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PROPERTIES OF MULTIPLIERS ON SPECIAL ALGEBRAS WITH APPLICATION TO SIGNAL PROCESSING

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Abstract: *In many cases, signal processing methods cannot be circumscribed to existing models but require some extensions of mathematical theories (operator theory, functional analysis) to enable the widening of more general spaces or the modeling using more general functions but with fewer properties satisfying theorems used in similar models.*

In this context, one can expand the Hilbert space L_2 to a locally convex space, sequentially complete, extension required in terms of types of data that must be processed. Hence, there is a need to obtain some medium ergodic theorems which can characterize operators endowed with a certain type of limitation. In 1943, Dunford obtained ergodic theorems using analytical calculation of the operational functions of a complex Banach space operator. In 1974, M. Lin added an equivalent condition to the uniform ergodic theorem .In this paper, a study on some equivalent conditions for T (where T is a multiplier on a locally convex algebra A which has a closed range) is performed.

Key words: *multipliers, locally convex algebra, relatively regular operator, barreled space*

1. INTRODUCTION

In signal and image treatment, input data are generally finite samples of the form:

$$\{x1, \dots, xn\} \epsilon\ \mathbb{C}\ (1 - D\ case)$$

$$\big(x(i,j)\big), (i,j) \epsilon\ (1,1\ ), \dots, (\ m,n)(2 - D\ case).$$

In addition to stationarity in a broader sense, it is assumed that the ergodicity hypothesis is satisfied and thus the statistical moments up to second order in probability space can be addressed by their equivalents calculated in the space of achievements. Many areas of application today require treating a growing volume of digital data that appear important in a raw form or indexed by one or more parameters. These index data are subject to very general topics, including signal processing, field image or volume, with many applications in medicine and industry.

Depending on the number of parameters used in the natural description, these data can be classified as mono-dimensional (1D), where they depend on one parameter, or multidimensional (2D, 3D and nD), where the indexation depends on multiple parameters. 3D environment is illustrated by blocks of data field or 3D imagery such as geological, seismic, medical, radar, fluid dynamics, etc.

Mathematically, these multi-dimensional data are generally treated as a function of one or more discrete or continuous variables - proper indexing parameters. The classes of models vary depending on the type of data and information considered relevant to these data as part of various applications.

According to these, models and associated estimation techniques are based on different areas of mathematics, such as harmonic analysis, probability theory, geometry and differential equations.

Three-dimensional coronary bypass models of the aortic-right coronary bypass systems are constructed using computational fluid-dynamics software. A larger framework to study flow dynamics at specific time instants of the cardiac cycle is the power-boundedness techniques for operators from special types of algebras which generalizes the Banach spaces case to the locally convex ones.

The general view of the stochastic context is to look at these not as scalars, but as a random variables in probability space.

As measurable functions, these variables are assumed to have mean zero and finite variance which means that they can be viewed as vectors in a Hilbert space L_2. This approach allows the use of the hilbertian theory, namely operators under the hypothesis of stationarity in the extended sense.

In addition to stationarity in a broader sense, it is assumed that the ergodicity hypothesis is satisfied and thus the statistical moments up to second order in probability space can be addressed by their equivalents calculated in the space of achievements.

In this context, expanding the Hilbert space L_2 to a locally convex sequentially complete space required in terms of types of data that must be processed, there is a need to obtain a theorem which can characterize ergodic medium operators endowed with a certain type of limitation.

In 1943, N. Dunford [6, 7] obtained certain ergodic theorems using operational calculus for analytic functions of an operator on a complex Banach space. In 1974, M. Lin [17] added an equivalent condition in the uniform ergodic theorem. Papers dedicated to the study of the class of universally bounded operators acting on a locally convex space have been elaborated by R.T.Moore [19] in 1969, A. Chilana in 1970 [4], G. Joseph in 1977 [12], F. Pater and T. Binzar in 2010 [20]. This class of operators was also taken into consideration by J.R. Giles, G. Joseph, D.O. Koehler, B. Sims [8] and by A. Michael in [18].

