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FORMULATION ON THE PRINCIPLES OF ANALYTICAL MECHANICS

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Abstract: *The differential and integral principles of the analytical mechanics are based on fundamental concepts of newtonian mechanics, in keeping with differential character and the linking type between the component bodies of a mechanical system. Among the fundamental concepts, with an essential role, is the kinetic energy as a central function in the Lagrange-Euler type equations, Hamilton equations, and variational principles. But differential equations of motion for mechanical systems with several degrees of freedom can be determined using acceleration energy, as central function, whose implementation in differential and variational principles will be the main objective of this paper work. To develop equations of motion with the above mentioned concepts, the paper will contain as sections: kinematic study of multibody systems, matrix exponential function, the expression of the general definition for acceleration energy, and its implementation in differential and variational principles of analytical mechanics*

Key words: *matrix exponentials, dynamics, acceleration energy, differential principle.*

1. KINEMATIC ANALYSIS

In this section, will be presented in matrix form, a geometrical study concerning the structure of multibody systems. As shown in Fig.1, there will be considered a linked system of bodies, denoted (S_i), where $i=1,2,...n=1\to n$.

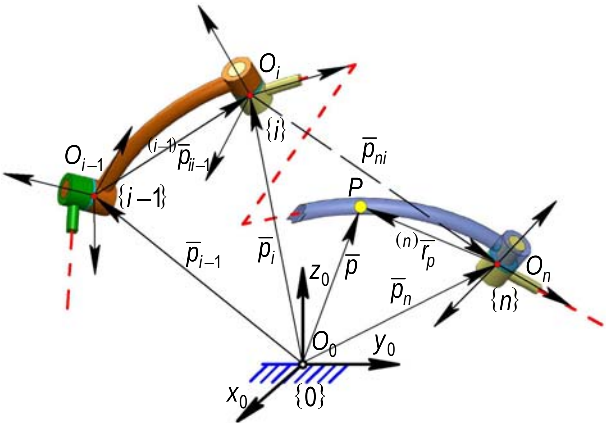

Fig. 1. Sequence form a mechanical robot structure (MRS)

Assuming, that the (n) bodies would be free, the whole system would have $6\cdot n$ degrees of freedom. The, geometric and kinematic study of each (S_i) body, is made using a reference system, invariably linked to the body with its origin, in an arbitrary point of it. There is entered the hypothesis that mechanical system consisting of (n) rigid bodies is covered in the direct sense from $(0)\to(S_n)$. For the geometrical study, also is considered, the sequence of two bodies $(S_{i-1})\to(S_i)$ connected, having attached the reference frames $\{i-1\}$ and $\{i\}$. Since the connections between the bodies that composing the mechanical system are allowing translations and/or rotations to the body (S_i), besides (S_{i-1}) is characterized by the number of degrees of freedom called, in analytical mechanics, generalized coordinates. They can be linear and/or angular coordinates, symbolized q_j, and are included in the symbol $\bar{\theta}_{ii-1}$ having, according with [1], [2], the signification of a column vector:

$$\bar{\theta}_{ii-1}=\left(q_j,\ j=\left(N_{i-1}+1\right)\to N_i\right)^T,\ \ i=1\to n \quad (1)$$

$$\bar{\theta}_i=\left[\bar{\theta}^T_{jj-1},\ \ j=1\to i\right]^T \quad (2)$$

where, according to matrix algebra, T means the transposed of a matrix, and $\bar{\theta}_i$ is a column vector which characterizing the degrees of freedom the same (S_i) body, with respect to the fixed reference system. There is introduced the operator Δ_i, having two possible values:

$$\Delta_i=1,\ \textit{for rotation};\ \Delta_i=0,\ \textit{for translation} \quad (3)$$

1.1 Orientation of the system bodies

Taking into account the issues covered in works such as [1], [2], [3] the orientation of the body (S_{i-1}), with respect to the fixed reference system is defined by the orientation matrix:

$${}^{0}_{i-1}(R) = R\left[\bar{\theta}_{i-1\Delta}(t)\right] \tag{4}$$

which is a matrix function of the column vector $\bar{\theta}_{i-1\Delta}(t)$ components, represented by independent angular coordinates:

$$\bar{\theta}_{i-1\Delta}(t) = \left[\bar{\theta}^{T}_{jj-1\Delta},\ j = 1 \to i-1\right]^{T} \tag{5}$$

where $\bar{\theta}_{jj-1\Delta} = \left[q_m \cdot \Delta_m,\ m = \left(N_{j-1}+1\right) \to N_j\right]$ (6)

Therefore, the orientation of (S_i) body, with respect to $\{i-1\}$ and $\{0\}$ frames is:

$${}^{i-1}_{i}(R) = R\left[q_j \cdot \Delta_j,\ j = \left(N_{i-1}+1\right) \to N_i\right] \tag{7}$$

$${}^{0}_{i}(R) = {}^{0}_{i-1}(R) \cdot {}^{i-1}_{i}(R) \tag{8}$$

The previous expression is rewritten as:

$${}^{0}_{i}(R) = {}^{0}_{1}(R) \cdot \ldots \cdot {}^{j-1}_{j}(R) \cdot \ldots \cdot {}^{i-1}_{i}(R) = \prod_{j=1}^{i} {}^{j-1}_{j}(R) \tag{9}$$

For $i = n$, the orientation of each axis of $\{n\}$ frame, attached to (S_n) body, with respect to $\{0\}$ fixed reference frame, is expressed with the rotation matrix:

$$\begin{aligned} {}^{0}_{n}(R) &= \prod_{i=1}^{n} {}^{i-1}_{i}(R) = {}^{0}_{n-1}(R) \cdot {}^{n-1}_{n}(R) = \\ &= {}^{0}_{n}(R)\left(q_j \cdot \Delta_j,\ j = 1 \to k\right) \end{aligned} \tag{10}$$

