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ELEMENTS OF FUZZY LOGIC NUMBERS

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Abstract: *This paper synthesizes the main properties of fuzzy numbers and fuzzy logic operators also. We considered this study timely because; in many applications easily pass over some theoretical aspects related to this area, which has applications in front of theoretical studies. This process, in which applications run on mathematical theories, is natural. This paper brings its modest contribution to the crystallization properties of the theory of fuzzy logic numbers. We put in evidence the fundamental aspect of the embedding functions from theory of fuzzy numbers, but not only, embedding functions appear as fuzzification and defuzzification operations.* ***Key words:*** *fuzzy logic, fuzzy numbers, embedding functions, fuzzification, defuzzyfication*

1. DEFINITION OF FUZZY NUMBERS

In vector spaces is defined notion of "base", which is a subset of special items. For example, consider V_2 the set of vectors from the plan, its base consists of unit vector $\{\vec{i},\vec{j}\}$, which has the property that any vector $\vec{V}\in V_2$ is expressed with the base, i.e. there is unique $a,b\in\Re$ so that $\vec{V}$ can write:

$$\vec{V}=a\cdot\vec{i}+b\cdot\vec{j} \tag{1}$$

This idea was taken and used in the fuzzy structures (set). Be G_A a fuzzy set and $\hat{G}_A$ a fuzzy subset were $\hat{G}_A\subset G_A$. $\hat{G}_A$ is convex if membership function $\hat{\mu}_A$ which describing the fuzzy set $\hat{G}_A$ is concave. If $\hat{\mu}_A$ is twice differentiable, it is concave, if and only if: $\hat{\mu}_A''(x)<0,\forall x\in\hat{A}$ .Where $\hat{A}\subset A$ is the interval of definition (the universe of discourse) of membership function $\hat{\mu}_A$ .i.e. there $a_1,a_2\in\Re$, unique so $\hat{A}=[a_1,a_2]$ and $\forall x\in\hat{A},\hat{\mu}_A(x)$ is concave on $\hat{A}$, figure 1.

An equivalent definition of convex fuzzy sets is that sets of the plan $(\Re^2)$, bounded by lines: $x=a_1$, $x=a_2$, $y=0$, $y=\hat{\mu}_A(x)$ $x\in[a_1,a_2]$ to be convex in the classical sense. We also can define, convex fuzzy set using the concept fuzzy cut. Fuzzy set $\hat{G}_A\subset G_A$ is convex if the cut $A_\alpha=[a_1,a_2]$, then $\forall\alpha'>\alpha$ result that $A_{\alpha'}\subset A_\alpha$. Geometrical image of this function is given in figure 2.

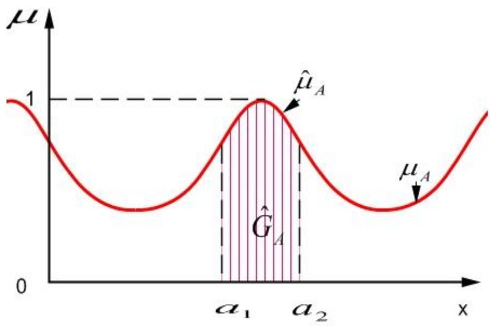

Fig. 1. *Geometrical image of convex fuzzy set*

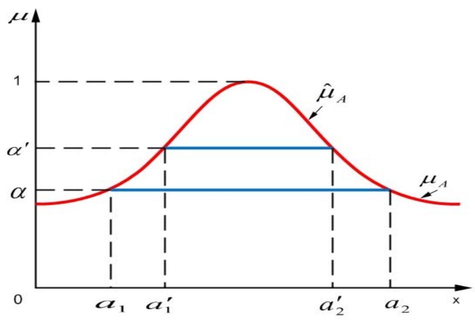

Fig. 2. *Geometrical image of cut* $A_\alpha=[a_1,a_2]$, $A_{\alpha'}=[a'_1,a'_2]$

A fuzzy subset $N_A \subset G_A$ is named "*fuzzy number*" if it satisfies the following conditions:

(i) N_A it is convex, the cut $[a_1, a_2]$.

(ii) $\exists x_0 \in [a_1, a_2]$ unique, so $\hat{\mu}_A(x_0) = 1$, i.e. N_A is unique normal.

Since the cut of a fuzzy number N_A is unique, i.e. $A_\alpha = [a_1, a_2]$ interval is unique, and has the property that $\forall x \in A_\alpha$, results $\mu_A(x) \geq \alpha$. And vice versa, $\forall (x, \hat{\mu}_A(x)) \in \hat{G}_A$ results $x \in [a_1, a_2]$. By this embedding, may agree that the fuzzy numbers N_A be confused with his projection on the axis O_x, i.e. in the followings will consider that the fuzzy number N_A is given by:

$$N_A = [a_1, a_2] \tag{2}$$

Figure 3 presents the geometric image of a fuzzy number. Using mathematical convention (embedding) given by (2), in figure 4 can be seen the image of conventional fuzzy numbers, i.e. that embedded of fuzzy number.