In 1958, V. Ptak in [22] introduces the B−completeness of a locally convex space, that allows an extension of the fundamental principles of functional analysis to such spaces. For such an extension, a result of D. Van Dulst [5] regarding the heredity property of a barreled space is needed.
Based on some known results obtained by Allan [1], F. G. Bonales and R. V. Mendoza [2], [3], develop a version of spectral theory for bounded linear operators on barreled spaces.

2. PRELIMINARIES

All locally convex spaces will be assumed Hausdorff and over the complex field $\boldsymbol{C}$. A calibration for the locally convex space $\boldsymbol{X}$ is a family $\boldsymbol{P}$ of semi-norms generating its topology, in the sense that the topology of $\boldsymbol{X}$ is the coarsest with respect to which all the semi-norms in $\boldsymbol{P}$ are continuous. A calibration $\boldsymbol{P}$ is characterized by the property that the sets
$E(p,\epsilon) = \{x \in X : p(x) < \epsilon\}, \epsilon > 0, p \in \boldsymbol{P}$
constitute a neighborhood sub-base at 0.

We denote by $(\boldsymbol{X}, \boldsymbol{P})$ a locally convex space $\boldsymbol{X}$ with a calibration $\boldsymbol{P}$.

A locally convex algebra is algebra with a locally convex topology in which the multiplication is separately continuous. Such algebra is locally m-convex (lmc) if it has a neighborhood base $\boldsymbol{U}$ at 0 such that each $U \in \boldsymbol{U}$ is convex, balanced ($\lambda U \subseteq U$ for $|\lambda| < 1$) and satisfies the semi-group property $U^2 \subseteq U$. Any algebra with identity will be called unital. A unital lmc algebra $\boldsymbol{A}$ is characterized by the existence of a calibration $\boldsymbol{P}$ such that each $p \in \boldsymbol{P}$ is sub-multiplicative $(p(xy) \leq p(x)p(y))$ and satisfies $p(1) = 1$. We recall that the dual $\boldsymbol{X}^*$ of a locally convex space $\boldsymbol{X}$ is endowed with the topology of uniform convergence on finite subsets of $\boldsymbol{X}$, denoted by $\sigma(\boldsymbol{X}^*, \boldsymbol{X})$. We also mention that U^0 denotes the polar of U. According to V. Ptak [22], a locally convex space $\boldsymbol{X}$ is B-complete if a linear sub-space $Y \subset \boldsymbol{X}^*$ is $\sigma(\boldsymbol{X}^*, \boldsymbol{X})$ closed whenever $Y \cap U^0$ is closed for each 0-neighborhood $U \subset \boldsymbol{X}$. Let $\boldsymbol{A}$ be a complex algebra of functions. By a locally multiplicative convex algebra (lmc) we understand a topologically algebra whose topology is given by the family of sub-multiplicative semi-norms $\{p_\alpha\}$, i.e.
$p_\alpha(xy) \leq p_\alpha(x)p_\alpha(y)$ *for any* $x, y \in \boldsymbol{A}$ *and* $\alpha \in I$.

An algebra $\boldsymbol{A}$ is called proper if only the zero element annihilates the whole algebra $\boldsymbol{A}$, i.e. if $x\boldsymbol{A} = \boldsymbol{A}x = \{0\}$ then $x = 0$. It is obvious that any semi-simple algebra is proper.

