



TECHNICAL UNIVERSITY OF CLUJ-NAPOCA

ACTA TECHNICA NAPOCENSIS

Series: Applied Mathematics and Mechanics

Vol. 54, Issue IV, 2011

## BENDING CALCULUS FOR CIRCULAR TAPPING PLATES WITH CIRCUMFERENTIAL EXTERIOR CONCENTRATED LOAD BY TRANSFER-MATRIX METHOD

Mihaela SUCIU

**Abstract:** The bending calculus for the circular tapping plates is very important for its applications in the mechanical, robotic, medical, military and aerospace industries. We present an application for this calculus: a circular tapping plate, embedded at the interior boundary and free at the exterior boundary, charged with a circumferential exterior concentrated load, using the Transfer-Matrix Method. That is an important possibility to result the circular plates, with the opportunity to program this algorithm and to obtain a simply code of this calculus. **Key words:** tapping plate, circumferential load, state vector, Transfer-Matrix, Dirac's function, Heaviside's function.

### 1. INTRODUCTION

The bending calculus for the circular tapping plates is very important for its applications in the mechanical, robotic, medical, military and aerospace industries. We present an application for this calculus: a circular tapping plate, embedded at the interior boundary and free at the exterior boundary, charged with a circumferential exterior concentrated load, using the Transfer-Matrix Method. This method is based on the theory of Dirac's and Heaviside's and the calculus with their functions and operators. After the analytical calculus we obtain the four elements of the exterior circumference state vector and for the interior circumference state vector of a tapping plate. Now, we can know for all the sections  $r_0 < r < R$ , all the elements for the all state vectors. That is an important possibility to result the circular plates, with the opportunity to program this algorithm and to obtain a simply code of this calculus.

For our studies, we define a transfer-matrix, that we should integrate in a general manner for the differential equation, witch gives the deformed of some plates loaded with an exterior density  $q(x,y)$ , written by means of the Dirac's and Heaviside's functions and

operators. After, we can express, for each case, the boundary conditions for the displacements calculus or for the stresses calculus in any plate points.

### 2. CALCULUS PREMISES

The bending plates theory has some bases: the curvature of the plate varies when the exterior load is perpendicular to the median plan; this curvature variation is made in two plans and they form one surface with double curvature, called elastic surface; the elastic surface form is characterized by the arrow variation low,  $w(x,y)$ , in Cartesian coordinates; we suppose that the numerical values of the function  $w(x,y)$  are very small compared with the plate thickness,  $h$ . The some simplifying work assumptions [1] are: we admit that the points aligned on the same normal line at the median surface before deformation remains aligned on the same normal line at the deformed surface after deformation; the normal stresses in the parallel sections at the median plan are negligible compared with the bending stresses; it not exist crush for the plate layers some more than others, besides that local crush under a concentrated load; the sign convention is as follows: the positive arrow is to up and the

positive angle is counterclockwise; when the arrow  $w$  decreases, the angle  $\omega$  is negative and  $\frac{dw}{dr}$  is negative too.

### 3. THE GENERAL CIRCULAR PLATE EQUATIONS

For the circular plates loaded with an exterior density  $q(x,y)$ , the general equations for the deformed of any plate are:

$$\Delta\Delta w(x, y) = \frac{1}{D} q(x, y) \tag{1}$$

$\Delta\Delta$  is the double Laplacian operator, with the developed expression:

$$\Delta\Delta = \frac{\partial^4}{\partial x^4} + \frac{2\partial^4}{\partial x^2\partial y^2} + \frac{\partial^4}{\partial y^4} \tag{2}$$

and:

$$D = \frac{Eh^3}{12(1-\nu^2)} \tag{3}$$

with  $E$  as longitudinal elasticity modulus,  $\nu$  is the Poisson's coefficient and  $h$  is the plate thickness.  $D$  is called also plate bending rigidity. The contour integration domain is essential for the integration of the equation (1). It is useless to determine a transfer-matrix for some plates, because the problem had not explicit solutions. The theoretical solution can be completed only for the field of the circular plates loaded with axially symmetric charges. We can be inferred the equations which allow to calculate the efforts and the deformations in all points of the circular plate, using the Transfer-Matrix Method. We note the arrow  $w(r)$  of the circular plate at the distance  $r$  from the axis and  $\omega(r)$ -the angle around which it spins the normal line (Fig. 1. and Fig. 2.), given by the relation (4):