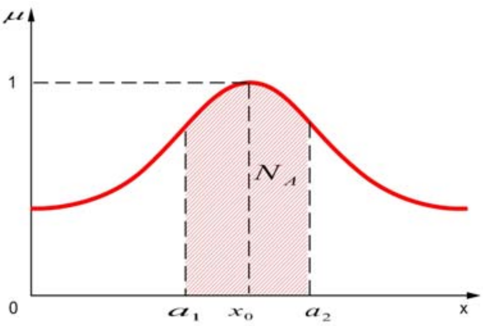

Fig. 3. *Geometrical image of a fuzzy number*

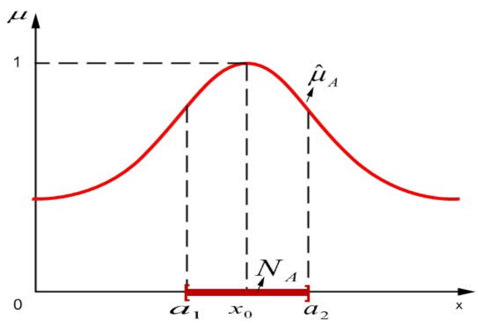

Fig. 4. *Image of conventional fuzzy number*

The types of fuzzy numbers most frequently used in practical application are: triangular fuzzy numbers, trapezoidal fuzzy numbers, Gauss fuzzy numbers, Bell fuzzy numbers, and others fuzzy numbers like: Z-shape fuzzy numbers, S-shape fuzzy numbers, Π-shape fuzzy numbers.

a) Triangular fuzzy numbers

Whether $a_1, a_2, a_3 \in \Re, a_1 < a_2 < a_3$, three real numbers which have the geometric interpretation given in figure 5. In this manner, triplet (a_1, a_2, a_3) defines a triangular fuzzy number, corresponding membership function is:

$$\mu_{tri}(x; a_1, a_2, a_3) = \begin{cases} 0, x \leq a_1 \\ \dfrac{x - a_1}{a_2 - a_1}, a_1 \leq x \leq a_2 \\ \dfrac{a_3 - x}{a_3 - a_2}, a_2 \leq x \leq a_3 \\ 0, a_3 \leq x \end{cases} \tag{3}$$

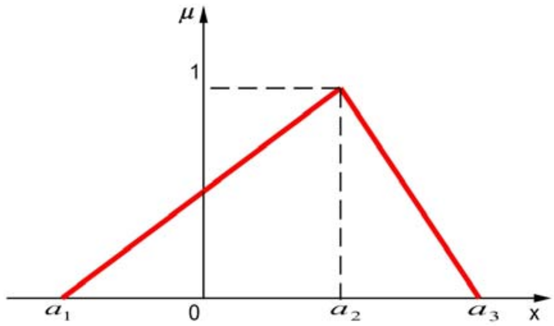

Fig. 5. *Geometrical image of triangular fuzzy number*

For $\alpha \in (0,1)$, cut $A_\alpha = [a_1(\alpha), a_3(\alpha)]$ and a_2 so that $\mu_A(a_2) = 1$, fuzzy numbers A_α is uniquely determined, as shown in figure 6.

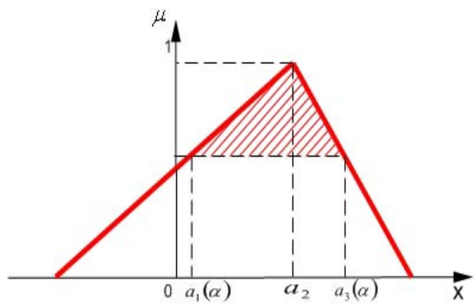

Fig. 6. *Triangular fuzzy number generated by cut*

b) Trapezoidal fuzzy numbers

Trapezoidal fuzzy numbers are defined by quartet (a_1, a_2, a_3, a_4), presented in figure 7. Trapezoidal membership function is:

$$\mu_{Tr}(x; a\ , a_2, a_3, a_4) = \begin{cases} 0, x \leq a_1 \\ \dfrac{x - a_2}{a_2 - a_1}, a_1 \leq x \leq a_2 \\ \dfrac{a_4 - x}{a_4 - a_3}, a_3 \leq x \leq a_4 \\ 0, a_4 \leq x \end{cases} \tag{4}$$

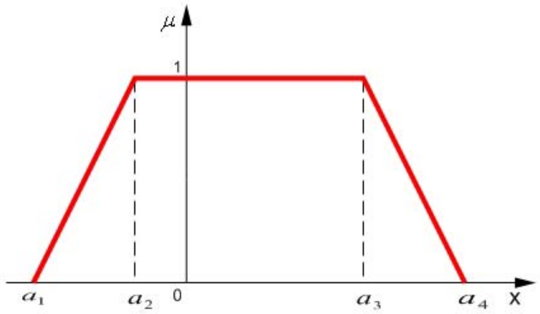

Fig. 7. *Trapezoidal fuzzy numbers*

c) Gaussian fuzzy number

In figure 8 is presented the Gaussian fuzzy numbers. The symmetric Gaussian function depends on two parameters: σ and c as given by:

$$\mu_{Gauss} = e^{\frac{-(x-c)^2}{2\sigma^2}} \tag{5}$$

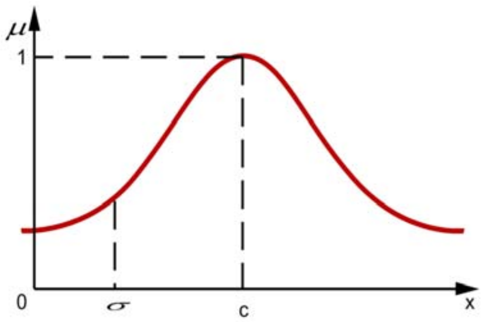

Fig. 8. *Gaussian fuzzy numbers*

d) Bell type fuzzy number

In figure 9 is presented Bell type fuzzy number. The generalized Bell function depends on three parameters a, b and c as given by:

$$\mu_{Bell} = \frac{1}{1+\left|\frac{x-c}{a}\right|^{2b}} \tag{6}$$

Where the parameter b is usually positive, the parameter c locates the center of the curve.

