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THE E.D.M. PROCESS STUDIED USING THE CUBIC SPLINE OF REGRESSION TYPE STEFAN'S PROBLEM

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Abstract: *The abbreviation E.D.M. means "ELECTRICAL DISCHARGE MACHINING" and refers to the materials processing (micro-smelting) by electrical discharge of spark or flame type. The numerous applications of this process are in domain of the polishing, micro-penetration, micro-cutting operations. The de-crystallization in this case is made quickly and at very high temperature. This means that we deal with a STEFAN problem of jump type. The boundary condition of Neumann type refers to the thermal flux, traversing the processed surface. Therefore, at the crater's free boundary, produced by EDM, it requires that jump of the thermal flux to not transcend a given value. In most of cases is used as a distribution of the thermal field, the GAUSS distribution. Because of the high grade instability of the solution, representing the thermal field, the boundary condition must not contain rough approximations. The Gauss function becomes null only if the crater radius tends to infinity. But is well-known that the crater radius is very small, therefore the flux approximation by the Gauss function can't be null. We eliminate this drawback replacing the Gauss function with a cardinal cubic Spline of regression, having the Gauss function's carriage and which becomes null with its derivative at the crater's boundary. Although the obtained mathematical model is a laborious one, its use not contains difficulties, considering the possibility to use some performance numerical software..*

Key words: *.D.M. process, flux of thermal field, Stefan problem, boundary value problem with free frontier, jump of a function, Spline function, regression function, Gauss function, Neumann type boundary condition.*

1. INTRODUCTION

The thermal field determination produced in EDM process, which generally is unsteady, is very important, because, first of all, permits to establish the shape of the crater, its dimensions, the eliminated material's volume by micro-smelting etc.

The scheme of EDM process is given in fig. 2.1.

If processed material is homogeneous and isotropic we denote:

$$a^2 = \frac{k_t}{\rho \cdot c_p} \tag{1.1}$$

the thermal diffusion of the material $a=a[m^2/s]$, where:

K_t – represents the thermal conductivity of the material, $K_t = K_t$ [J/mKs]

ρ - material density, $\rho = \rho$ [kg/m^3]

C_p - specific heat $C_p = C_p$ [J/kgK]

T – thermal field, $T : [0, +\infty) \times \mathcal{R}^3 \to \mathcal{R}$, $T = T(t, x, y, z)$, where $t = t[s]$ represents the time and (x, y, z) are the special coordinates of the material point. In this hypothesis, the Cartesian equation of Fourier – Kirchhoff is given by:

$$\frac{\partial T}{\partial t} = a^2 \Delta T \tag{1.2}$$

where ΔT is the thermal field's Laplace:

$$\Delta T = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} \tag{1.3}$$

The determination of the general solution for (2.2) will be done transforming this equation in cylindrical coordinates.

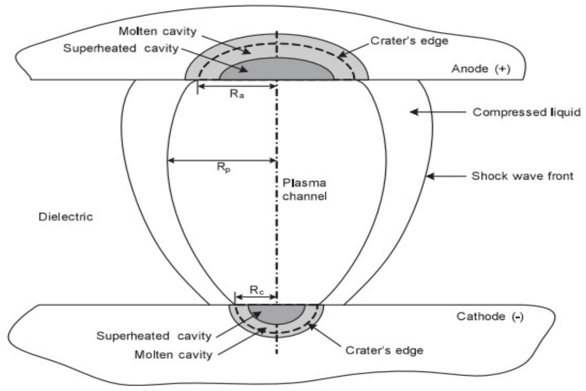


Fig. 1.1. Schematic diagram of micro-EDM process [6]

2. FOURIER–KIRCHHOFF'S EQUATION IN CYLINDRICAL COORDINATES

Using the hypothesis formulated over the processed material and the cylindrical coordinates:

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta, \quad \theta\in[0,2\pi, \text{ with} \\ z = z, \end{cases} \qquad (2.1)$$

$$r > 0, \; z \in \mathbb{R}$$

the equation (1.2) becomes:

$$\frac{\partial T}{\partial t} = a^2\left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2}\right) \qquad (2.2)$$

