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NUMERICAL INTEGRATION AND STABILITY PROBLEMS IN THE STUDY OF LORENZ SYSTEM

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Abstract: *The differential equations systems with chaotic behavior have been studied in the last twenty years due to their deep applications in many engineering-oriented applied fields, such as nonlinear circuits, synchronization, lasers or secure communications. The first canonical chaotic attractor was found by Lorenz in 1963 when he studied the atmospheric convection phenomenon in three-dimensional. The goal of our paper is to study this system from the mechanical geometry point of view: nonlinear stability problems via energy-Casimir method, periodical orbits, numerical integration via Poisson integrator and numerical simulation.*

Key words: *Lorenz system, Hamilton-Poisson realization, nonlinear stability, Lie-Trotter integrator, Kahan integrator.*

1. INTRODUCTION

The Lorenz system is described by the following equations:

$$\begin{cases} \dot{x} = a(y - x) \\ \dot{y} = c\,x - y - x\,z \\ \dot{z} = -b\,z + x\,y \end{cases} \qquad (1)$$

where a, b, c are positive real parameters. In 1993 H. Gumral and Y. Nutku have proved (see [4] for details) that for some values of its parameters, Lorenz system admits a Hamilton-Poisson realization. More exactly, if

$$a = \frac{1}{2}, b = 0, c = 1$$

the system admits

$$R = \frac{y^2 + z^2}{(x^2 - z)^2} \qquad (2)$$

as a conserved quantity.

In order to exhibit the bi-Hamiltonian structure of the equations (1) we need two time independent constants of motion. But extensive searches have failed to yield such a conserved quantity. The clue to the bi-Hamiltonian structure of the system (1) lies in the fact that the system admits two time-dependent conserved quantities

$$H_1 = e^t(x^2 - z)$$

and

$$H_2 = e^{2t}(y^2 + z^2)$$

and the time-independent expression (2) results from the elimination of t from the above equations.

To find a bi-Hamilton-Poisson structure for the system (1), H. Gumral and Y. Nutku have made a transformation of the dynamical variables and time given by:

$$x = \frac{1}{2}\tau u$$
$$y = \frac{1}{4}\tau^2 v$$
$$z = \frac{1}{4}\tau^2 w$$

and $t = -\log\left(\frac{\tau^2}{4}\right)$.

Now, the system (1) takes the form:

$$\begin{cases} \dot{u} = \frac{1}{2}v \\ \dot{v} = -u\,w \\ \dot{w} = u\,v \end{cases} \qquad (3)$$

The new conserved quantities

$$H_1 = w - u^2 \tag{4}$$

and

$$H_2 = v^2 + w^2 \tag{5}$$

are indeed time-independent conserved quantities.
Using now (4) and (5) one can be proved (see [4] for details):

Proposition 1.1. The system (3) has the following Hamilton-Poisson realizations: $(R^3, \Pi_i, H_i), i = 1,2$, where:

$$\Pi_1 = \begin{pmatrix} 0 & -\frac{1}{2}w & \frac{1}{2}v \\ \frac{1}{2}w & 0 & 0 \\ -\frac{1}{2}v & 0 & 0 \end{pmatrix},$$

is the first Poisson structure with the Hamiltonian $H_1 = w - u^2$, and

$$\Pi_2 = \begin{pmatrix} 0 & \frac{1}{4} & 0 \\ -\frac{1}{4} & 0 & -\frac{1}{2}u \\ 0 & \frac{1}{2}u & 0 \end{pmatrix},$$

is the second Poisson structure with the Hamiltonian $H_2 = v^2 + w^2$.
The Casimir of the first configuration is the function

$$C_1 = v^2 + w^2$$

and for the second one:

$$C_2 = w - u^2.$$

The goal of this paper is to study the geometrical and dynamical properties of the system (3) using the Hamilton-Poisson realizations given by the Proposition 1.1.
For details on Poisson geometry and Hamiltonian dynamics, see, e.g. [8], [9].