Let $\boldsymbol{A}$ be a semi-simple commutative algebra. Then from [9] a multiplier is a mapping $T : \boldsymbol{A} \to \boldsymbol{A}$ which verifies $Tx \cdot y = x \cdot Ty$ *for all* $x, y \in \boldsymbol{A}$. We denote the set of all multipliers on $\boldsymbol{A}$ by $\boldsymbol{M(A)}$. We observe that the identity $x \cdot Ty = T(xy) = Tx \cdot y$ holds for some $x, y \in \boldsymbol{A}$ and thus the range $T\boldsymbol{A}$ and the kernel *ker T* of *T* are both bilateral ideals of A. Using the above identities it can be shown easily that ***M(A)*** is a commutative algebra having an identity element (i.e. the unit operator) (see for instance [10]). It is well known that each $T \epsilon M(A)$ is a continuous and bounded operator ([16] Corollary 2.3). Moreover, ***M(A)*** is a closed sub-algebra of the algebra of all continuous, linear and bounded operators endowed with the strong operator topology ([15] Theorem 2.4).

Laursen and Mbekta [16] determined, in the context of Banach algebras some necessary and sufficient conditions for a multiplier, to have closed range.

Our objectives are to study this problem in the context of locally convex algebras. Also, we will investigate some equivalent conditions for a closed range multiplier *T* of a certain type on a locally convex algebra. Moreover, we will prove that $TA \oplus ker\ T = A$ if and only if there exist a generalized inverse which commutes with *T*. These conditions are equivalent with the existence of a factorization $T = PB$, where *P* is an idempotent operator and *B* is an invertible multiplier. Now, we address the problem to a more general case, if *T* is a linear and continuous operator on a barreled, ***B***—complete space ***X***, then there exists a factorization $T = PB$, where *P* and *B* commute, *B* is invertible, *P* is idempotent and $X = TX \oplus ker\ T$ (see also [9], [11], [13], [14], [15], [21]). Moreover, if ***X*** is a decomposable space in this mode, then ***TX*** is necessarily closed.

3. SOME CHARACTERIZATIONS FOR CLOSED RANGE MULTIPLIERS ON LOCALLY CONVEX ALGEBRAS

Let ***X*** a barreled ***B***—complete space, such that the co-dimension of *TX* is Ψ_0, with the family of seminorms $\{p_n\}_{n \in N}$ generating its topology and let $T \epsilon C(X)$. Consider $T \epsilon C(X)$, where *TX* and *ker T* denote the range, respectively the kernel of *T*.

Theorem 3.1. *Suppose that TX* $\cap$ *ker T* = {0} *and TX + ker T is closed for any* $T \epsilon C(X)$. *Then* $T^n(X)$ *is closed for some n* $\in$ *N.*

Proof. First, we show that *T(X)* is closed in the topology of ***X***. By hypothesis $X = TX \oplus ker\ T$ is a ***B***—complete, barreled space. Moreover, it is easy to verify that ***TX*** is a ***B***—complete barreled space, endowed with the topology generated by the family of semi-norms $\{q_n\}$ given by

$$q_n(y) = p_n(y) = \inf_{x \in X,\ y = Tx} p_n(x),$$

$n \epsilon N$. Furthermore, since $p_n(y) < q_n(y)$ for some $y \epsilon TX$, the canonical injection $TX \to X_0$ is continuous.

Define $\psi: TX \times ker\ T \to X_0$ by $\psi(y, x) = y + x$. Thus ψ is a continuous bijection and from the open mapping theorem (see [9]) we obtain that ψ is bi-continuous and thus $TX = \psi(TX \times \{0\})$ is closed in X_0, hence it is closed in **X**.

We've obtained that *T* has a closed range. Because $ker\ T \cap TX = \{0\}$, $ker\ T^2 = ker\ T$, it results that $ker\ T^n = ker\ T$, for all $n \epsilon N$.

Now we can complete the proof by an inductive reasoning. If T^n has closed range for $n \epsilon N$, then because $TX \oplus ker\ T = TX \oplus ker\ T^n$ is closed, from hypothesis we get that

$T^{n+1}X = T^n(TX \oplus ker\ T) = T^n(TX \oplus ker\ T^n)$

is closed. This completes the proof.