The equation (3.2) is named Fourier-Kirchhoff's equation in cylindrical coordinates. Have to mention that the composed function $T = T(t, r, z)$ not depends on θ, because of the isotropy even if:

$$T(t, r, z) = T(t, r\cdot\cos\theta, r\cdot\sin\theta, z) \qquad (2.3)$$

This transformation of (1.2) into (2.2) is necessary because in (2.2) we can apply the method of separation of the variables, obtaining the general solution for Fourier – Kirchhoff equation, in both of cylindrical and Cartesian coordinates. In the specialty literature, as in [9], this operation is known, results that we assume as such, because the interest of this paper is to propose the spline function as the distribution of the thermal flux.

Therefore,

$$T(t,r,z)=R_0(L,M)\cdot Z_0(M)\cdot T_0(L)\cdot$$
$$\cdot J_0(r\cdot\sqrt{L^2+M^2})\cdot e^{-M\cdot z-a^2\cdot L^2\cdot t} \qquad (2.4)$$

where $L, M > 0$ are constant of integration as well as R_0, Z_0, T_0.

These integration constants are determined from the initial condition:

$$T(0, r, z) = T_0 \qquad (2.5)$$

where $r > 0,$, $z \in \mathbb{R}$ *and* T_0 is the thermal field value of the environment.

In the general solution given by (2.4) appears the Bessel's function, J_0 and its representation as a series is:

$$J_0(x) = \sum_{n\geq 0}\frac{(-1)^n}{(n!)^2}\cdot\left(\frac{x}{2}\right)^n \qquad (2.6)$$

respectively the integral representation of J_0 is given by the following formula:

$$J_0(x) = \frac{2}{\pi}\cdot\int_0^{\frac{\pi}{2}}\cos(x\cdot\sin\tau)d\tau \qquad (2.7)$$

In this paper we'll use the representation given in (2.6).

3. OPTIMAL CUBIC SPLINE OF REGRESSION

Let Δn be an arbitrary discretization of a closed bounded interval [a, b] and $I_k = [x_k, x_{k+1}]$, $k = 0 \div n$ a generic subinterval of a discretization Δn. Organize I_k as a geometrical finite element with five knots, x_k and x_{k+1} being double knots, respectively:

$$x_k' = \frac{x_k + x_{k+1}}{2} \text{ a simple knot.}$$

Making an offline transformation,

$$T: \quad t = \frac{x - x_k}{x_{k+1} - x_k}, \quad t \in [0, 1] \qquad (3.1)$$

the geometrical finite element I_k transform into a canonical finite element $\hat{I} = [0, 1]$, having the knots $t_0=0$ and $t_1=1$ double ones and t_α, simple knot, $t_\alpha \in (0, 1)$.

The finite element base $\{\varphi_k\}_{k=1}^{5}$ attached to the canonical element $\hat{I}$ is defined by the following condition:

$$\varphi_1(t_0) = 1\,, \quad \varphi_1(t_\alpha) = 0 \qquad (3.2)$$

$$\varphi_0'(t_0) = \varphi_1'(t_1) = 0 \qquad (3.3)$$

Analogously for the rest of the elements from the base, that is

$$\varphi_2'(t_0) = 1, \quad \varphi_2'(t_1) = 0 \text{ and} \qquad (3.4)$$

$$\varphi_2(t_0) = \varphi_2(t_\alpha) = \varphi_2(t_1) = 0 \qquad (3.5)$$

Then we put conditions for the simple knot t_α, that means:

$$\varphi_3(t_\alpha)=1, \varphi_3(t_0)=\varphi_3(t_1)=\varphi_2(t_1)=0 \qquad (3.6)$$

$$\varphi_3'(t_0) = \varphi_3'(t_1) = 0 \qquad (3.7)$$

The conditions for the double knot t_1 are as in the cases (4.2) and (4.3), that is:

$$\varphi_4(t_1) = 1, \quad \varphi_4(t_0) = \varphi_4(t_\alpha) = 0 \qquad (3.8)$$

$$\varphi_4'(t_0) = \varphi_4'(t_1) = 0 \qquad (3.9)$$

$$\varphi_5'(t_1) = 1, \quad \varphi_5'(t_0) = 1 \qquad (3.10)$$

$$\varphi_5(t_0) = \varphi_5(t_\alpha) = \varphi_5(t_1) = 0 \qquad (3.11)$$

In this paper we'll take $t_\alpha = \frac{1}{2}$.