According to [1], [2] the orientation of the (S_n) rigid body is expressed by a set of three independent orientation angles as:

$$\begin{aligned} &{}^{0}_{n}(R) = R\left(\alpha_A - \beta_B - \gamma_C\right), \\ &A = \{x;y;z\};\ B = \{y;z;x\};\ C = \{z;x;y\} \end{aligned} \tag{11}$$

where A, B, C is the type of axis around which is performed the rotation. For orientation, there are known 12 sets of angles, one of these, being the set of Euler angles, included in the orientation vector: $\underset{(3\times1)}{\Omega} = \left[\psi\ \ \theta\ \ \varphi\right]^{T}$. (12)

1.2 Position of the system bodies

Knowing the position vector, between the $\{i\}-\{i-1\}$ frames, projected on $\{i-1\}$ reference frame, symbolized with ${}^{i-1}\bar{p}_{ii-1}$, and the rotation matrix (8), the position of $\{i\}$ frame with respect to the fixed frame $\{0\}$ is:

$${}^{i-1}\bar{p}_{i\,i-1} = {}^{i-1}\bar{p}^{(0)}_{i\,i-1} + \sigma_i \cdot {}^{i-1}\bar{p}_{i\,i-1}(t) \tag{13}$$

$$\bar{p}_{ii-1} = {}^{0}_{i}(R) \cdot {}^{i-1}\bar{p}_{ii-1} \tag{14}$$

$$\bar{p}_i = \sum_{j=1}^{i} p_{jj-1} = \sum_{j=1}^{i} {}^{0}_{j-1}(R) \cdot {}^{j-1}\bar{p}_{jj-1} \tag{15}$$

where the operator $\sigma_i = \{1;0\}$, has the signification: $\left(1, \mathit{if}\ {}^{i-1}\bar{p}_{ii-1} = {}^{i-1}\bar{p}_{i\,i-1}(t)\right)$, respectively $\left(0, \mathit{if}\ {}^{i-1}\bar{p}_{i\,i-1} = cst.\right)$.

Going through the system of bodies: $(S_1),\ldots,(S_i),\ldots,(S_n)$ in direct sense, and replacing $i = n$ in (15) results:

$$\bar{p}_n = \sum_{i=1}^{n} \bar{p}_{ii-1} = \sum_{i=1}^{n} {}^{0}_{i-1}(R) \cdot {}^{i-1}\bar{p}_{ii-1} = \bar{p}_n\left[q_j(t);\ j = 1 \to k\right] \tag{16}$$

the position of $\{n\}$ reference frame, affixed to (S_n), regard the fixed reference frame.

1.3 Angular velocities and accelerations

According to [2], the starting expression in determining of the angular velocity and acceleration is:

$${}^{0}_{i}(R) = {}^{0}_{i-1}(R)(t) \cdot {}^{i-1}_{i}(R) \tag{17}$$

There is applied the property ${}^{0}_{i}(\dot{R}) \cdot {}^{0}_{i}(R)^{T} = \left(\bar{\omega}_i \times\right)$, representing the skew symmetric matrix associated to angular velocity vector. On the expression (17) there are applied the absolute first order and second order time derivatives. By successive transformations, there are obtained the definition expressions for absolute angular velocities and accelerations of the (S_i) rigid body as follows:

$$\bar{\omega}_i = \bar{\omega}_0 + \sum_{j=1}^{i} \Delta_j \cdot \bar{\omega}_{jj-1} \tag{18}$$

$$\bar{\varepsilon}_i = \bar{\varepsilon}_0 + \sum_{j=1}^{i} \Delta_j \cdot \left(\bar{\omega}_{j-1} \times \bar{\omega}_{jj-1} + \bar{\varepsilon}_{jj-1}\right) \tag{19}$$

where $\bar{\omega}_0$ and $\bar{\varepsilon}_0$ are expressing the absolute rotation of the fixed frame, having the values: $\bar{\omega}_0 = 0$, $\bar{\varepsilon}_0 = 0$, while the vectors $\bar{\omega}_{jj-1}$, $\bar{\varepsilon}_{jj-1}$ are characterizing the relative rotation of (S_i) body with respect to (S_{i-1}).

The projection of $\overline{\omega}_i$ and $\overline{\varepsilon}_i$ defined with (18) and (19), on its reference frame axes $\{i\}$, are established by the following expression:

$$ {}^{i}\overline{\omega}_i = {}^{0}_{i}(R)^T \cdot \overline{\omega}_i; \qquad {}^{i}\overline{\varepsilon}_i = {}^{0}_{i}(R)^T \cdot \overline{\varepsilon}_i \tag{20} $$

Going through the cinematic chain of the system bodies: $(S_1),\ldots,(S_i),\ldots,(S_n)$, namely for $i = 1 \rightarrow n$, the expressions (18) and (19), turns into:

$$ \overline{\omega}_n = \overline{\omega}_0 + \sum_{i=1}^{n} \Delta_i \cdot \overline{\omega}_{ii-1} \tag{21} $$

$$ \overline{\varepsilon}_n = \overline{\varepsilon}_0 + \sum_{i=1}^{n} \Delta_i \cdot \left(\overline{\omega}_{i-1} \times \overline{\omega}_{ii-1} + \overline{\varepsilon}_{ii-1} \right) \tag{22} $$

Therefore, the absolute rotation of the frame $\{n\}$, attached to the (S_n) body is completely defined by the rotation matrix (10), respectively by the absolute angular velocity (21) and absolute angular acceleration (22).