Remark 3.2. *If* T^2X *is closed, then TX + ker T is closed, without the assumption of having direct sum.*

Indeed, in fact, if T^2X is closed and suppose $Ta_n + b_n \to c$, where $b_n \epsilon ker\ T$, then $T^2a_n \to Tc$, so from hypothesis there exists $x \epsilon X$ such that $Tc = T^2x$. If we put $z = c - Tx \epsilon ker\ T$ we obtain $c = Tx + z \epsilon TX + ker\ T$. Now the proof is immediate following **Theorem 3.1.** These leads to the following result.

Theorem 3.3. *Let T* $\in$ *C(**X**) with the property ker T* $\cap$ *TX = {0}. Then the following assertions are equivalent:*

(1) *TX + ker T is closed;*
(2) T^nX *is closed for all* $n \epsilon N$;
(3) T^2X *is closed;*
(4) *The mapping* $T^\Diamond : X/\ ker\ T \to X/\ ker\ T$ *has closed range.*

Proof. Using the **Theorem 3.1** it remains to prove only the equivalence (1) ↔ (4). Let $\pi : X \to X/\ ker\ T$ be the canonical surjection. Then

$T^{\lozenge}\ (\boldsymbol{X}/\ ker\ T) = \pi\ (T\boldsymbol{X} + ker\ T)$

and thus we have

$\pi^{-1}(T^{\lozenge}\ (\boldsymbol{X}/\ ker\ T)) = T\boldsymbol{X} + ker\ T$. So we have obtained $T^{\lozenge}\ (\boldsymbol{X}/\ ker\ T)$ is closed if and only if $T\boldsymbol{X} + ker\ T$ is closed. □

4. CONCLUSIONS

Most of the computing processes that occur in the medical field, as in many other fields, require the existence and efficient implementation of parallel processing.

Modern imaging methods: classical radiography, CT scan, MRI, ultrasounds have the shortcoming of two dimensions limitation, often providing insufficient information in complex deformation cases. Modern tomography offers three-dimensional models but these are static, do not permit operations on them, can be analyzed only from standard angles, do not emphasize several tissues simultaneously, do not permit virtual dissection and especially do not permit their further use in specialized software, on usual computers.

Appealing to computer modeling and 3D graphics based on multipliers technique (described in this article) as main algorithmic tool, this domain of information technology which is practically the core of modern CAD/CAM software engineering improves the medical act of facial reconstruction in all its aspects: preoperative investigation, study model in surgery preparation, modeling of osteosynthesis and bone replacement implants, analysis method of the postoperative results.