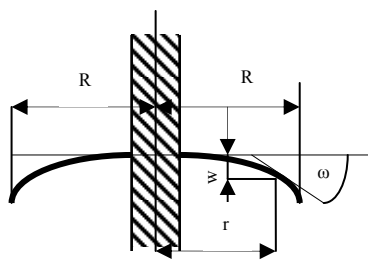


Fig. 1. Circular deformed plate

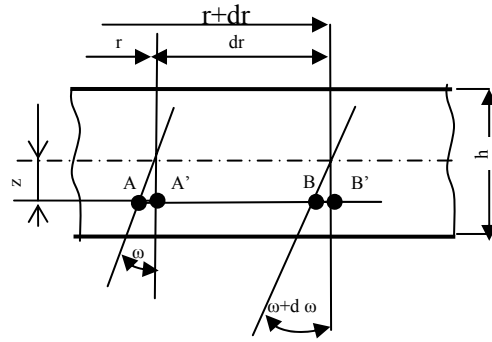


Fig. 2. Axial section of the plate

$$\omega = \frac{dw}{dr} \tag{4}$$

In the Fig. 2. shows one axial section of the plate and two normal sections at distance between them of  $r$ -before deformation and  $r+dr$ -after deformation. We have the relative elongation of segment  $AB$  at the rate  $z$  of the middle fiber:

$$A'B' - AB = z(\omega + d\omega) - z\omega = zd\omega \tag{5}$$

and the relative radial elongation is:

$$\varepsilon_r = z \frac{d\omega}{dr} \tag{6}$$

The point  $A'$  is on the circle of radius  $r+z\omega$ , and the tangential relative elongation is:

$$\varepsilon_t = \frac{2\pi \cdot z\omega}{2\pi \cdot r} = z \frac{\omega}{r} \tag{7}$$

The elasticity theory links the deformations with the radial and tangential stresses by the formulas (8) and (9), ([2] and [3]):

$$\begin{cases} \varepsilon_r = \frac{1}{E} (\sigma_r - \nu\sigma_t) \\ \varepsilon_t = \frac{1}{E} (\sigma_t - \nu\sigma_r) \end{cases} \tag{8}$$

$$\begin{cases} \sigma_r = \frac{E}{1-\nu^2} (\varepsilon_r + \nu\varepsilon_t) \\ \sigma_t = \frac{E}{1-\nu^2} (\varepsilon_t + \nu\varepsilon_r) \end{cases} \tag{9}$$

With (6) and (7), the expression (9) becomes:

$$\begin{cases} \sigma_r = \frac{Ez}{1-\nu^2} \left( \frac{d\omega}{dr} + \nu \frac{\omega}{r} \right) \\ \sigma_t = \frac{Ez}{1-\nu^2} \left( \frac{\omega}{r} + \nu \frac{d\omega}{dr} \right) \end{cases} \quad (10)$$

For a sectoral prism element on the plate [1], we write the balance equations. The moments are applied on the element faces. They are taken per unit length after one radial axis.  $M_r$  is the radial moment and  $M_t$  is the tangential moment. If we known the stresses  $\sigma_r$  and  $\sigma_t$ , is easy to calculate the resultant of these moments on the faces:

$$\begin{cases} M_r r d\varphi = r d\varphi \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_r z dz = \frac{Eh^3}{12(1-\nu^2)} \left( \frac{d\omega}{dr} + \nu \frac{\omega}{r} \right) r d\varphi \\ M_t dr = dr \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_t z dz = \frac{Eh^3}{12(1-\nu^2)} \left( \frac{\omega}{r} + \nu \frac{d\omega}{dr} \right) dr \end{cases} \quad (11)$$

or, with the notation (3):

$$\begin{cases} M_r = D \left( \frac{d\omega}{dr} + \nu \frac{\omega}{r} \right) \\ M_t = D \left( \frac{\omega}{r} + \nu \frac{d\omega}{dr} \right) \end{cases} \quad (12)$$