1.4 Linear velocities and accelerations

In order to establish the definition expressions for absolute linear velocities and accelerations, according to [2], the starting equation is:

$$ \overline{p}_i(t) = \overline{p}_{i-1}(t) + \overline{p}_{ii-1}(t) \tag{23} $$

On the expression (23) there are applied the absolute first and second order derivatives with respect to time. By successive transformations there are obtained the absolute linear velocities and accelerations of (S_i) body, as:

$$ \overline{v}_i = \overline{v}_0 + \sum_{j=1}^{i} \left[\overline{\omega}_{j-1} \times \overline{p}_{jj-1} + \sigma_j \cdot \overline{v}_{jj-1} \right] \tag{24} $$

$$ \begin{aligned} \overline{a}_i = \overline{a}_0^* &+ \sum_{j=1}^{i} \left(\overline{\varepsilon}_{j-1} \times \overline{p}_{jj-1} + \overline{\omega}_{j-1} \times \overline{\omega}_{j-1} \times \overline{p}_{jj-1} \right) + \\ &+ \sum_{j=1}^{i} \sigma_j \cdot \left(2 \cdot \overline{\omega}_j \times \overline{v}_{jj-1} + \overline{a}_{jj-1} \right) \end{aligned} \tag{25} $$

where $\overline{v}_{jj-1}$ and $\overline{a}_{jj-1}$ are representing the relative linear velocity and acceleration of the origin $O_i \in \{i\}$ with respect to $\{i-1\}$ reference frame. The Kinematical parameters $\overline{v}_0$ and $\overline{a}_0^*$ from (24) and (25), are characterizing, alongside $\overline{\omega}_0$ and $\overline{\varepsilon}_0$, the absolute motion of the fixed frame, having the values: $\overline{v}_0 = 0$, $\overline{a}_0^* = 0$, and in the studies concerning the dynamics of the material systems $\overline{a}_0^* = \tau \cdot g \cdot \overline{k}_0$, representing a vector, equal in modulus and in opposite sense of the gravitational acceleration $(\tau = \pm 1)$. [1]

In order to project the vectors $\overline{v}_i$ and $\overline{a}_i$ defined with (24) and (25), on its $\{i\}$ reference frame axes, there are established by the following transfer matrix expression:

$$ {}^{i}\overline{v}_i = {}^{0}_{i}(R)^T \cdot \overline{v}_i; \qquad {}^{i}\overline{a}_i = {}^{0}_{i}(R)^T \cdot \overline{a}_i \tag{26} $$

Going through the cinematic chain of the system bodies: $(S_1),\ldots,(S_i),\ldots,(S_n)$, namely for $i = 1 \rightarrow n$, the expressions (24) and (25), turns into:

$$ \overline{v}_n = \overline{v}_0 + \sum_{i=1}^{n} \left(\overline{\omega}_{i-1} \times \overline{p}_{ii-1} + \sigma_i \cdot \overline{v}_{ii-1} \right) \tag{27} $$

$$ \begin{aligned} \overline{a}_n = \overline{a}_0^* &+ \sum_{i=1}^{n} \left(\overline{\varepsilon}_{i-1} \times \overline{p}_{ii-1} + \overline{\omega}_{i-1} \times \overline{\omega}_{i-1} \times \overline{p}_{ii-1} \right) + \\ &+ \sum_{i=1}^{n} \sigma_i \cdot \left[2 \left(\overline{\omega}_0 + \sum_{j=1}^{i} \Delta_j \cdot \overline{\omega}_{jj-1} \right) \times \overline{v}_{ii-1} + \overline{a}_{ii-1} \right] \end{aligned} \tag{28} $$

Therefore, the absolute rotation of the $\{n\}$ frame, attached to the (S_n) body is completely defined by the position and orientation parameters, by the angular velocity and acceleration (21) - (22), respectively by the linear velocity and acceleration, defined by (27)-(28).

2 MATRIX EXPONENTIALS

The kinematical parameters, developed in previous section, can be expressed, according to [3], by using of matrix exponential functions. First, there is expressed the rotation matrix and the position vector between $\{j\}$ and $\{0\}$ reference frames as:

$$ \begin{aligned} R_{j0} &= \left\{ exp \left\{ \sum_{i=1}^{j} \left\{ \overline{u}_i^{(0)} \times \right\} q_i \Delta_i \right\} \right\} \cdot R_{j0}^{(0)} = \\ &= \prod_{i=1}^{j} exp \left\{ \left\{ \overline{u}_i^{(0)} \times \right\} q_i \Delta_i \right\} \cdot R_{j0}^{(0)} \end{aligned} \tag{29} $$

$$ \overline{p}_j = \sum_{i=1}^{j} \left\{ exp \left\{ \sum_{k=0}^{i-1} \left\{ \overline{u}_k^{(0)} \times \right\} q_k \cdot \Delta_k \right\} \right\} \cdot \overline{b}_i \tag{30} $$

where $R_{j0}^{(0)}$ corresponds to initial configuration of the multibody system. The column vector $\overline{b}_i$ from the expression (30), according to [1], is:

$$\bar{b}_j = \left\{ I_3 \cdot q_j + \left\{ \bar{u}_j^{(0)} \times \right\} \left[1 - \cos\left(q_j \cdot \Delta_j \right) \right] + \right.$$
$$\left. + \bar{u}_j^{(0)} \cdot \bar{u}_j^{(0)T} \cdot \left[q_j - \sin\left(q_j \cdot \Delta_j \right) \right] \right\} \cdot \bar{v}_j^{(0)}. \tag{31}$$