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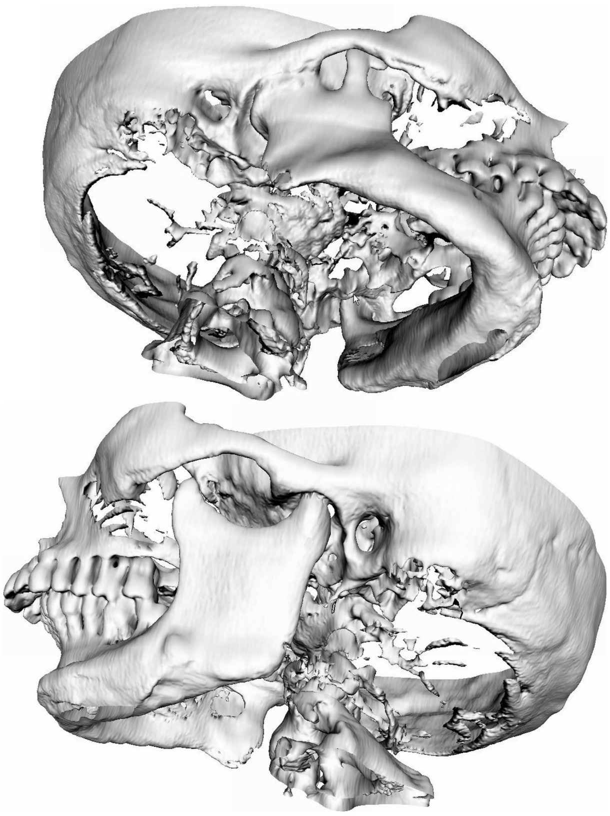
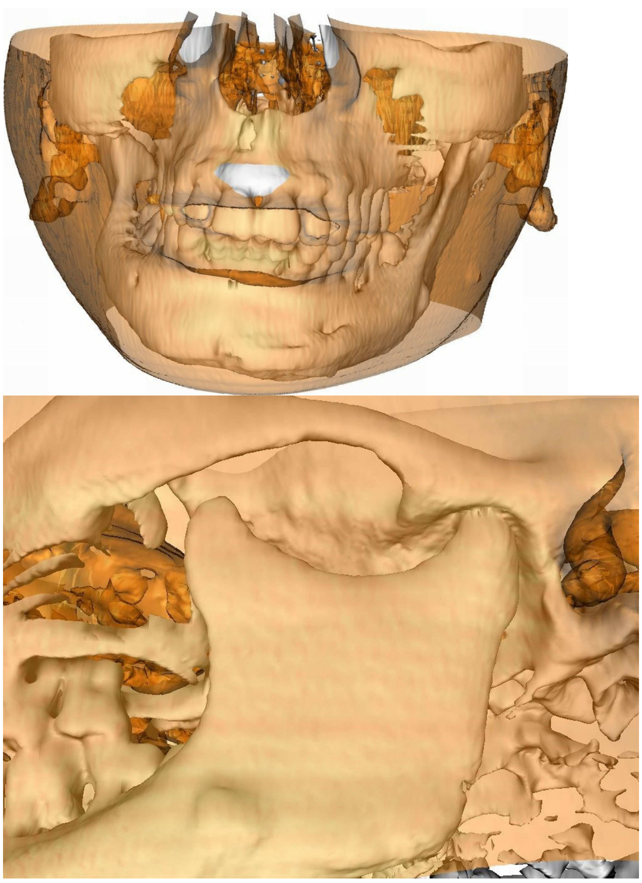


Fig. 1. – temporomandibular joint modeling-right joint ankilosis



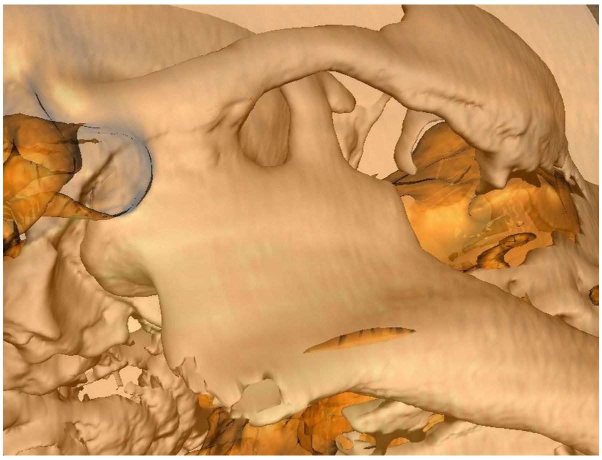


Fig. 2. – Temporomandibular joint and adjacent tissues modeling

References

[1] Allan, G.R., *A spectral theory for locally convex algebras*, Proc. London Math. Soc. (3) **15** (1965), 399-421

[2] Bonales, F., G., Mendoza, R. V., *A formula to calculate the spectral radius of a compact linear operator*, Internat. J. Math. Sci., **20**, (3), (1997), 585-588

[3] Bonales, F., G., Mendoza, R. V.,, *Extending the formula to calculate the spectral radius of an operator*, Proc. Amer.Math. Soc., **126**, (1), (1998), 97-103