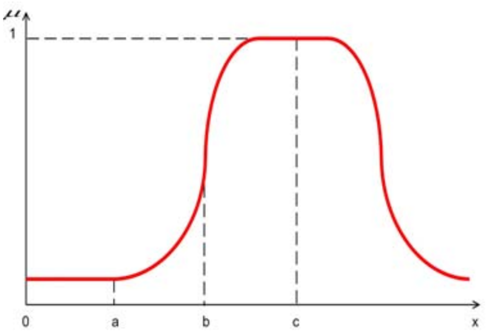

Fig. 9. *Bell type fuzzy numbers*

e) S-shape fuzzy numbers

This spline based curve is mapping on the vector x, and it is named because of its S-shape, presented in figure 10. The parameters *a* and *b* locate the extremes of the sloped portion of the curve. The membership function is:

$$\mu_S = \begin{cases} 0, x \le a \\ 2\left(\frac{x-a}{b-a}\right)^2, a \le x \le \frac{a+b}{2} \\ 1-2\left(\frac{x-b}{b-a}\right)^2, \frac{a+b}{2} \le x \le b \\ 1, x \ge b \end{cases} \tag{7}$$

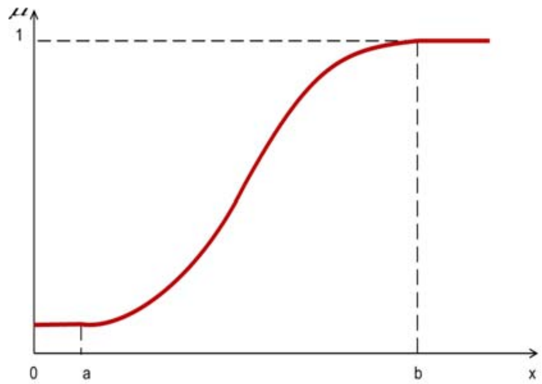

Fig. 10. *S-shape fuzzy numbers*

f) Z-shape fuzzy numbers

This spline based function of x is so named because of its Z-shape, presented in figure 11. The parameters *a* and *b* locates the extremes of the sloped portion of the curve as given by:

$$\mu_Z = \begin{cases} 1, x \le a \\ 1-2\left(\frac{x-a}{b-a}\right)^2, a \le x \le \frac{a+b}{2} \\ 2\left(\frac{x-a}{b-a}\right)^2, \frac{a+b}{2} \le x \le b \\ 0, x \ge b \end{cases} \tag{8}$$

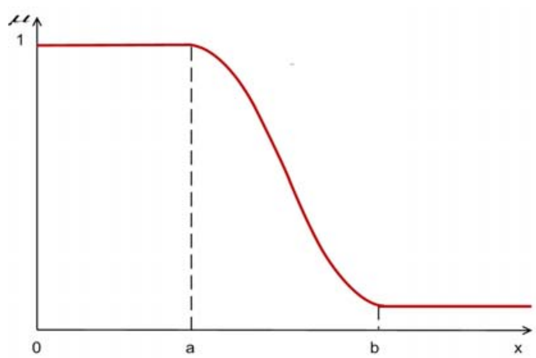

Fig. 11. *Z-shape fuzzy numbers*

g) Π-shape fuzzy numbers

This spline based curve is so named because of its Π shape. The membership function is evaluated at the points determined by the vector x. The parameters *a* and *d* locate the “feet” of the curve, while b and c locate its “shoulders”, presented in figure 12. The membership function is a product of S and Z membership function, and is given by:

$$\mu_{\Pi} = \begin{cases} 0, x \le a \\ 2\left(\frac{x-a}{b-a}\right)^2, a \le x \le \frac{a+b}{2} \\ 1-2\left(\frac{x-b}{b-a}\right)^2, \frac{a+b}{2} \le x \le b \\ 1, b \le x \le c \\ 1-2\left(\frac{x-c}{d-c}\right)^2, c \le x \le \frac{c+d}{2} \\ 2\left(\frac{x-d}{d-c}\right)^2, \frac{c+d}{2} \le x \le d \\ 0, x \ge d \end{cases} \quad (9)$$