In the previous equations, are used the symbols $\bar{u}_j = \left\{ \bar{x}_j; \bar{y}_j; \bar{z}_j \right\}$ and $\bar{v}_j = \left\{ \bar{p}_j \times \right\} \bar{u}_j \cdot \Delta_j + \left(1 - \Delta_j \right) \cdot \bar{u}_j$, together expressing the screw parameters or the homogeneous parameters of the oriented axis $\{j\}$ around or along which are achieved the generalized coordination. In accordance with [2], the defining expressions for angular velocities and accelerations (18), (19) can be established on the basis of matrix exponential thus:

$${}^0\bar{\omega}_j = \left\{ \sum_{i=1}^{j} \left\{ exp \left\{ \sum_{k=1}^{i-1} \left\{ \bar{u}_k^{(0)} \times \right\} q_k \Delta_k \right\} \right\} \bar{u}_i^{(0)} \dot{q}_i \Delta_i \right\}; \tag{32}$$

$${}^0\dot{\bar{\omega}}_j = \left\{ \sum_{i=1}^{j} \left\{ Mexp\left\{ V_{i1} \right\} \ddot{q}_i + Mexp\left\{ \dot{V}_{i1} \right\} \dot{q}_i \right\} \bar{u}_i^{(0)} \Delta_i \right\} \tag{33}$$

$$\underset{(3x3)}{ME}\left(V_{j1} \right) = \exp\left\{ \sum_{i=0}^{j-1} \left\{ \bar{u}_i^{(0)} \times \right\} q_i \cdot \Delta_i \right\} \tag{34}$$

$$ME\left\{ \dot{V}_{j1} \right\} = \sum_{i=1}^{j-1} \left\{ e^{V_1^*} \right\} \left\{ \bar{u}_i^{(0)} \times \right\} \dot{q}_i \cdot \Delta_i \left\{ e^{V_2^*} \right\}$$
$$where \;\; V_1^* = \sum_{k=0}^{i-1} \left\{ \bar{u}_k^{(0)} \times \right\} \cdot q_k \cdot \delta_{ik\cdot} \cdot \Delta_k; \tag{35}$$
$$V_2^* = \sum_{m=i}^{j-1} \left\{ \bar{u}_m^{(0)} x \right\} \cdot q_m \cdot \delta_m \cdot \Delta_m$$

The velocity $\bar{v}_j$ and the acceleration $\bar{a}_j$ are expressed as:

$$\bar{v}_j = \sum_{i=1}^{j} \left\{ \sum_{m=1}^{i-1} \left\{ \exp\left\{ V_1^* \right\} \right\} \left\{ \bar{u}_m^{(0)} \times \right\} \dot{q}_m \Delta_m \left\{ \exp\left\{ V_2^* \right\} \right\} \right\} +$$
$$+ \sum_{i=1}^{j} \left\{ \exp\left\{ V_3^* \right\} \right\} \cdot \dot{\bar{b}}_i$$
$$where \; V_1^* = \sum_{k=0}^{m-1} \left\{ \bar{u}_k^{(0)} \times \right\} q_k \delta_{mk\cdot} \Delta_k; \tag{36}$$
$$V_2^* = \sum_{l=m}^{j} \left\{ \bar{u}_l^{(0)} \times \right\} q_l \delta_l \Delta_l;$$
$$V_3^* = \sum_{k=0}^{i-1} \left\{ \bar{u}_k^{(0)} \times \right\} q_k \cdot \Delta_k$$

$$\bar{a}_j = \frac{d}{dt} \left\{ \sum_{i=1}^{j} \left\{ \sum_{m=1}^{i-1} \left\{ \exp\left\{ V_1^* \right\} \right\} \left\{ \bar{k}_m^{(0)} \times \right\} \dot{q}_m \Delta_m \left\{ \exp\left\{ V_2^* \right\} \right\} \right\} + \right.$$
$$\left. + \sum_{i=1}^{j} \left\{ \exp\left\{ V_3^* \right\} \right\} \cdot \dot{\bar{b}}_i \right\}, where$$
$$V_1^* = \sum_{k=0}^{m-1} \left\{ \bar{k}_k^{(0)} \times \right\} q_k \delta_{mk\cdot} \Delta_k; V_2^* = \sum_{l=m}^{j} \left\{ \bar{k}_l^{(0)} \times \right\} q_l \delta_l \Delta_l; \tag{37}$$
$$V_3^* = \sum_{k=0}^{i-1} \left\{ \bar{u}_k^{(0)} \times \right\} q_k \cdot \Delta_k$$

$$\delta_{ik} = \left\{ (0; i > j-1), (1; i \leq j-1) \right\}, \;\; \delta_{mk} = \left\{ (0; m > j); (1; m \leq j) \right\}$$

$$where \;\; \dot{\bar{b}}_j = \left\{ I_3 + \left\{ \bar{u}_j^{(0)} \times \right\} \sin\left(q_j \cdot \Delta_j \right) + \right.$$
$$\left. + \left\{ \bar{u}_j^{(0)} \times \right\}^2 \left[1 - \cos\left(q_j \cdot \Delta_j \right) \right] \right\} \cdot \bar{v}_j^{(0)} \cdot \dot{q}_j.$$

The using of matrix exponentials apparently seems to be complicatedly, has the advantage of not using reference systems. This observation is visible in the above equations, by the occurrence of homogeneous coordinates, specific to initial configuration.

3 THE ACCELERATION ENERGY

The dynamics equations for a mechanical system, having $(n\, d.o.f.)$, subjected to olonomous or nonolonomous links can be determined by a central function in dynamics, known as acceleration energy, as shown in [1], [2] and [4]-[6]. In Fig. 2, there is considered a rigid body, denoted (j), divided into an infinity of elementary masses (dm), continuously distributed in all body volume.