[4] Chilana, A., *Invariant subspaces for linear operators on locally convex spaces*, J. London. Math. Soc **2** (1970), 493-503

[5] Dulst, Van D., *Barreledness of subspaces of countable co-dimension and the closed graph theorem*, Compositio Mathematica **tom 24** , (2), (1972), 227-234

[6] Dunford, N., *Spectral theory I. Convergence to projection*, Trans. Amer. Math. Soc. **54** (1943), 185-217

[7] Dunford, N., Schwartz, J.T., *Linear Operators I: General theory, Pure and Appl. Math.*, vol. 7, Interscience Publishers, Inc., New-York 1958

[8] Giles, J.R., Joseph, G.A., Koehler, D.O. and Sims, B., *On numerical ranges of operators on locally*

[9] Heuser, H. *Functional Analysis.* Wiley, New York, 1982.

[10] Husain, T., *The open mapping and closed graph theorems in toplogical vector spaces*, Oxford Mathematical Monographs, (1965*).*

[11] Husain, T., *Multipliers of topological algebras,* Dissertationes Mathematicae CCLXXXV, (1989), 1 - 36

[12] Joseph, G.A., *Boundedness and completeness in locally convex spaces and algebras*, J. Austral. Math. Soc. **24** (Series A) (1977)

[13] Kramar, E., *Invariant subspaces for some operators on locally convex spaces*, Comment. Math. Univ. Carolinae 38, **3** (1997), 635-644.

[14] Karol, J., Sliwa, W., Wojtowicz, M., *Cs-barreled spaces*, Collect. Math. 45, **3** (1994), 271-276.

[15] Larsen, R., *An introduction to the theory of multipliers*, Springer, Berlin, 1971.

[16] Laursen, K. B., Mbekhta, M., *Closed range multipliers and generalized inverses*, Studia Math. 107 **2,** (1993), 127-135.

[17] Lin, M., *On the uniform ergodic theorem*, Proc. Amer. Math. Soc. **43** (2) (1974), 337-340

[18] Michael, A., *Locally multiplicatively convex topological algebras*, Mem. Amer. Math. Soc., 11, 1952.

[19] Moore, R.T., *Banach algebras of operators on locally convex spaces*, Bull. Am. Math. Soc. **75** (1969), 69-73.

[20] Pater, F., Binzar, T., *On some ergodic theorems for a universally bounded operator*, Carpathian Journal of Mathematics, (26) (2010), 1, 97-102

[21] Perez-Carreras, P., Bonet, J., *Barrelled locally convex spaces*, Mem. Amer. Math. Soc., **11,** 1952.

[22] Ptak, V., *Completeness and the open mapping theorem*, Bulletin de la S.M.F., tome **86** (1958), 41-74.

Proprietatile multiplicatorilor in spatial algebric cu aplicatie in prelucrarea semnalelor

Rezumat: *În cele mai multe cazuri, metodele folosite în prelucrarea semnalelor nu pot fi circumscrise unor modele deja existente, ci, este necesară aplicarea unor extensii ale teoriilor matematice (teroia operatorilor, analiză funtională) pentru a generaliza fie spatiile fie modelele, folosind functii mai generale dar cu mai putine proprietăti satisfăcând teoreme folosite în modele similare.*

În acest context, spatiul Hilbert L_2 *poate fi extins la un spatiu local convex secvential complet, extensie necesară în termeni de tipuri de date care urmează a fi prelucrate. Astfel, este nevoie să se obtină teoreme ergodice de medie care pot caracteriza operatorii cu anumite tipuri de limitări. În 1943, Dunford a obtinut teoreme ergodice folosind calculul analitic al functiilor operationale asociate unui operator care actionează pe un spatiu Banach complex. În 1974, M. Lin a adăugat o conditie echivalentă teoremei uniform ergodice. În această lucrare, se studiază conditii echivalente pentru T (T fiind un multiplicator pe o algebră local convexă A cu range închis).*

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