Determine the base functions of cubic Spline type:

$$s_3(t) = \alpha_0 + \alpha_1 \cdot t_+ + \alpha_2 \cdot t_+^2 + \alpha_3 \cdot t_+^3 + \alpha_4 \cdot \left(t - \tfrac{1}{2}\right)_+^3 \qquad (3.12)$$

$t \in [0, 1] = \hat{I}$.

Using the conditions (3.2) – (3.11), determine every element from the base, and we have:

$$\begin{cases} \varphi_1(t) = 1 - 9t_+^2 + 10t_+^3 - 16\left(t - \frac{1}{2}\right)_+^3 \\ \varphi_2(t) = t_+ - \frac{7}{2}t_+^2 + 3t_+^3 - 4\left(t - \frac{1}{2}\right)_+^3 \\ \varphi_3(t) = 12t_+^2 - 16t_+^3 + 32\left(t - \frac{1}{2}\right)_+^3 \\ \varphi_4(t) = -3t_+^2 + 6t_+^3 - 16\left(t - \frac{1}{2}\right)_+^3 \\ \varphi_5(t) = \frac{1}{2}t_+^2 - t_+^3 + 4\left(t - \frac{1}{2}\right)_+^3 \end{cases} \qquad (3.13)$$

Here we used the truncated function:

$$(t - t_k)_+^n = \begin{cases} (t - t_k)^n\,, & t > t_k \\ 0\,, & t = t_k \\ 0\,, & t < t_k \end{cases} \qquad (3.14)$$

Define the vector space Σ_3 by: $\Sigma_3 =$ Span $\{\varphi_k\}_{k=1}^{5}$ and the projection operator: $\Pi_3 : \Sigma_3 \to \Sigma_3$.

$$\Pi_3(y)(x) = \sum_{k=1}^{5} y_k \cdot \varphi_k(x)\,,$$

$$y = (y_1, \ldots, y_5) \qquad (3.15)$$

We can easily verify that Π_3 is invariant for 1, x, x^2 and x^3.

Determine an element from Σ_3, which has to be a regression with respect to two empirical random variables.

$$X : \begin{pmatrix} x_1 & x_2 & \cdots & x_n \\ \frac{1}{n} & \frac{1}{n} & \cdots & \frac{1}{n} \end{pmatrix}, \quad Y : \begin{pmatrix} y_1 & y_2 & \cdots & y_n \\ \frac{1}{n} & \frac{1}{n} & \cdots & \frac{1}{n} \end{pmatrix}$$

Therefore, determines β_k, $k = 1 \div 5$, such that:

$$s(x) = \sum_{k=1}^{5} \beta_k \cdot \varphi_k(x)\,, \qquad (3.16)$$

Minimizes the function

$$L(\beta) = \sum_{k=1}^{n} [y_k - \sum_{j=1}^{5} \beta_j \cdot \varphi_j(x_k)\,]^2 \qquad (3.17)$$

The form (3.17) of the function $L(\beta)$, $\beta = (\beta_1, \dots, \beta_5)$ assures the existence and oneness of the minimum point $\beta^* = (\beta_1^*, \dots, \beta_5^*)$.

Therefore, β^* is the critical point for $L(\beta)$, that is, β^* is the solution of linear system:

$$\frac{\partial L}{\partial B_i} = 0 \; , i = 1 + 5 \qquad (3.18)$$

This system is equivalent with:

$$\sum_{k=1}^{n} [y_k - \sum_{j=1}^{5} \beta_j \cdot \varphi_j(x_k) \;] \cdot \varphi_i(x_k) = 0,$$

$$i = 1 + 5 \qquad (3.19)$$

In this equation, the Spline functions $\varphi_k(x)$, $k = 1 + 5$ are obtained from (3.13), determining t from (3.1), that is: $\varphi_k^{(j)}(x) = \varphi_k \left(\frac{x - x_j}{x_{j+1} - x_j}\right)$