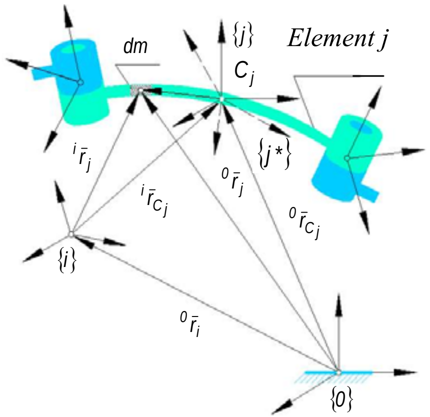

Fig. 2. The element j belonging to a MRS

The position of elementary mass (dm) with respect to its reference frame having the origin in the center of masses C_j is given by position vector ${}^j\bar{r}_j$. According to [1],[2], [4], the starting expression for acceleration energy is:

$$E_A = \left\{ \frac{1}{2} \int \dot{v}_j^2 \cdot dm = \frac{1}{2} \int {}^j\dot{\bar{v}}_j^T \cdot {}^j\dot{\bar{v}}_j \cdot dm \right\}; \tag{38}$$

where: ${}^j\dot{\bar{v}}_j = {}^j\dot{\bar{v}}_{C_j} + {}^j\dot{\bar{\omega}}_j \times {}^j\bar{r}_j + {}^j\bar{\omega}_j \times {}^j\bar{\omega}_j \times \bar{r}_j$. (39)

$${}^j\dot{\bar{v}}_{C_j} = \bar{a}_0^* + \sum_{j=1}^{i} \left(\bar{\varepsilon}_{j-1} \times \bar{p}_{jj-1} + \bar{\omega}_{j-1} \times \bar{\omega}_{j-1} \times \bar{p}_{jj-1} \right) +$$
$$+ \sum_{j=1}^{i} \sigma_j \cdot \left(2 \cdot \bar{\omega}_j \times \bar{v}_{jj-1} + \bar{a}_{jj-1} \right) + \bar{\varepsilon}_j \times \bar{r}_{C_j} + \bar{\omega}_j \times \bar{\omega}_j \times \bar{r}_{C_j} \tag{40}$$

In above equations, the parameters $\left\{{}^j\dot{\bar{v}}_{C_j};{}^j\bar{\omega}_j;{}^j\dot{\bar{\omega}}_j\right\}$ are the acceleration of the mass centre, the angular velocity and acceleration, which are expressing the absolute rotation of rigid body. Substituting (39) in (38), the acceleration energy in integral form is written as:

$$\begin{aligned}E_A=\frac{1}{2}\cdot\int\Big({}^j\dot{\bar{v}}_{C_j}+{}^j\dot{\bar{\omega}}_j\times{}^j\bar{r}_j+\\+{}^j\bar{\omega}_j\times{}^j\bar{\omega}_j\times\bar{r}_j\Big)^T\cdot\left({}^j\dot{\bar{v}}_{C_j}+{}^j\dot{\bar{\omega}}_j\times{}^j\bar{r}_j+{}^j\bar{\omega}_j\times{}^j\bar{\omega}_j\times\bar{r}_j\right)\cdot dm\end{aligned}\tag{41}$$

By performing some differential and integral transformation in (38) there is obtained the acceleration energy corresponding to a rigid body, in performing of a general movement:

$$\begin{aligned}&E_A^j\left(q_k;\dot{q}_k;\ddot{q}_k\ k=1\rightarrow j\right)=\\&=\left(-1\right)^{\Delta M}\cdot\frac{1-\Delta_M}{1+3\cdot\Delta_M}\cdot\left\{\frac{1}{2}\cdot M_j\cdot{}^j\dot{\bar{v}}_{C_j}^T\cdot{}^j\dot{\bar{v}}_{C_j}\right\}+\\&+\Delta_M^2\left\{\frac{1}{2}\cdot{}^j\dot{\bar{\omega}}_j^T\cdot\left\{{}^jI_j^*\cdot{}^j\dot{\bar{\omega}}_j+\left[{}^j\bar{\omega}_j\times{}^jI_j^*\cdot{}^j\bar{\omega}_j\right]\right\}+\right.\\&+\frac{1}{2}\cdot{}^j\dot{\bar{\omega}}_j^T\cdot\left[{}^j\bar{\omega}_j\times{}^jI_j^*\cdot{}^j\bar{\omega}_j\right]+\\&\left.+\frac{1}{2}\cdot\bar{\omega}_j^T\cdot\left[{}^j\bar{\omega}_j^T\cdot Trace\left({}^jI_{pj}\right)\cdot{}^j\bar{\omega}_j-{}^j\bar{\omega}_j^T\cdot{}^jI_{pj}\cdot{}^j\bar{\omega}_j\right]\cdot{}^j\bar{\omega}_j\right\}\end{aligned}\tag{42}$$

where $\mathit{\Delta}_M$ is an operator which express the following types of motion for a rigid body:

$\Delta_M=\left(-1\ \mathit{general};\ 0\ \mathit{translation};\ 1\ \mathit{rotation}\right)$;

${}^jI_j^*$ is the inertial axial-centrifugal tensor, and ${}^jI_{pj}$ inertial planar-centrifugal tensor; the both tensors belonging to the (j) rigid body are established with respect to $\{j\}$ reference frame, applied in the mass center C_j.

4 FORMULATIONS CONCERNING THE DIFFERENTIAL PRINCIPLES

It is known that the Lagrange-Euler equations can be determined by differential calculus applied to D'Alembert-Lagrange principle, or by integral calculus based on variational principles [1], [6]. By similarity of the issues above, further, in order to implement the accelerations energy in the equations of motion, according to [1]-[6], there is considered the D'Alambert-Lagrange principle for system of bodies, whose general form is:

$$\begin{aligned}&\sum_{i=1}^{n}M_i\cdot{}^i\bar{a}_{C_i}\cdot\delta\,{}^i\bar{r}_{C_i}+\sum_{i=1}^{n}\Big({}^i\bar{r}_{C_i}\times M_i\cdot{}^i\bar{a}_{C_i}+\\&+{}^iI_i^*\cdot{}^i\bar{\varepsilon}_i+{}^i\bar{\omega}_i\times{}^iI_i^*\cdot{}^i\bar{\omega}_i\Big)\cdot\delta\,{}^i\bar{\Omega}_i\\&=\sum_{i=1}^{n}{}^i\bar{F}_i^*\cdot\delta\,{}^i\bar{r}_{C_i}+\sum_{i=1}^{n}\left({}^i\bar{r}_{C_i}\times{}^i\bar{F}_i^*+{}^i\bar{N}_i^*\right)\cdot\delta\,{}^i\bar{\Omega}_i.\end{aligned}\tag{43}$$

