while respecting the physical constraints of the system. A state system is completely controllable if the controllability matrix has rank *n*, where the rank of the matrix is the number of independent columns (or rows). The system controllability matrix is of the form:

$$\mathrm{Co} = \left[\mathrm{B} \mid \mathrm{AB} \mid \mathrm{A}^2 B \mid \ldots \mid A^{n-1} B \right] \quad [14]$$

Number *n* is the number of state variables of the system. Since the controllability matrix has size 4x4, rank of the matrix must be 4. To build this matrix and verify rank matrix, the following Matlab commands can be used:

```
co=ctrb(Pendul);
controllability = rank(co)
```

It has been verified that the system is controllable and therefore can design a controller to meet performance requirements. To determine the gain matrix K we use the linear quadratic regulator method. The Matlab *LQR* function allows a choice of two parameter matrices R and Q which weights the effects of the control signal and the deviation based on performance.

In the simplest case one can choose R = 1 and Q = C'C. Corresponding cost function will weigh equally the importance of the control signal and states that are outputs (pendulum angle, cart position). *LQR* method allows control of both outputs. In this case, this is accomplished quite easily. The controller will be tuned by changing the non-zero elements of the matrix Q in order to achieve the desired response. To obtain the structure of the matrix Q, the equation Q = C'C can be used.

It is obtained:

```
Q = [ 1 0 0 0; 0 0 0 0; 0 0 1 0; 0 0 0 0]
```

Element Q[1,1] weights the angular position of the pendulum and the element Q [3,3] weights the cart position. Input weight R = 1 can be selected and the weight between control signal and state signals can be done by the Q matrix.

Based on these parameters choice and considering the desired performances, K matrix can be chose experimentally which leads to the best controller that meets the performance requirements.

Use the following Matlab code:

```
R = 1;
K = lqr(A,B,Q,R)
Ac = [(A-B*K)];
Bc = [B];
Cc = [C];
Dc = [D];
Pendul = ss(Ac,Bc,Cc,Dc);
t = 0:0.01:5;
r =0.2*ones(size(t));
[y,t,x]=lsim(Pendul,r,t);
[AX,H1,H2] = plotyy(t,y(:,2),t,y(:,1),'plot');
set(get(AX(1),'Ylabel'),'String', 'pozitie cart (m)')
set(get(AX(2),'Ylabel'),'String', 'unghi pendul (radiani)')
title('Raspuns la treapta cu control LQR')
```

Examples:

1. Q(1,1)=1; Q(3,3)=1; Results:
K =[-37.9520 -2.5039 -1.0000 -20.3583]

System response to a step signal of 0.2 amplitude (cart movement) is presented in figure 2. Even if the pendulum angle tends to zero (with overshot), from the point of view of the position of the cart response is slow and there is a stationary error. In 2, 3, 4 examples, the Q(1,1) and Q(3,3) parameters will be increased thus leading to reduced stationary error.

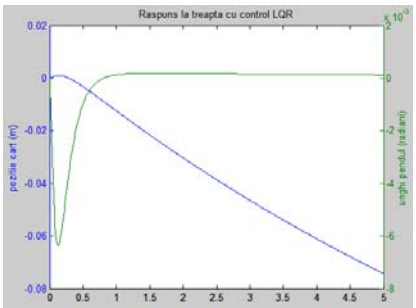

Fig. 2 State Space Controller. Example 1.

2. Q(1,1)=100; Q(3,3)=100 ; Results:
K=[-45.7437 -2.9227 -10.0000 -23.7412]

System outputs are presented in figure 3.

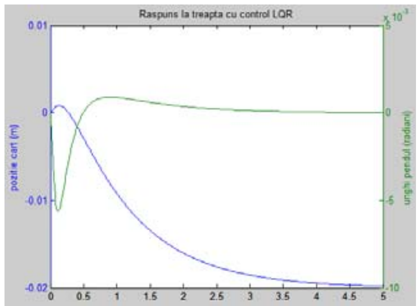

Fig. 3 State Space Controller. Example 2.

3. Q(1,1)=1000; Q(3,3)=1000; Results:
K=[-74.9689 -4.2474 -31.6228 -33.8597]
System outputs are presented in figure 4.

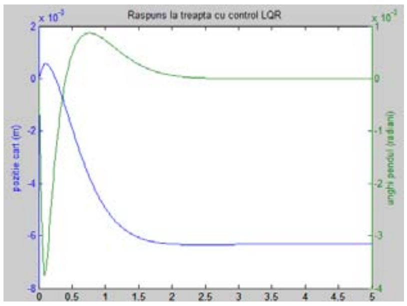

Fig. 4 State Space Controller. Example 3.