$$\sum_{j=1}^{5} K_{ij} \beta_j = B_i \, , \quad i = 1 + 5 . \qquad (3.20)$$

with

$$B_i = \sum_{k=1}^{n} \varphi_i^{(k)}(x_k) y_k \qquad (3.21)$$

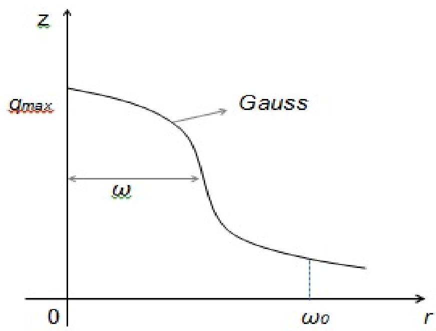
$$K_{ij} = \sum_{k=1}^{n} \varphi_i^{(k)}(x_k) \cdot \varphi_j^{(k)}(x_k) \qquad (3.22)$$

With β_j^*, $j = 1 + 5$ determined we obtain the optimal cubic Spline of regression, $s(x)$:

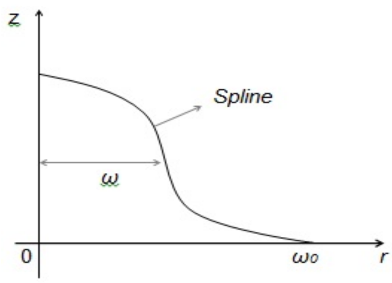
$$s(x) = \sum_{k=1}^{5} \beta_k^* \varphi_k(x) \qquad (3.23)$$

4. STEFAN'S PROBLEM OF JUMP TYPE FOR THE EDM PROCESS

Is known that in the EDM process obtains craters with very small diameters and in the micro-smelting process these craters enlarge their volume in time. If $S(t)$ denotes the crater's frontier, that is a surface separating the liquid and solid parts of the materials, then $S(t)$ modifies in time displacing from the liquid part to the solid one. The surface $S(t)$ is named "free frontier" or "variable frontier". Solving the Stefan problem it means to find out the free frontier and the solution of the Fourier – Kirchhoff equation's too. Moreover this solution must satisfy the initial, the classical boundary value and the thermal flux jump's conditions in the points of the free frontier.



a).



b).

Fig. 4.1 The geometrical image of thermal flux distribution for the Gauss a). and cubic Spline b). of regression models.

In the followings we present the three types of conditions over the general solution given by (3.4). These conditions allow to determine uniquely the integration constants from the general solution, that is the unique solution of Fourier – Kirchhoff's equation. The Stefan problems contain two categories:

(A). Stefan problems of continuous type, corresponding to crystallization (solidification) respectively de-crystallization (smelting) of materials, slowly and in long time. From mathematical point of view that means to require that the thermal flux from the free frontier to be a continuous function in its points.

(B). Jump type Stefan problems, which appear when the crystallization, respectively the smelting produces quickly, in very short time and at very high temperature, that is the thermal flux is discontinuous in the frontier's points, that supposes to admits a jump.

The EDM process is a (B) category process. From this reason the three conditions follow the formulation given in (B) case.

a) Initial condition

This condition shows the thermal field's, $T(t,x,y,z)$ behavior at the t = 0 moment. Therefore, we have to know a function $w(x, y, z)$ such that the restriction of T at $t = 0$ coincides with w, that is:

$$T(t,x,y,z)|_{t=0} = w(x,y,z) \quad (4.1)$$

Generally, w is the environment's temperature.

b) Boundary value condition

The boundary value condition for EDM process is of Neumann type. If q^* is the thermal flux, that is:

$$q^* = -\lambda < \nabla T, \vec{n} >_{\Re^3} \quad (4.2)$$

$q^* : [0,\infty) \times S^* \to \Re$, *where* S^* denotes the contact surface of the processed pieces with the thermal flux, specified in fig. 5.1.

Therefore, is given a continuous function $s : [0,\infty) \times S^* \to \Re$, such that the Neumann type boundary value condition is satisfied:

$$q^* = -\lambda < \nabla T, \vec{n} >_{\Re^3} = s, \quad (4.3)$$

where:

λ – is a given constant

∇T – is the thermal field's gradient

$\vec{n}$ - denotes the outward normal at S^*

$<,>_{\Re^3}$ - the scalar product in $\Re^3$

The theory of stability requires boundary value conditions, and these conditions have to be as accurate as possible relatively to the studied problem. For this reason we introduce spline functions.