According to [1], [2] the virtual variation of the position vector $\bar{r}_{C_i}$ corresponding to the mass center and the orientation vector Ω_i, symbolized by (12), is expressed as follows:

$$\bar{r}_{C_i}=\bar{r}_{C_i}\left[q_j(t)\right];\ \Omega_i=\Omega_i\left[q_j(t)\cdot\Delta_j\right],\ j=1\rightarrow k\tag{44}$$

$$\delta\bar{r}_{C_i}=\sum_{j=1}^{k}\frac{\partial\bar{r}_{C_i}}{\partial q_j}\cdot\delta q_j,\ \ \delta\bar{\Omega}_i=\sum_{j=1}^{k}\Delta_j\cdot J_\Omega^i\cdot\frac{\partial\bar{\Omega}_i}{\partial q_j}\cdot\delta q_j.\tag{45}$$

where J_Ω^i is the angular transfer matrix between the time derivatives of the orientation angles (eg Euler angles), included in $\dot{\Omega}$ and the components of the angular velocity vector $\bar{\omega}$ on the axis of the fixed reference frame. The vectors ${}^i\bar{F}_j^*$ and ${}^i\bar{N}_j^*$, according to [1], [2] are the outside forces and moments of the outside forces, with respect to the system having the origin in the center of mass, vectors exerted on each body, belonging to the multibody system.

On the left side of (43), in accordance with (45) there are performed a series of matrix and differential transformations, which finally leading to the expression:

$$\begin{aligned}&\sum_{j=i}^{n}\frac{\partial}{\partial\ddot{q}_j}\left(\frac{1}{2}\cdot M_j\cdot{}^j\dot{\bar{v}}_{C_j}^T\cdot{}^j\dot{\bar{v}}_{C_j}\right)+\\&+\sum_{j=i}^{n}\frac{\partial}{\partial\ddot{q}_j}\left[\frac{1}{2}\cdot{}^j\dot{\bar{\omega}}_j^T\cdot\left({}^jI_j^*\cdot{}^j\dot{\bar{\omega}}_j+{}^j\bar{\omega}_j\times{}^jI_j^*\times{}^j\bar{\omega}_j\right)+\right.\\&\left.+\frac{1}{2}\cdot{}^j\dot{\bar{\omega}}_j^T\cdot\left({}^j\bar{\omega}_j\times{}^jI_j^*\times{}^j\bar{\omega}_j\right)\right]\\&=\sum_{i=1}^{n}{}^i\bar{F}_i^*\cdot\delta\,{}^i\bar{r}_{C_i}+\sum_{i=1}^{n}\left({}^i\bar{r}_{C_i}\times{}^i\bar{F}_i^*+{}^i\bar{N}_i^*\right)\cdot\delta\,{}^i\bar{\Omega}_i\end{aligned}\tag{46}$$

Comparing with (42), it can be seen that in the left side of (46) is found the component of acceleration energy specific to current movements (there aren`t the $\dot{q}^4$ components). The right side, after transformations takes the following definition form:

$$\begin{aligned}&\sum_{i=1}^{n}{}^i\bar{F}_i^*\cdot\delta\,{}^i\bar{r}_{C_i}+\sum_{i=1}^{n}\left({}^i\bar{r}_{C_i}\times{}^i\bar{F}_i^*+{}^i\bar{N}_i^*\right)\cdot\delta\,{}^i\bar{\Omega}_i=\\&=-Q_g^i+Q_m^i\,;\end{aligned}\tag{47}$$

where

$$Q_g^i = \sum_{j=i}^{n} M_j \cdot \bar{g}^T \cdot \frac{\partial \bar{r}_{C_j}}{\partial q_i},\ \bar{g}^T = \bar{a}_0^* = \tau \cdot g \cdot \bar{k}_0 \quad (48)$$

Thus, the equation (46) takes the final form:

$$\frac{\partial E_A}{\partial \ddot{q}_i} + Q_g^i = Q_m^i \qquad (49)$$

In concordance to [1], [2], results the identity:

$$Q_{i\mathscr{F}}^i = \frac{d}{dt}\left(\frac{\partial E_C}{\partial \dot{q}_i}\right) - \frac{\partial E_C}{\partial q_i} = \frac{\partial E_A}{\partial \ddot{q}_i}. \qquad (50)$$

where $Q_{i\mathscr{F}}^i$ represents the inertial generalized force. In conclusion, according to (0.49), the partial derivative with respect to generalized accelerations of the accelerations energy for olonomous mechanical systems, is equivalent to Euler-Lagrange equations.

5 FORMULATION CONCERNING THE VARIATIONAL PRINCIPLES

The dynamic study of multibody mechanical systems, generally evidenced by differential equations of motion, can be reduced to the calculation of variations, namely of the values for that the integral of a particular function has a stationary value (it is minimum). The principle which leads to the calculus of the variations is called Hamilton – Ostrogradski variational principle [7]. For the interpretation of this principle, according to [1], is considered a discrete system of (n) material particles M_i, (see Fig.3), subjected to a number of $(p \le 3 \cdot n)$ ideal linkages. The system is characterized by the masses (m_i), the position vectors $(\bar{r}_i)$ to a fixed reference frame, and $(k = 3 \cdot n - m)$ degrees of freedom. The external force $(\bar{F}_i)$ is acting on the material particle, causing the particle movement on the trajectory (Γ_i), in finite time interval $[t_0 \ \ t_1]$. The movement of the material system, (see Fig. 3, is expressed by the generalized coordinates included in $\bar{\theta}(t) = (q_j(t); j = 1 \to k)^T$ by the velocities $(\bar{v}_i)$and accelerations $(\bar{a}_i)$. The imaginary point (M_i') is defined by the position vector $(\bar{r}_i + \delta\bar{r}_i)$ at the $(t + \delta t)$. To imaginary point (M_i') is associated a velocity $(\bar{v}_i')$ and an acceleration $(\bar{a}_i')$.