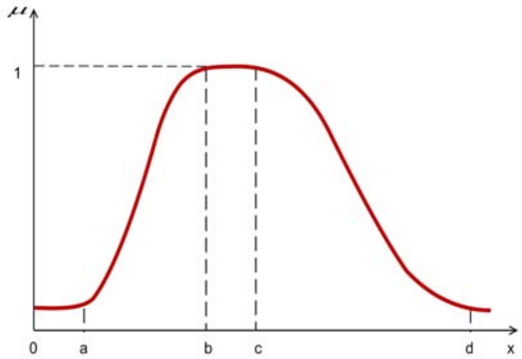

Fig. 12. Π*-shape fuzzy numbers*

2. RANKING AND EMBEDDING FUNCTION OF FUZZY NUMBERS

For triangular fuzzy numbers, ranking and embedding function F is defined at $F:(NFT) \rightarrow \Re$:

a) Yager’s index of order I

If $a,b \in N_A$ are two triangular fuzzy numbers, $a=(a_1,a_2,a_3)$ and $b=(b_1,b_2,b_3)$, then:

$$F_1(a)=\frac{a_1+a_2+a_3}{3}, F_1(b)=\frac{b_1+b_2+b_3}{3} \quad (10)$$

If $F_1(a)<F_1(b)$, result that $a<b$, which is equivalent to: $a_1+a_2+a_3<b_1+b_2+b_3$, result that triangular fuzzy numbers *a* is smaller than triangular fuzzy number *b*.

b) Yager’s index of order II

$$F_2(a)=\frac{a_1+2a_2+a_3}{4}, F_2(b)=\frac{b_1+2b_2+b_3}{4} \quad (11)$$

For trapezoidal fuzzy numbers, embedding functions can be defined by:

c) Yager’s index of order III

$$F_3(a)=\frac{a_1+a_2+a_3+a_4}{4}, a=(a_1,a_2,a_3,a_4) \quad (12)$$

d) Adamo ranking and embedding function

It is used for cuts, when the fuzzy number $A_\alpha=[a_1(\alpha),a_3(\alpha)]$, is generated by cutting A_α, in a fuzzy set with μ_A membership function, presented in figure 13.

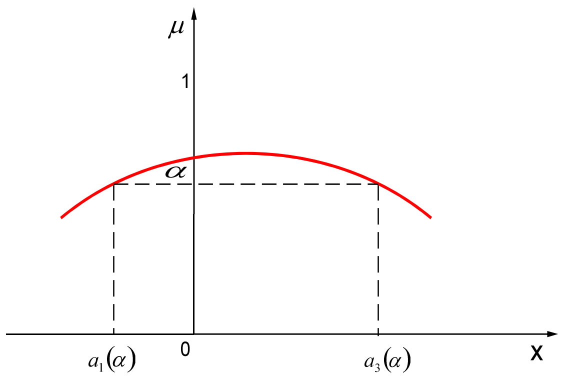

Fig. 13. *Fuzzy number generated by* A_α *cut, not necessarily normal*

For this class of fuzzy numbers, embedding functions is actually a family of functions $\{F_\alpha\}_{\alpha\in(0,1)}$, for which generic element F_α, is defined by:

$$F_\alpha(A_\alpha)=a_1+\alpha(a_3-a_1) \quad (13)$$

Where A_α is defined by cut:

$$A_\alpha=[a_1(\alpha),a_3(\alpha)], \alpha\in(0,1) \quad (14)$$

Observe that fuzzy number defined by cut its uniquely defined by μ_A and α.

e) Gonzales ranking and embedding function

If triangular fuzzy numbers is generated by the triplet $a=(a_1,a_2,a_3)$, then ranking and embedding function $F_{\alpha}^{t}:(NFT)\to\Re$ named Gonzales function, which depends on two parameters $\alpha\in(0,1)$ and $t>0$. Parameters α and t have specific meanings. Parameter α shows the degree of optimism when $\alpha\to1$ and degree of pessimism when $\alpha\to0$. Parameter $t>0$ indicates negotiation time. Gonzales ranking and embedding function is defined by:

$$F_{\alpha}^{t}(a)=a_2+\frac{a_2-a_1}{t+1}+\alpha\cdot\frac{a_3-a_1}{t+2},\ a=(a_1,a_2,a_3),a\in(NFT) \quad (15)$$

In conclusion, if we have two fuzzy numbers $a,b\in(NTF)$, i.e. two triangular fuzzy numbers: $a=(a_1,a_2,a_3)$ and $b=(b_1,b_2,b_3)$, parameters $\alpha\in(0,1)$ and $t>0$ are determined empirically. "*a*" number is lower than "*b*" number. Note „$a<b$" when done in $\Re$ inequality:

$$a_2+\frac{a_2-a_1}{t+1}+\alpha\cdot\frac{a_3-a_1}{t+2}<b_2+\frac{b_2-b_1}{t+1}+\alpha\cdot\frac{b_3-b_1}{t+2} \quad (16)$$

Inequality can be written in a simpler form, if it occurs in $\Re$ the inequality:

$$\frac{\alpha}{t+2}(b_3-a_3)+\frac{t+2}{t+1}(b_2-a_2)\geq\frac{t+\alpha+2}{t+2}(b_1-a_1) \quad (17)$$

then „$b\geq a$". Of course that is abstract and conventional ordering, agreement was given the choice of empirical Gonzales membership function and parameters α and t.