4. Q(1,1)=10000; Q(3,3)=10000; Results:
K =[-189.8932 -9.2054 -100.0000 -72.6659]
System outputs are presented in figure 5.

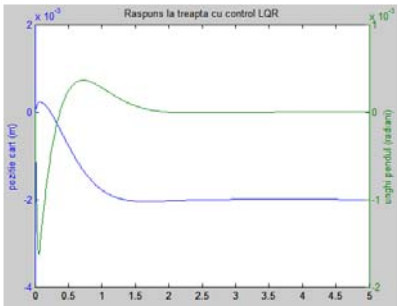

Fig. 5. State Space Controller. Example 4.

It is observed that if the Q values increase, the transitory answer improves but it is necessary to increase the *u* force, which is a disadvantage. On the other hand it is necessary to satisfy the requirements regarding stationary error. Unlike other design methods where the measured output is compared with the reference thus obtaining the error, a complete state controller will use all the states. For the calculation of the input signal it is necessary to calculate the desired value of the steady states, to be multiplied like the K gain matrix and to be used as new reference value. This can be accomplished by adding a reference amplification factor [5]. Figure 6 show the response obtained using this technique.

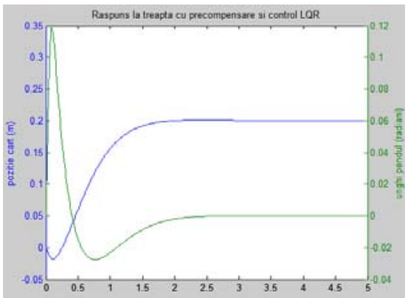

Fig. 6 State Space Controller with pre compensation

It is noted that in this case the stationary error is canceled.

4. STATE SPACE CONTROL BASED ON A SEARCH METHOD

The method presented in the previous section needs a linear model of the process; therefore it can be applied only for small values of the pendulum's angle.

A possible solution is the direct use of the nonlinear model of the process. In this case, the previous presented methods can no longer be applied. However, we can adopt a simple method of searching the parameters of the vector $K=(k1,k2,k3,k4)$. We assume that based on experiments the possible range of the four parameters is established as follows:

$$k_{i\,\min,i=1..4} \le k_{i,i=1..4} \le k_{i\,\max,i=1..4} \qquad (15)$$

For each parameter k_i we will divide the range (K_{imin}, k_{imax}) in n equal ranges. Therefor we can choose the vector K in n^4 modes. For each possible combination of the vector K, the behavior of the system is on-line simulated for a sufficient number of sampling steps (using the model and the history of the process) and a performance index is calculated (the index penalize the angular error of the pendulum from the vertical position and the cart error from the programmed position). It is taken into account the limited value of the control signal and the limited value of the cart movement. In the end the K vector that gives the optimum considered trajectory is chose. Of course, if the calculus is repeated at each sample step, the vector K can be variable in time. The main disadvantage of the proposed method is the large value of the computational time. But, for the majority of the n^4 possible combinations the on-line simulation will be quickly stopped due to the distancing of the pendulum from the instable equilibrium position or the movement limits of the cart will be reached.

The following parameters of the process are considered: g=9.81; l=0.11; f=0; M=0.2; m=0.022; b=0; u_{max}=5;. In these conditions the proposed algorithm has generated the signals presented in the figures 7 (control signal), 8 (pendulum position, speed) and 9 (cart position, speed) and the obtained parameters of the vector K are:

$$K=[-0.8\ -6\ -36\ -20].$$

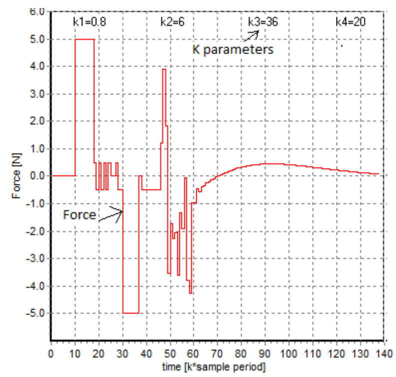

Fig. 7 Control signal and K parameters

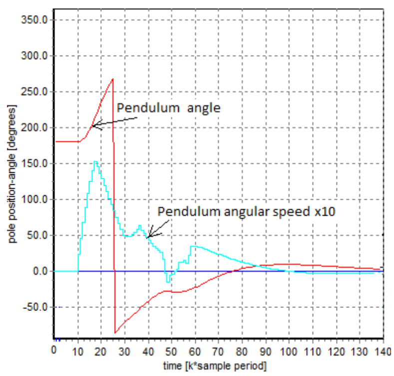

Fig. 8 Pendulum angle and speed

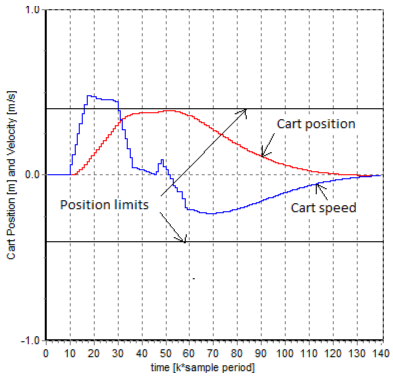

Fig. 9 Cart Position and speed

6. CONCLUSIONS

The paper presents aspects regarding the control of a non-linear process, respectively the inverse pendulum on a cart. It is shown that there are different approaches needed for the swing-up phase, respectively for the stabilization phase. For the case of stabilization there are presented control algorithms in state space which utilizes linear control methods. Finally, a control method is presented where based on on-line simulation the considered optimal parameters of the gain matrix *K* are established. Although the computational time is significant, the obtained results validate the proposed method and the research efforts of new types of control algorithms.

ACKNOWLEDGEMENTS

This work has been supported by the project "Q-DOC - Creşterea calităţii studiilor doctorale în stiin□e inginereşti pentru sprijinirea dezvoltării societăţii bazate pe cunoaştere" contract no. **POSDRU**/107/1.5/S/78534, project co-funded from European Social Fund through Sectorial Operational Program Human Resources 2007-2013

7. REFERENCES

[1] K. J. Åström, K. Furuta, Swinging up a pendulum by energy control, IFAC 13th World Congress, San Francisco, California, 1996, pp 37-42;

[2] G. Kerschena, K.Wordenb, A. F. Vakakisc, J.C. Golinval, Past, present and future of nonlinear system identification in structural dynamics, Mechanical Sys-tems and Signal Processing 20 (2006) 505–592

[3] http://en.wikipedia.org/wiki/Inverted_pendulum

[4] http://www.control.isy.liu.se/student/tsrt15/file/invpendpmenglish.pdf

[5]http://ctms.engin.umich.edu/CTMS/index.php?example=InvertedPendulum&section=ControlStateSpace

Stabilizarea pendulului invers pe un cart bazată pe utilizarea unei metode de căutare

Rezumat: *Pendulul invers pe un cart este utilizat uzual pentru exemplificarea metodelor de control destinate proceselor neliniare instabile. Prin intermediul unui pivot pendulul este ataşat unui cart ce se poate mişca în plan orizontal. Ca urmare a mişcării cartului pendulul va începe să se rotească în jurul punctului de sprijin. Utilizând un sistem de control adecvat a mişcării cartului, pendulul poate fi adus în poziţia verticală superioară instabilă. Sunt două etape esenţiale în controlul acestui proces neliniar: faza iniţială de ridicare a pendulului şi faza finală de stabilizare în poziţia vericală superioară instabilă. O metodă uzuală de control pentru faza finală este utilizarea controlului bazat pe reacţie după stare. Lucrarea prezintă o modalitate de calcul a legii de control pentru faza finală, lege a cărui parametri sunt obţinuţi printr-un procedeu de căutare în spaţiul soluţiilor posibile.*

Adina BĂLAN, drd.prof.ing., Technical University of Cluj-Napoca, Mechatronics Department, E-mail: adinabalan@yahoo.com, Tel.: 0264401756

Alin PLEŞA, dr.ing., Technical University of Cluj-Napoca, Mechatronics Department, E-mail: plesa.alin@gmail.com, Tel.: 0264401756

Radu DONCA, dr.ing., Technical University of Cluj-Napoca, Mechatronics Department, E-mail: doncaradu@yahoo.com , Tel.: 0264401756

Mihai DRĂGHICI, drd.ing., Technical University of Cluj-Napoca, Mechatronics Department, E-mail: mikiduta.dr@gmail.com , Tel.: 0264401756

Vistrian MĂTIEŞ, prof.dr.ing., Technical University of Cluj-Napoca, Mechatronics Department, E-mail: vistrian.maties@yahoo.com , Tel.: 0264401756