We note:

$$M_{res} = \frac{1}{D(1+\nu)} (M_r + M_t) \quad (13)$$

and we obtain:

$$M_{res} = \frac{\omega}{r} + \frac{d\omega}{dr} = \frac{1}{r} \frac{d\omega}{dr} + \frac{d^2\omega}{dr^2} = \frac{1}{r} \frac{d}{dr} (r\omega) \quad (14)$$

The cut force  $T$  is per unit of length on a circumference of radius  $r$ .  $M_{res}$  is not a bending moment, but is proportional at a bending moment. The balance equation after the vertical axis for the radial prism element is:

$$(T + dT)(r + dr)d\varphi - T r d\varphi + q(r) \cdot r dr \cdot d\varphi = 0 \quad (15)$$

where:

$$\frac{d}{dr} (rT) = -q(r) \cdot r \quad (16)$$

With writing the sum of the moments in report with the tangential axis of the arc circle of radius  $r$ , at the level of median plane, the balance equation for the element is:

$$\begin{aligned} (M_r + dM_r)(r + dr)d\varphi - M_r r d\varphi + q(r) \cdot r dr \cdot d\varphi - \frac{dr}{2} \\ - M_t dr \cdot d\varphi + (T + dT)(r + dr) dr \cdot d\varphi = 0 \end{aligned} \quad (17)$$

Neglecting the small higher order term, we obtain:

$$M_t - \frac{d}{dr} (rM_r) = rT \quad (18)$$

and replacing  $M_r$  and  $M_t$  with (13):

$$\frac{1}{D} rT = \frac{\omega}{r} - r \frac{d^2\omega}{dr^2} - \frac{d\omega}{dr} = -r \frac{dM_{res}}{dr} \quad (19)$$

We note:

$$\tau = \frac{1}{D} rT \quad (20)$$

The differential base equations assess the deformations and the efforts in a circular plate loaded with an axially symmetrical force:

$$\begin{cases} \frac{d\tau}{dr} = -\frac{1}{D} r q(r) \\ \frac{dM_{res}}{dr} = -\frac{\tau}{r} \\ \frac{d(r\omega)}{dr} = r M_{res} \\ \frac{dw}{dr} = \omega \end{cases} \quad (21)$$

In the equations (21) are inserted four sizes [1]:  $w$  and  $\omega$ , which have a physic significance—they are the displacements,  $w$  is the arrow and  $\omega$  is the angle;  $M_{res}$  and  $\tau$ , which are:  $M_{res}$  is proportional at a bending moment and  $\tau$  is related at the cut force by the relation (20). After integration, the equation system (21) give four constants:  $w_R$ ,  $\omega_R$ ,  $M_{resR}$  and  $\tau_R$  and:

$$\begin{cases} \tau = \tau_R - q_1(r) \\ M_{res} = M_{resR} - \tau_R \log \frac{r}{R} + q_2(r) \end{cases} \quad (22)$$

with:

$$\begin{cases} q_1(r) = \int_R^r \frac{\rho}{D} q(\rho) d\rho \\ q_2(r) = \int_R^r \frac{1}{\rho} q_1(\rho) d\rho \end{cases} \quad (23)$$

We can write:

$$\frac{d(r\omega)}{dr} = M_{resR} r - \tau_R r \log \frac{r}{R} + r q_2(r) \tag{24}$$

After integration, we have:

$$\omega = \omega_R \frac{R}{r} + \frac{1}{2} M_{resR} \left( r - \frac{R^2}{r} \right) - \frac{1}{2} \tau_R \left( r \log \frac{r}{R} + \frac{R^2}{2r} - \frac{r}{2} \right) + \frac{1}{r} q_3(r) \tag{25}$$

and the arrow w is:

$$w = w_R + \omega_R R \log \frac{r}{R} + \frac{1}{2} M_{resR} \left( \frac{r^2 - R^2}{2} - R^2 \log \frac{r}{R} \right) - \frac{1}{2} \tau_R \left( \frac{r^2 + R^2}{2} \log \frac{r}{R} + \frac{r^2 - R^2}{2} \right) + q_4(r) \tag{26}$$