3. OPERATION WHITH EMBEDDER OF FUZZY NUMBERS

Trough N_A will be noted fuzzy sets numbers defined on the universe of discourse $A\subset\Re$. On set $N_A X N_A$ can define the following operation:

a) Sum of fuzzy number

If $N_A=[a_1,a_2]$ and $N'_A=[a'_1,a'_2]$, then:

$$N_A(+)N'_A=[a''_1,a''_2],\ a''_1=a_1+a'_1,\ a''_2=a_2+a'_2 \quad (18)$$

It can be observed that the sum from N_A its similar with the sum from $\Re^2$. The following will use fuzzy logic operations: conjunction "$\wedge$" and disjunction "$\vee$", which unlike classical logic is defined by:

$$x\wedge y=\min\{x,y\},\ x,y\in G_A$$
$$x\vee y=\max\{x,y\},\ x,y\in G_A \quad (19)$$

b) Product of fuzzy numbers

If $\lambda\in\Re$ and $N_A\in\tilde{N}_A, N_A=[a_1,a_2]$, then:

$$\lambda(\cdot)N_A=[a'_1,a'_2],\ a'_1=\lambda\cdot a_1\wedge\lambda\cdot a_2,\ a'_2=\lambda\cdot a_1\vee\lambda\cdot a_2 \quad (20)$$

For $\lambda=-1$ is obtained the opposite of number N_A, noted with $-N_A$, which has the formula:

$$-N_A=[-a_2,-a_1] \quad (21)$$

If $\lambda\in(0,1)$, then $\lambda(\cdot)N_A$ is condensate of N_A, respectively for $|\lambda|>1$, then $\lambda(\cdot)N_A$ represent the dilatation of fuzzy numbers N_A.

c) Subtraction of fuzzy numbers

Whether $N_A, N`_A\in N_A$ two fuzzy numbers, defined by: $N_A=[a_1,a_2]$ and $N`_A=[a'_1,a'_2]$, subtraction is defined by:

$$N_A(-)N'_A=[a''_1,a''_2],\ a''_1=a_1-a'_2,\ a''_2=a_2-a'_1 \quad (22)$$

because $N_A(-)N'_A=N_A(+)(-N'_A)$.

d) Product of fuzzy numbers

If $N_A, N`_A\in N_A$, $N_A=[a_1,a_2]$ and $N`_A=[a'_1,a'_2]$, then:

$$N_A\otimes N'_A=[a''_1,a''_2] \quad (23)$$

where,
$a''_1=a_1a'_1\wedge a_1a'_2\wedge a_2a'_1\wedge a_2a'_2,\ a''_2=a_1a'_1\vee a_1a'_2\vee a_2a'_1\vee a_2a'_2$

These details, on "$\wedge$" and "$\vee$" are absolutely necessary at all fuzzy logic, since in any interval $[x,y]$, $x<y$.

e) Division of fuzzy numbers

Be $N_A, N`_A\in N_A$, $N_A=[a_1,a_2]$ and $N`_A=[a'_1,a'_2]$, with $a'_1\neq0$ and $a'_2\neq0$. Division of fuzzy numbers N_A and $N`_A$ is defined as:

$$N_A(:)N'_A=[a''_1,a''_2] \quad (24)$$

where:

$$a_1'' = \frac{a_1}{a_1'} \wedge \frac{a_1}{a_2'} \wedge \frac{a_2}{a_1'} \wedge \frac{a_2}{a_2'},\ a_2'' = \frac{a_1}{a_1'} \vee \frac{a_1}{a_2'} \vee \frac{a_2}{a_1'} \vee \frac{a_2}{a_2'}$$

f) Inverse of fuzzy numbers

If $N_A = [a_1, a_2]$ and $a_1 \neq 0$, $a_2 \neq 0$, then inverse of N_A,noted with N_A^{-1} is defined by:

$$N_A^{-1} = [a_1', a_2'] \qquad (25)$$

where: $a_1' = \frac{1}{a_1} \wedge \frac{1}{a_2}$, $a_2' = \frac{1}{a_1} \vee \frac{1}{a_2'}$

4. OPERATORS OF FUZZY NUMBERS SET

In theory of topological vector spaces meet two main types of convergent, namely “uniformly convergent” studied from the perspective of the notion of uniform rule, or T-norm, and “weak convergent” studied from perspective of S-norm. It is known that if a array (x_n) from a topological vector space converges hard, then converges week too. Reverse implication is not generally true. In terms of specific notations we have:

$$x_n \xrightarrow{T} x \Rightarrow x_n \xrightarrow{S} x\ ,\ x_n \xrightarrow{S} x \not\Rightarrow x_n \xrightarrow{T} x \qquad (26)$$

This property of the two norms can be interpreted as a suggestion: S-norm is more generous than the T-norm.

a) *T-norm (norm of intersection*)

Be G_A and G_B two fuzzy structures, define operator $T:[0,1]X[0,1] \to [0,1]$, actually $T: G_A X G_B \to [0,1]$ by:

$$T(\mu_A(x), \mu_B(x)) = \mu_{A \cap B}(x), \forall x \in A \qquad (27)$$

Satisfying the conditions:

(i) T is associative

$$T(\mu_A(x), T(\mu_B(x), \mu_C(x))) = T(T(\mu_A(x), \mu_B(x), \mu_C(x))), \forall x \in A \qquad (28)$$

(ii) T is symetric

$$T(\mu_A(x), \mu_B(x)) = T(\mu_B(x), \mu_A(x)), \forall x \in A \qquad (29)$$

(iii) T is monotonically increasing, if $\mu_A(x) \le \mu_B(x)$ and $\mu_{A^*}(x) \le \mu_{B^*}(x)$, $\forall x \in A$, when occurs relationship:

$$T(\mu_A(x), \mu_{A^*}(x)) = T(\mu_B(x), \mu_{B^*}(x)), \forall x \in A \qquad (30)$$

(iv) T is normalized.