2. ALTERNATIVE HAMILTON-POISSON STRUCTURES

As the first consequence of the Proposition 1.1 we obtain:
Remark 2.1 The phase curves of the dynamics (3) are the intersections of the surfaces:

$$H_1 = const.$$

and

$$C_1 = const.$$

see the Figure 2.1.

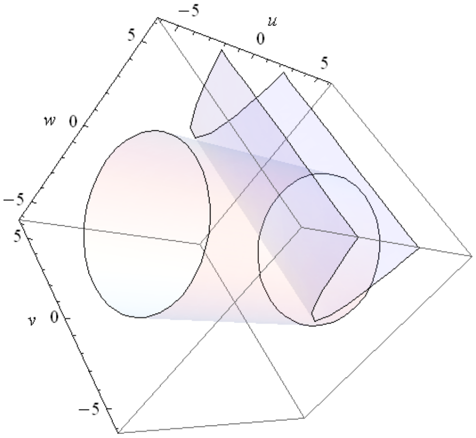

Figure 2.1. The phase curves of the system (3)

Proposition 2.2. The system (3) may be modeled as a Hamilton-Poisson system in an infinite number of different ways, i.e. there exist infinite many different (in general non isomorphic) Poisson structures on R^3 so that the system (3) is induced by an appropriate Hamiltonian.
Proof: The triples $(R^3, \{\cdot,\cdot\}_{a,b}, H_{c,d})$, where:

$$\{f, g\}_{a,b} = -\nabla C_{a,b} \cdot (\nabla f \times \nabla g), \forall f, g \in C^\infty(\mathbb{R}^3, \mathbb{R}),$$
$$C_{a,b} = aC + bH, H_{c,d} = cC + dH,$$
$$a, b, c, d \in R, ad - bc = 1,$$
$$H = \frac{1}{2}(w - u^2), C = \frac{1}{2}(v^2 + w^2)$$

define Hamilton-Poisson realizations of the dynamics (3).
Indeed, we have:

$$\{u, H_{c,d}\}_{a,b} = -\begin{vmatrix} -bu & av & aw + \frac{b}{2} \\ 1 & 0 & 0 \\ -du & cv & cw + \frac{d}{2} \end{vmatrix}$$

$$= \frac{1}{2}v$$

$$= \dot{u};$$

$$\{v, H_{c,d}\}_{a,b} = -\begin{vmatrix} -bu & av & aw + \frac{b}{2} \\ 0 & 1 & 0 \\ -du & cv & cw + \frac{d}{2} \end{vmatrix}$$

$$= -(ad - bc)uw$$

$$= \dot{v};$$

$$\{w, H_{cd}\}_{a,b} = -\begin{vmatrix} -bu & av & aw + \frac{h}{2} \\ 0 & 0 & 1 \\ -du & cv & cw + \frac{d}{2} \end{vmatrix}$$

$$= (ad - bc)uv$$

$$= \dot{w}.$$

3. STABILITY PROBLEMS

Let us continue now with a discussion concerning the nonlinear stability of the equilibrium states of the system (3).
It is obvious to see that the equilibrium points of the dynamics (3) are given by:

$$e_1^M = (M, 0, 0), \quad M \in R;$$
$$e_2^M = (0, M, 0), \quad M \in R.$$

Let A be the matrix of the linear part of the system (3), i.e.

$$A = \begin{pmatrix} 0 & \frac{1}{2} & 0 \\ -w & 0 & -u \\ v & u & 0 \end{pmatrix}.$$

Then the characteristic roots of $A(e_1^M)$ are given by:
$\lambda_1 = 0, \lambda_{2,3} = \pm iM$

so we can conclude that:

Proposition 3.1 The equilibrium states e_1^M are spectrally stable, for any $M \in R$.
The characteristic roots of $A(e_2^M)$ are given by:

$$\lambda_1 = 0, \lambda_{2,3} = \pm i\sqrt{\frac{M}{2}},$$

so we can conclude that:

Proposition 3.2 The equilibrium states e_2^M are spectrally stable, for any $M \in R$.