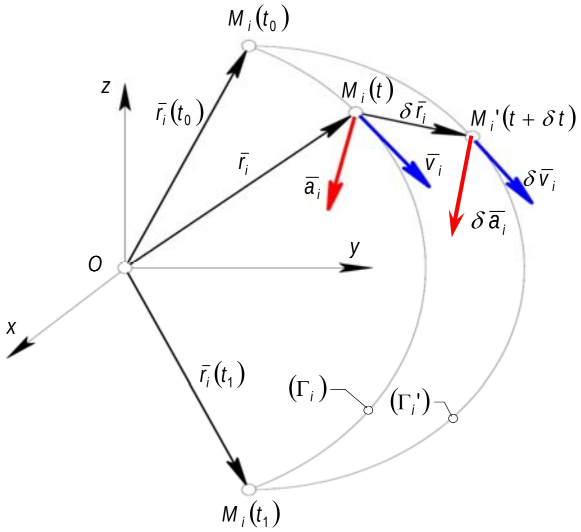

Fig. 3. System of material points (actual and virtual state)

For the two extremities of the material point`s trajectory, (beginning and end of movement), are imposed the conditions:

$$(\delta\bar{r}_i)_{t_0} \equiv (\delta\bar{r}_i)_{t_1} \equiv 0;\ (\delta t)_{t_0} \equiv (\delta t)_{t_1} \equiv 0 \qquad (51)$$

By a series of matrix and differential transformations, according to [1],[4]-[6] results the integral principle in differential form:

$$\int_{t_0}^{t_1}\left[\delta E_C + \sum_{i=1}^{n}\bar{F}_i \cdot \delta\bar{r}_i - \left(\dot{E}_C + \sum_{i=1}^{n}\bar{F}_i \cdot \bar{v}_i\right)\cdot \delta t\right]\cdot dt = 0 \, (52)$$

In the case of olonomous systems, the integral principle (52), takes a particular form, due to the restrictive conditions $\delta t = 0$, as seen:

$$\int_{t_0}^{t_1}\left(\delta E_C + \sum_{i=1}^{n}\bar{F}_i \cdot \delta\bar{r}_i\right)\cdot dt = 0,\ \delta t = 0. \qquad (53)$$

By integrating this principle, there are obtained the Euler-Lagrange equations for olonomous mechanical systems, known in literature as:

$$\frac{d}{dt}\left(\frac{\partial E_C}{\partial \dot{q}_j}\right) - \left(\frac{\partial E_C}{\partial q_j} + Q_j\right) = 0,\ j = 1 \to k. \qquad (54)$$

Taking into account (50), it can be seen that the acceleration energy, as central function is of interest in differential equations of motion only by components which are containing the generalized accelerations $(\ddot{q}_i)$. By performing differential and integral transformations, on identity (0.49), according to [1], finally results the expression between kinetic energy and acceleration energy, as:

$$E_A\left(\overline{\theta};\dot{\overline{\theta}};\ddot{\overline{\theta}}\right)=\frac{1}{2}\cdot\sum_{i=1}^{n}\frac{\partial^2 E_C}{\partial\dot{q}_i^2}\cdot\ddot{q}_i^2+ \\ +\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\frac{\partial^2 E_C}{\partial\dot{q}_i\partial\dot{q}_j}\cdot\ddot{q}_i\cdot\ddot{q}_j+ \\ +\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\partial^2 E_C}{\partial\dot{q}_i\partial q_j}\cdot\ddot{q}_i\cdot\dot{q}_j-\sum_{i=1}^{n}\frac{\partial E_C}{\partial q_i}\cdot\ddot{q}_i; \tag{55}$$

To implement the acceleration energy in the integral principle, there is established the virtual variation for a system as:

$$\delta E_A\left(\ddot{q}_j\right)=\sum_{j=1}^{k}\frac{\partial E_A}{\partial\ddot{q}_j}\cdot\delta\ddot{q}_j=\sum_{j=1}^{k}\frac{\partial E_A}{\partial\ddot{q}_j}\cdot\frac{d}{dt}\delta\dot{q}_j \\ \delta E_A\left(\ddot{q}_j\right)=\sum_{j=1}^{k}\left[\frac{d}{dt}\left(\frac{\partial E_A}{\partial\ddot{q}_j}\cdot\delta\dot{q}_j\right)-\frac{d}{dt}\left(\frac{\partial E_A}{\partial\ddot{q}_j}\right)\cdot\delta\dot{q}_j\right] \tag{56}$$

On this expression is applied the time integral on the finite interval $\left[t_0;t_1\right]$, taking into account the boundary conditions (51):

$$\int_{t_0}^{t_1}\delta E_A\left(\ddot{q}_j\right)\cdot dt=\sum_{j=1}^{k}\int_{t_0}^{t_1}\frac{\partial E_A}{\partial\ddot{q}_j}\cdot\delta\ddot{q}_j\cdot dt=\sum_{j=1}^{k}\left(\frac{\partial E_A}{\partial\ddot{q}_j}\right)\cdot\delta\dot{q}_j \tag{57}$$

Further, the expression (57) is integrated as:

$$\int_{t_0}^{t_1}\left[\int_{t_0}^{t_1}\delta E_A\left(\ddot{q}_j\right)\cdot dt\right]\cdot dt=\sum_{j=1}^{k}\int_{t_0}^{t_1}\frac{\partial E_A}{\partial\ddot{q}_j}\cdot\delta\dot{q}_j\cdot dt= \\ =\int_{t_0}^{t_1}\frac{\partial E_A}{\partial\ddot{q}_j}\cdot\delta\dot{q}_j\cdot dt=-\frac{\partial E_A}{\partial\ddot{q}_j}\cdot\delta q_j \tag{58}$$