$$T(0,0) = 0 \text{ and } T(1, \mu_A(x)) = T(\mu_A(x), 1) = 1, \forall x \in A \qquad (31)$$

(v)If in addition T is strictly idempotent, then T-norm is Arhimedian norm. For $\forall x \in Supp(\mu_A)$, we have:

$$T(\mu_A(x), \mu_A(x)) < \mu_A(x) \qquad (32)$$

b) S-norm (norm of reunion)

S-norm, weak operator will be defined by: $S:[0,1]X[0,1] \to [0,1]$ actually $S: G_A X G_B \to [0,1]$ which, through the membership function $\mu_A(x)$ is given by:

$$S(\mu_A(x), \mu_B(x)) = \mu_{A \cup B}(x), \forall x \in A \qquad (33)$$

And satisfying the axioms (i),(ii) and (iii) of T-norm. Axiom of operator norm, in this case is defined by:

(iv)* $S(1, \mu_A(x)) = S(\mu_A(x), 1) = 1, \forall x \in A$

$$S(0, \mu_A(x)) = S(\mu_A(x), 0) = \mu_A(x), \forall x \in A \qquad (34)$$

Normalization of relations (29), respectively (34) results immediately. Also from the definition of the two norms and from $G_A \cap G_B \subset G_A \cup G_B$, we conclude that the S-norm is more generous, i.e. is more tolerant at the behavior of fuzzy variables. Relationships that occur between the T and S norm are:

$$S(\mu_A(x), \mu_B(x)) = 1 - T(\overline{\mu_A}(x), \overline{\mu_B}(x)), \forall x \in A$$
$$T(\mu_A(x), \mu_B(x)) = 1 - S(\overline{\mu_A}(x), \overline{\mu_B}(x)), \forall x \in A \qquad (35)$$

Indicating that $\overline{\mu_A}(x) = 1 - \mu_A(x)$ and $\overline{\mu_B}(x) = 1 - \mu_B(x)$.

c) Neutral operator (complementary operator)

Be G_A a fuzzy set, the operator $n: G_A \to [0,1]$ named neutral operator if his expression was defined using elements of the universe of discourse A, is:

$$n(\mu_A(x)) = \overline{\mu_A}(x) \quad (36)$$

And if it satisfies the axioms:
(i) n is idempotent, i.e.:

$$n^2(\mu_A(x)) = \mu_A(x) \quad (37)$$

(ii) n is monotonically decreasing, i.e.: $\forall \mu_A(x) \le \mu_B(x)$, results:

$$n(\mu_A(x)) \ge n(\mu_B(x)), \forall x \in A \quad (38)$$

(iii) n is norm, i.e.:

$$n(0) = 1 \quad (39)$$

It may be observed that the notation $n^2(\mu_A)$ is taken from theory operators, and means:

$$n^2(\mu_A(x)) = (n \cdot n)(\mu_A(x)) = n(n(\mu_A(x))) \quad (40)$$

d) T_w - the most pessimistic fuzzy operator

If G_A and G_B are fuzzy structures defined over the same universe of discourse A, then $T_w : G_A X G_B \to [0,1]$, is defined by:

$$T_w(\mu_A(x), \mu_B(x)) = \begin{cases} \mu_A(x), if\_\mu_B(x) = 1 \\ \mu_B(x), if\_\mu_A(x) = 1 \\ 0, if\_\mu_A(x) \ne 1 și \mu_B(x) \ne 1 \end{cases} \quad (41)$$

e) H_λ - Hamacher's operator

Be G_A and G_B two fuzzy structures, $\lambda \ge 1$ Hamacher's operator is noted with $H_\lambda : G_A X G_B \to [0,1]$, and its defined by:

$$H_\lambda(\mu_A(x), \mu_B(x)) = \frac{\mu_A(x) \cdot \mu_B(x)}{\lambda + (1-\lambda)[\mu_A(x) + \mu_B(x) - \mu_A(x) \cdot \mu_B(x)]}, \forall x \in A \quad (42)$$

For $\lambda = 0$ and $\lambda = 1$ are obtained the next operators:

$$H_0(\mu_A(x), \mu_B(x)) = \frac{\mu_A(x) \cdot \mu_B(x)}{\mu_A(x) + \mu_B(x) - \mu_A(x) \cdot \mu_B(x)}, \forall x \in A,$$

$$H_1(\mu_A(x), \mu_B(x)) = \mu_A(x) \cdot \mu_B(x), \forall x \in A$$

f) T_p - Non-Poincare generalized operator

Be G_A and G_B two fuzzy sets, will note with T_p, $p \ge 0$ Non-Poincare generalized operator, defined by:

$$T_p(\mu_A(x), \mu_B(x)) = 1 - \left[(1-\mu_A(x))^p + (1-\mu_B(x))^p - (1-\mu_A(x))^p \cdot (1-\mu_B(x))^p\right]^{\frac{1}{p}} \quad (43)$$

If $p = 1$, result:

$$T_1(\mu_A(x), \mu_B(x)) = \mu_A(x) \cdot \mu_B(x) \quad (44)$$

If $p \to 0$, result:

$$\lim_{p \to 0} T_p(\mu_A(x), \mu_B(x)) = 1 - \lim_{p \to 0} \left[(1-\mu_A(x))^p + (1-\mu_B(x))^p - (1-\mu_A(x))^p \cdot (1-\mu_B(x))^p\right]^{\frac{1}{p}} \quad (45)$$