We can now proceed to discuss the Lyapunov stability of the equilibrium states e_1^M.

Proposition 3.3 The equilibrium states e_1^M are Lyapunov stable, for any $M \in R$.

Proof: We shall make the proof using the energy-Casimir method (see [1] and [5]). Let:

$$H_\varphi(u, v, w) = H_2 + \varphi(C_2)$$
$$= v^2 + w^2 + \varphi(w - u^2)$$

be the energy-Casimir function, where $\varphi: R \to R$ is a smooth real valued function defined on *R*. Now, the first variation of H_φ is given by:

$$\delta H(u, v, w) = 2v\delta v + 2w\delta w + \varphi'(w - u^2)$$
$$\cdot (\delta w - 2u\delta u)$$

At the equilibrium of interest:

$$\delta H(M, 0, 0) = \varphi'(-M^2) \cdot (\delta w - 2M\delta u).$$

This equals zero if and only if:

$$\varphi'(-M^2) = 0, \qquad (5)$$

The second variation of H_φ is given by:

$$\delta^2 H_\varphi(u, v, w) = 2(\delta v)^2 + 2(\delta w)^2 + \varphi''(w - u^2) \cdot (\delta w - 2u\delta u)^2.$$

At the equilibrium of interest we have, using (5), the second variation becomes:

$$\delta^2 H_\varphi(M, 0, 0) = 2(\delta v)^2 + 2(\delta w)^2 + \varphi''(-M^2) \cdot (\delta w - 2M\delta u)^2$$
$$- 4M^2\varphi''(-M^2)(\delta u)^2 + 2(\delta v)^2 + [2 + \varphi''(-M^2)](\delta w)^2 - 4M\varphi''(-$$

If we choose now the function φ such that:

$$\varphi'(-M^2) = 0, \quad \varphi''(-M^2) > 0$$

we can conclude that the second variation of H_φ at the equilibrium of interest is positively defined and thus e_1^M is Lyapunov stable.

Proposition 3.4 If $M > 0$ then the equilibrium states e_2^M are Lyapunov stable.
Proof: We shall make the proof using the energy-Casimir method (see [1] and [5]) for details). Let:

$$H_\varphi(u, v, w) = H_2 + \varphi(C_2)$$
$$= v^2 + w^2 + \varphi(w - u^2)$$

be the energy-Casimir function, where $\varphi: R \to R$ is a smooth real valued function defined on *R*. Now, the first variation of H_φ at the equilibrium of interest is given by:
$$\delta H(0, 0, M) = (2M + \varphi'(M)) \cdot (\delta w)$$

This equals zero if and only if

$$\phi(M) = -2M. \qquad (6)$$

Using (6), the second variation of H_φ at the equilibrium of interest is given by:

$$\delta^2 H_\varphi(0,0,M) =$$
$$= 4M(\delta u)^2 + 2(\delta v)^2 + [2 + \phi(M)](\delta w)^2.$$

If we choose now the function φ such that:

$$\phi(M) = 0,\ \dot{\phi}(M) > 0$$

we can conclude that the second variation of H_φ at the equilibrium of interest is positively defined if $M > 0$ and thus e_2^M is Lyapunov stable.

4. PERIODICAL ORBITS

In this section we are making use of the property that the dynamics described by a Hamilton–Poisson system takes place on the symplectic leaves of the Poisson configuration manifold, to prove the existence of periodic orbits by looking for periodic orbits of the symplectic Hamiltonian completely integrable system obtained by the restriction of the Lorenz system to the regular coadjoint orbits of $(R^3)^*$. This procedure will be implemented around nonlinearly stable equilibrium states.

The procedure is the following: we consider the system restricted to a regular coadjoint orbit of $(R^3)^*$ that contains a nonlinearly stable equilibrium and then we will get the existence of periodic solutions for the restricted system. These periodic solutions are periodic solutions also for the unrestricted system.