In figure 4.1 a) and b) we represent these two Gauss and respectively spline functions and we can observe that for $r = \omega_0$ can appear perturbation phenomena, of high level in Gauss case and very small ones in Spline case.

In these figures appear ω and ω_0, representing the distribution radius of the thermal field, respectively its biggest radius. Therefore we have:

$$q^*|_{\omega_0} = 0 \quad (4.4)$$

c) Stefan's boundary value condition of jump type (unconventional)

As we already mentioned before, on the variable frontier $S(t)$, fig. 4.2, requires boundary value conditions, which are neither Dirichlet nor Neumann ones.

If $q^*(t,x,y,z)$, $(x,y,z) \in S(t)$ *and* $[q^*]$ denotes the jump of q^*, then the condition on $S(t)$ is:

$$[q^*](x,y,z) = \vartheta \quad (4.5)$$

where:

$\vartheta > 0$ is an empiric constant, depending on the type of micro-smelted material and the intensity of the electric power applied in EDM process.

In further applications following this new model we'll stand out the possibilities to specify the constants ω_0, λ and ϑ.

In the next figure, S^* denotes the contact surface of the flux q^* with the processed material, $S(t)$ is the free frontier, M_L represent the liquid material and M_L the solid material.

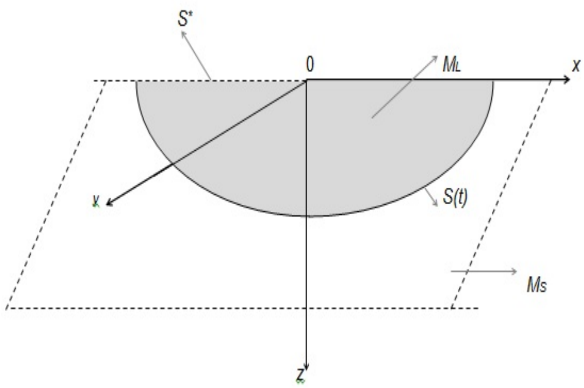


Fig. 4.2 The crater image with liquid and solid material

5. CONCLUSIONS

This paper constitutes base for further researches in domain of EDM. Taking account the calculus complexities, produced by numerical formulas and procedures it's obvious that they require performance software.

We refer here, first of all to the Bessel's function J_0, but to the numerical procedures used in determination of the integration constants from the general solution, using conditions (4.1), (4.3) and (4.5). This research we'll realize in further papers.

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Studierea procesului E.D.M. folosind curbele spline cubic a regresiei problemei de tip Stefan

Rezumat: Abrevierea EDM inseamna "Electrical Dischargte Machining" si se refera la indepartarea de material (micro topire) prin descarcari sub forma de scantei si plasma. Numeroasele aplicații ale acestui proces sunt in domeniul superfinisarii, micro gaurire, micro taieri. Decristalizarea in acest caz se face foarte rapid si la temperaturi foarte ridicate. Aceasta inseamna ca ne incadram in problema de tip Stefan cu salt.. De aceea la frontiera libera a craterului produs prin eroziune electrica, este necesar ca fluxul sa nu transceanda la o valoare data. In cele mai multe cazuri este utilizata pentru distributia fluxul termic, o curba Gauss. Din cauza gradului ridicat de instabilitate a solutiei reprezentand campul termic, conditiile la limita nu trebuie sa contina aproximari grosolane. Functia Gauss devine nula numai in cazul in care raza craterului tinde la infinit. Dar e bine stiut ca raza craterului este forte mi, de aceea aproximarea fluzului printr-o functie Gauss nu poate fi nula. Eliminam acest neajuns prin inlocuirea functiei Gauss cu o functie de regresie cubic ate tip Spline, avand ca si directoare functia Gauss care devine nula care e derivabila la limita craterului. Cu toate ca modelul matematic este laborios, nu contine dificultati, considerand posibilitatile unor soft-uri numerice performante.

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