Are distinguished the next cases:

- If $\mu_A(x) = 1$ and $\mu_B(x) \ne 1$, then the limit has the value $1 - \mu_B(x)$ and therefore:

$$\lim_{p \to 0} T_p(\mu_A(x), \mu_B(x)) = \mu_B(x) \quad (46)$$

- If $\mu_B(x) = 1$ and $\mu_A(x) \ne 1$, then:

$$T_0(\mu_A(x), \mu_B(x)) = \mu_A(x) \quad (47)$$

- If $\mu_A(x) \ne 1$ and $\mu_B(x) \ne 1$, result:

$$T_0(\mu_A(x), \mu_B(x)) = 1 - e^{\lim_{p \to 0} \frac{1}{p}\left[(1-\mu_A(x))^p + (1-\mu_B(x))^p - [(1-\mu_A(x))\cdot(1-\mu_B(x))]^p\right]} \quad (48)$$

i.e., $T_0(\mu_A(x), \mu_B(x)) = 0$.

- If $p \to \infty$, result:

$$T_\infty(\mu_A(x), \mu_B(x)) = \min_{x \in A}\{\mu_A(x), \mu_B(x)\} \quad (49)$$

i.e. $T_p(\mu_A(x), \mu_B(x)) = T_\infty$

g) σ_α - Non-Arhimedian generalized operator

This operator is noted with σ_α, $\alpha \geq 0$, is defined at $G_A X G_B$, trough membership function, therefore:

$$\sigma_\alpha(\mu_A(x), \mu_B(x)) = \frac{\mu_A(x) \cdot \mu_B(x)}{\max\limits_{x \in A}\{\alpha, \mu_A(x), \mu_B(x)\}}, \alpha \in [0,1] \quad (50)$$

It requires a discussion after $\alpha \in \Re_+$. If $\alpha = 0$, results:

$$\sigma_0(\mu_A(x), \mu_B(x)) = \begin{cases} \mu_A(x), if\_\mu_A(x) \leq \mu_B(x) \\ \mu_B(x), if\_\mu_A(x) \geq \mu_B(x) \end{cases} \quad (51)$$

or,

$$\sigma_0(\mu_A(x), \mu_B(x)) = \frac{\mu_A(x) \cdot \mu_B(x)}{\max\limits_{x \in A}\{\mu_A(x), \mu_B(x)\}} \quad (52)$$

If $\alpha = 1$, then σ_1 is product operator, i.e.:

$$\sigma_1(\mu_A(x), \mu_B(x)) = \mu_A(x) \cdot \mu_B(x) \quad (53)$$

because $\max\{1, \mu_A(x), \mu_B(x)\} = 1$.

If $\alpha \in (0,1)$, then σ_α can be written as:

$$\sigma_\alpha(\mu_A(x), \mu_B(x)) = \begin{cases} \frac{\mu_A(x) \cdot \mu_B(x)}{\alpha}, if\_\alpha \geq \mu_A(x), \mu_B(x) \\ \min\limits_{x \in A}\{\mu_A(x), \mu_B(x)\} in\_rest \end{cases} \quad (54)$$

h) S_λ - Sugeno's operator

If $\lambda \in [-1. + \infty)$, then Sugeno's operator is defined also on the Cartesian product $G_A X G_B$ by:

$$S_\lambda(\mu_A(x), \mu_B(x)) = \min\{1, (\mu_A(x) + \mu_B(x) + \lambda(\mu_A(x) \cdot \mu_B(x)))\} \quad (55)$$

Conclusive discussion in this case, it is for: $S_\infty(\mu_A(x), \mu_B(x)) = 1$, if $\mu_A(x) \cdot \mu_B(x) \neq 0$. Of course it can be obtained other cases, if the membership function is known and it is given effectively. The only affirmation that can be done on S_λ is that this operator is monotone decreasing, and it does admit S_w the most pessimistic operator:

$$S_w(\mu_A(x), \mu_B(x)) = \begin{cases} \mu_A(x), if\_\mu_B(x) = 0, \forall x \in A \\ \mu_B(x), if\_\mu_A(x) = 0, \forall x \in A \\ 1, if\_\mu_A(x) \neq 0 and \mu_B(x) \neq 0 \end{cases} \quad (56)$$

i) Y_q – Non-Sugero's generalized operator

If $q \geq 0$ and $Y_q : G_A X G_B \to [0,1]$, then non-Sugeno's generalized operator is:

$$Y_q(\mu_A(x), \mu_B(x)) = 1 - \min\limits_{x \in A}\left\{1, \left[(1 - \mu_A(x))^q + (1 - \mu_B(x))^q\right]^{\frac{1}{2}}\right\} \quad (57)$$

Discussion over Y_q, after parameter $q \geq 0$ can be made for cases $q = 0$ and $q = 1$. Indeed, if $q = 0$ easily shows that $Y_0 = Y_w$, i.e.:

$$Y_w(\mu_A(x), \mu_B(x)) = \begin{cases} \mu_A(x), if\_\mu_B(x) = 1 \\ \mu_B(x), if\_\mu_A(x) = 1 \\ 0, in\_rest \end{cases} \quad (58)$$