The expression (58) is integrated again. Therefore, after the previous differential and integral transformations, there is obtained:

$$\int_{t_0}^{t_1}\left[\int_{t_0}^{t_1}\left[\int_{t_0}^{t_1}\delta E_A\left(\ddot{q}_j\right)\cdot dt\right]\cdot dt\right]\cdot dt= \\ =-\sum_{j=1}^{k}\delta q_j\cdot\int_{t_0}^{t_1}\frac{\partial E_A}{\partial\ddot{q}_j}\cdot dt\equiv\int_{t_0}^{t_1}\delta E_C\cdot dt \tag{59}$$

The second term from the left side of (53) is:

$$\int_{t_0}^{t_1}\left(\sum_{i=1}^{n}\overline{F}_i\cdot\delta\overline{r}_i\right)\cdot dt=\sum_{j=1}^{k}\left[\int_{t_0}^{t_1}\left(\sum_{i=1}^{n}\overline{F}_i\cdot\frac{\partial\overline{r}_i}{\partial q_j}\cdot\delta q_j\right)\cdot dt\right]= \\ =\sum_{j=1}^{k}\left(\int_{t_0}^{t_1}Q_j\cdot\delta q_j\cdot dt\right)=\sum_{j=1}^{k}\left(\int_{t_0}^{t_1}Q_j\cdot dt\right)\cdot\delta q_j \tag{60}$$

where Q_j represents the generalized force.

As a result, the integral principle for olonomous systems and synchronous running $\delta t=0$, takes the form

$$\sum_{j=1}^{k}\left\{\int_{t_0}^{t_1}\left[-\frac{\partial E_A}{\partial\ddot{q}_j}+Q_j\right]\cdot dt\right\}\cdot\delta q_j=0,\ \delta t=0. \tag{61}$$

Applying the conditions:

$\left(\delta q_j\neq 0, j=1\rightarrow k\right);\left(\delta q_i\neq 0, i=1\rightarrow k, i\neq j\right)$, results:

$$\frac{d}{dt}\left\{\int_{t_0}^{t_1}\left[-\frac{\partial E_A}{\partial\ddot{q}_j}+Q_j\right]\cdot dt\right\}=0; \\ -\frac{\partial E_A}{\partial\ddot{q}_j}+Q_j=0;\quad j=1\rightarrow k \tag{62}$$

Thus, by integral calculation there is obtained the identity (0.49), shown in previous section, on the basis of differential approach.

$$\frac{d}{dt}\left(\frac{\partial E_C}{\partial\dot{q}_j}\right)-\frac{\partial E_C}{\partial q_j}=\frac{\partial E_A}{\partial\ddot{q}_j}. \tag{63}$$

The final expression of integral principle is written below as:

$$\int_{t_0}^{t_1}\left[\int_{t_0}^{t_1}\left(\int_{t_0}^{t_1}\delta E_A\cdot dt\right)\cdot dt+\sum_{i=1}^{n}\overline{F}_i\cdot\delta\overline{r}_i\right]\cdot dt= \\ =\int_{t_0}^{t_1}\left(\delta E_C+\sum_{i=1}^{n}\overline{F}_i\cdot\delta\overline{r}_i\right)\cdot dt=0,\ where\ \delta t=0 \tag{64}$$

Expression (64) highlights the integral principle based on the acceleration energy as central function. By integrating this there is obtained, finally, the moving differential functions, of the form (50).

6 CONCLUSIONS

The main objective of this work was the implementation of the acceleration energy as a central function in differential and integral principles of the analytical mechanics. The achieving of this goal required initially a laborious kinematic study, in matrix form, applied to multibody mechanical systems. In this study were highlighted relations concerning the position and orientation of multibody system and generalized defining relations for linear and angular velocities and accelerations. An important aspect was the using of matrix exponential functions, whose undeniable advantage to classical transformations, refers to not using of reference systems, which is the effect of applying specific homogeneous coordinates of the initial configurations of mechanical systems. Expressions which are the basis of the kinematic study were used to define acceleration energy in explicit form, specific to general movement of a body. After applying the differential principles D'Alambert-Lagrange, specific to multibody systems, it`s been shown the implementation of acceleration energy as central function in differential equations of motion. As is known in analytical mechanics, the equations of motion can be established by applying of the variational principles. As a result, the last part of the work, it has been presented the implementation of acceleration energy in the integral principles, specific to olonomous systems, and the equivalence of the two principles based on kinetic energy and acceleration energy.

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Formulări asupra principiilor din mecanica analitică

Rezumat: Principiile diferențiale şi integrale ale mecanicii analitice se bazează pe noțiuni fundamentale din mecanica newtoniană, ținând seama însă, de caracterul diferențial şi de tipul legăturilor dintre corpurile ce compun un sistem mecanic. Printre noțiunile fundamentale, cu un rol esențial, se afla energia cinetică ca funcție centrală în ecuațiile de tip Lagrange-Euler, ecuațiile Hamilton, precum şi în principiile variaționale. Dar, ecuațiile diferențiale de mişcare pentru sistemele mecanice cu mai multe grade de libertate pot fi determinate utilizând energia accelerațiilor, ca funcție centrală, a cărei implementare în principiile diferențiale şi variaționale, va constitui obiectivul principal al acestei lucrări. Pentru dezvoltarea ecuațiilor de mişcare cu ajutorul noțiunilor mai sus amintite, lucrarea va conține ca secțiuni principale: studiul cinematic al sistemelor multicorp, funcțiile exponențiale de matrice, expresia de definiție în formă generală a energiei accelerațiilor, iar apoi implementarea acesteia în principiile diferențiale şi variaționale ale mecanicii analitice.

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