It is clear that the reduction of the system (3) to the generic coadjoint orbit:

$$v^2 + w^2 = M^2$$

gives rise to a classical Hamiltonian system.

Then, we have:

Proposition 4.1 Near to e_2^M the reduced dynamics has for each sufficiently small value of the reduced energy at least 1-periodic solution whose period is close to $\frac{2\pi\sqrt{2}}{\sqrt{M}}$.

Proof: Indeed, we have successively:

(i) The matrix of the linear part of the reduced dynamics has purely imaginary roots. More exactly:

$$\lambda_1 = 0,\ \lambda_{2,3} = \pm i\sqrt{\frac{M}{2}}.$$

(ii) $span(\nabla C_1) = V_0 = span\begin{bmatrix}0\\0\\1\end{bmatrix}$,

where

$$V_0 = Ker(A(e_2^M)).$$

(iii) The smooth function

$$F_\lambda(u,v,w) = H_1 + \lambda H_2 = v^2 + w^2 + \lambda(w - u^2)$$

has the following properties:

- it is a constant of motion for the dynamics (3);
- $\nabla F_\lambda(e_2^M) = 0$ for $\lambda = -2M$;
- $\nabla^2 F_\lambda(e_2^M)\big/_{W \times W} > 0$

where:

$$W = Ker\, dC_2 = \left\{\begin{bmatrix}a\\b\\0\end{bmatrix},\ a,b \in R\right\}.$$

Then our assertion follows via the Moser-Weinstein theorem with zero eigenvalue, see [1] for details.

5. NUMERICAL INTEGRATION VIA LIE-TROTTER AND KAHAN INTEGRATORS

We shall discuss now the numerical integration of the dynamics (3) via the Lie–Trotter integrator and also via the Kahan integrator.

Let us observe that the Hamiltonian vector field X_H splits as follows:

$$X_H = X_H^1 + X_H^2$$

where

$$H_1 = v^2, H_2 = w^2.$$

Their corresponding integral curves are respectively given by:

$(u(t)\quad v(t)\quad w(t))^T = A_i(u(0)\quad v(0)\quad w(0))^T$, $i = 1,2$, where:

$$A_1 = \begin{pmatrix}1 & \frac{t}{2} & 0\\ 0 & 1 & 0\\ at & \frac{at^2}{4} & 1\end{pmatrix}, \qquad a = v(0),$$

and

$$A_2 = \begin{pmatrix} 1 & 0 & 0 \\ -bt & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad b = w(0);$$

Then, the Lie-Trotter integrator is given by:

$$(u^{n+1} \quad v^{n+1} \quad w^{n+1})^T = A_1 A_2 (u^n \quad v^n \quad w^n)^T$$

or explicitly:

$$\begin{cases} u^{n+1} = \left(1 - \frac{bt^2}{2}\right) u^n + \frac{t}{2} v^n \\ v^{n+1} = -bt\ u^n + v^n \\ w^{n+1} = \left(at - \frac{abt^2}{4}\right) u^n + \frac{at^2}{4} v^n + w^n, \end{cases} \quad (7)$$

$a = v(0), b = w(0).$

Some of its properties are sketched in the following proposition:

Proposition 5.1. The numerical integrator (7) has the following properties:

(i) It is a Poisson integrator.

(ii) It preserves the Hamiltonians (4) and (5).

(iii) Its restrictions to the coadjoint orbits:

$$v^2 + w^2 = M^2$$

gives rise to a symplectic integrator.

Poof: The proof consists of some long but straightforward computations.

Let us observe now that Kahan's integrator (see [6]) associated to the dynamics (3) that has the following form:

$$\begin{cases} u^{n+1} - u^n = \frac{h}{4}(v^{n+1} + v^n) \\ v^{n+1} - v^n = -\frac{h}{8}(u^{n+1}w^n + u^n w^{n+1}) \\ w^{n+1} - w^n - \frac{h}{8}(u^{n+1}v^n + u^n v^{n+1}) \end{cases} \quad (8)$$

Some of its properties are sketched in the following proposition:

Proposition 5.2. The Kahan integrator does not preserve the minus Lie–Poisson structures and also does not preserve the Hamiltonians H_1 and H_2.