The first two relationships are directly, for $\mu_A(x) \neq 1$ and $\mu_B(x) \neq 1$ results:

$$\lim_{q \to 0}\left[(1 - \mu_A(x))^q + (1 - \mu_B(x))^q\right]^{\frac{1}{q}} = 2^\infty = \infty$$

Consequently: $Y_0(\mu_A(x), \mu_B(x)) = 0$

For the case $q = 1$, we obtain: $Y_1(\mu_A(x), \mu_B(x)) = 0$

j) Logaritmic fuzzy operators

Be $a > 0$ and $a \neq 0$, then $LOG : G_A X G_B \to [0,1]$, is defined by:

$$LOG(\mu_A(x), \mu_B(x)) = \log_a\left[1 + \frac{(a^{\mu_A(x)} - 1) \cdot (a^{\mu_B(x)} - 1)}{a - 1}\right] \quad (59)$$

Comment required over this operator is: if $\mu_B(x) = 1$, then results: $LOG(\mu_A(x), 1) = \log_a\left[1 + (a^{\mu_A(x)} - 1)\right] = \mu_A(x)$. Similar: $LOG(1, \mu_B(x)) = \mu_B(x)$. If $\mu_A(x) \cdot \mu_B(x) = 0$, results: $LOG(\mu_A(x), \mu_B(x)) = 0$. If $\mu_A(x) = \mu_B(x) = 1$, obtain the maximum value for the logarithmic operator, i.e. $\log_a(1 + a - 1) = 1$.

k) Logical operators

Be G_A and G_B two fuzzy sets, defined over the Cartesian product $G_A X G_B$ through membership function is defined "conjunction" and "disjunction" operators, noted with "$\wedge$" respectively "$\wedge$" classic already.

$$\mu_A(x) \wedge \mu_B(x) = T(\mu_A(x), \mu_B(x)) + \frac{1}{2}(\mu_A(x) + \mu_B(x)) \quad (60)$$

respectively,

$$\mu_A(x) \vee \mu_B(x) = S(\mu_A(x), \mu_B(x)) + \frac{1}{2}(\mu_A(x) + \mu_B(x)) \quad (61)$$

Logical operators express with relationships (60) and (61) are named in scientific literature operators of mediation.

l) Linguistic fuzzy operators

Fuzzy logic theory use linguistic fuzzy variables. On the assumption the use of these linguistic fuzzy variables, involved "modified" operators, noted with $M: U(G_A) \to U(G_B)$, where $U(G_A)$ is universe of fuzzy sets.

Normalization operator:

$$Norm(G_A) = \frac{1}{\ln(G_A)} \cdot \mu_A(x) \quad (62)$$

Concentration operator:

$$Con(G_A) = \mu_A^2(x) \quad (63)$$

Dilatation operator:

$$Dil(G_A) = \sqrt{\mu_A(x)} \quad (64)$$

Contrast intensification operator:

$$IntC(G_A) = \begin{cases} 2\mu_A^2(x), \mu_A(x) \le 1/2 \\ 1 - 2(1 - \mu_A^2(x)), \mu_A(x) > 1/2 \end{cases} \quad (65)$$

„very (G_A)" operator mathematical point of view coincides with the concentration operator, but linguistically it has its own meaning.

$$Very(G_A) = \mu_A^2(x) \quad (66)$$

very (very) (G_A) is defined by:

$$(Very(Very))(G_A) = \mu_A^4(x) \quad (67)$$

It is clear that the concentration is doubled. Naturally, this operator can be generalized as follows:

$$Very^n(G_A) = \mu_A^{2n}(x), x \in A \quad (68)$$

„Plus" operator:

$$Plus(G_A) = \mu_A^{1,25}(x) \quad (69)$$

„Minus" operator:

$$Minus(G_A) = \mu_A^{0,75}(x) \quad (70)$$

The operator "more or less" is a fuzzy modification operator, being the correspondent of dilatation. This linguistic fuzzy operator is noted with (more/less):

$$(more / less)(G_A) = \sqrt{\mu_A(x)} \quad (71)$$

5. CONCLUSION

The fuzzy logic theory is in a continuous process of development and completion with new properties, because the applications are ahead of theory. This brief synthesis of the most important properties, but not the only ones is addressed to those which use fuzzy logic in applications and which will master better this instrument of work with these theories. Because of space restriction we omitted an also important chapter which will be object of a further paper, chapter named "fuzzy logic number inferences". Finally we underline that the fuzzy numbers, generated by optimal control problems have a very important role in application.

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ELEMENTE DE LOGICĂ A NUMERELOR FUZZY

Rezumat: În această lucrare se realizează o sinteză a principalelor proprietăţi ale numerelor fuzzy şi de asemenea a operatorilor logicii fuzzy. Am considerat oportună realizarea acestui studiu, deoarece, în multe aplicaţii se trece cu usurinţă peste unele aspecte teoretice relative la acest domeniu, care are aplicaţiile în faţa studiilor teoretice. Acest proces, în care, aplicaţiile conduc la teorii matematice, este unul natural. Lucrarea de faţă îşi aduce modestul aport la cristalizarea unor proprietăţi din teoria logicii numerelor fuzzy. Punem în evidenţă în acest sens aspectul fundamental al funcţiilor de scufundare din cadrul acestei teorii a numerelor fuzzy, dar nu numai, adică, funcţiile de scufundare apar ca şi operaţii de fuzzificare şi defuzzificare.

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