6. CONCLUSION

The Lorenz system belongs to electrical engineering fields and it was studied in the last

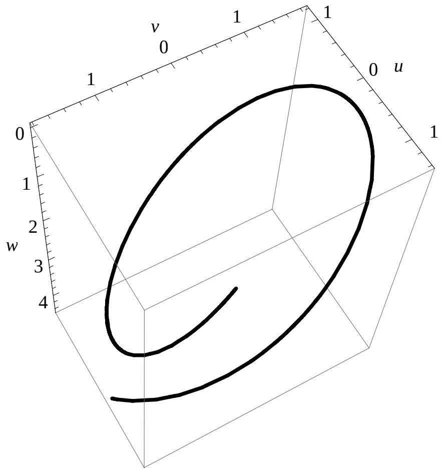

Figure 5.1. The Lie-Trotter integrator for $a = b = 1.$

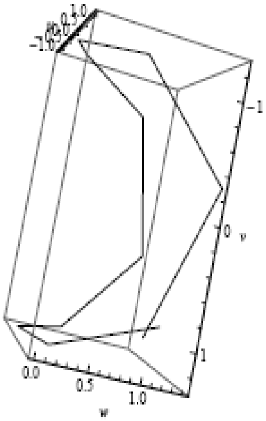

Figure 5.2. The Kahan's integrator for $u(0) = v(0) = w(0) = 1.$

twenty years from various points of view. In this paper we consider the system arisen from Lorenz system using a transformation of the dynamical variables and time; in this case the system admits Hamilton-Poisson realizations. This allows us to point out to some of its geometrical and dynamical properties using the specific tools of Hamilton-Poisson geometry, such as the stability of its equilibrium points (using energy-Casimir method), the existence of the periodic solutions (using Moser theorem) and numerical integration via Poisson and non-Poisson integrators. It is easy to see that Kahan's integrator (Fig. 5.2) gives us a good enough approximation of the solution being very close to the solution obtained via Runge-

Kutta 4th steps integrator (Fig. 5.3). This time, Lie-Trotter's integrator has failed despite its properties (Lie-Trotter's integrator is a Poisson one).

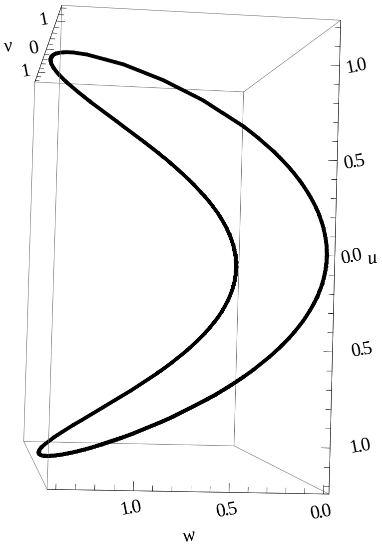

Figure 5.3. The Runge-Kutta 4th steps integrator for $u(0) = v(0) = w(0) = 1$.

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Integrare numerica si probleme de stabilitate in studiul sistemului lui Lorenz

Sistemele de cuatii diferentiale cu comportament haotic au fost studiate in ultimii douazeci de ani datorita aplicatiilor pe care aceste sisteme le au in multe domenii inginerești, cum ar fi circuite neliniare, sincronizare, lasere si securitatea comunicatiilor. Primul atractor haotic a fost descoperit de Lorenz in 1963 in cadrul studiului fenomenului de convective atmosferica 3-dimensionala. Obiectul lucrarii il constituie studiul sistemului lui Lorenz din punctual de vedere al mecanicii geometrice: stabilitate neliniara folosind metoda energie-Casimir, existenta orbitelor periodice, integrare numerica prin integratori Poisson si simulari numerice.

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