**4. THE CIRCULAR PLATE AND ITS TRANSFER-MATRIX**

At the radius r, we define for a circular plate, a state vector with four elements:

$$\{V(r)\}_r = \{M_{res}(r), \tau(r), w(r), \omega(r)\}^{-1} \tag{27}$$

The state vector at the radius r<sub>0</sub> is:

$$\{V(r_0)\}_0 = \{M_{res0}, \tau_0, w_0, \omega_0\}^{-1} \tag{28}$$

and at the radius R is:

$$\{V(R)\}_R = \{M_{resR}, \tau_R, w_R, \omega_R\}^{-1} \tag{29}$$

We can write for the radius r:

$$\{V(r)\}_r = [T]_r \{V(R)\}_R + \{V_e\}_r \tag{30}$$

with: [T]<sub>r</sub> is the Transfer-Matrix for the circular plate at the radius r:

$$[T]_r = \begin{bmatrix} 1 & -\log \frac{r}{R} & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \frac{1}{2} \left( \frac{r^2 - R^2}{2} - R^2 \log \frac{r}{R} \right) & -\frac{1}{4} \left[ (r^2 + R^2) \log \frac{r}{R} + (R^2 - r^2) \right] & 1 & R \log \frac{r}{R} \\ \frac{1}{2} \left( r - \frac{R^2}{r} \right) & -\frac{1}{2} \left( r \log \frac{r}{R} + \frac{R^2}{2r} - \frac{r}{2} \right) & 1 & \frac{R}{r} \end{bmatrix} \tag{31}$$

{V<sub>e</sub>}<sub>r</sub> is the vector for the exterior loads:

$$\{V_e\}_r = \left\{ q_2(r), -q_1(r), q_4(r), \frac{1}{r} q_3(r) \right\}^T \tag{32}$$

**5. APPLICATION: TAPPING CIRCULAR PLATE EMBEDDED AT THE INTERIOR**

**CIRCUMFERENCE AND FREE AT THE EXTERIOR CIRCUMFERENCE, WITH CIRCUMFERENTIAL CONCENTRATED EXTERIOR LOAD**

We consider a tapping circular plate, embedded at the interior circumference and free at the exterior circumference (interior charged with a circumferential concentrated exterior load (Fig. 3.)).

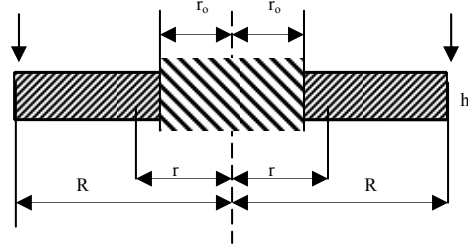


Fig. 3. A circular tapping plate embedded at the interior boundary, with a concentrated vertical exterior load.

After the calculus with Dirac's and Heaviside's functions and operators, for this charge, the density associated function is:

$$q(r) = -F \delta(r - R) \tag{33}$$

We calculate the functions q<sub>i</sub>(r), i=1,4, for the uniform load at the interior circumference:

$$q_1(r) = \frac{FR}{D} [1 - Y(r - R)] \tag{34}$$

$$q_2(r) = -\frac{FR}{D} [1 - Y(r - R)] \log \frac{R}{r} \tag{35}$$

$$q_3(r) = \frac{FR}{4D} [1 - Y(r - R)] \left[ R^2 - r^2 \left( 1 + \log \frac{R}{r} \right) \right] \tag{36}$$

$$q_4(r) = \frac{FR}{4D} [1 - Y(r - R)] \left[ R^2 - r^2 - (R_0^2 + r^2) \log \frac{R}{r} \right] \tag{37}$$

For r=r<sub>0</sub>, the matrix relation (33) becomes:

$$\{V(r_0)\}_0 = [T]_{r_0} \{V(R)\}_R + \{V_e\}_{r_0} \tag{38}$$

with:

$$[T]_{r_0} = \begin{bmatrix} 1 & -\log \frac{r_0}{R} & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \frac{1}{2} \left( \frac{r_0^2 - R^2}{2} - R^2 \log \frac{r_0}{R} \right) & -\frac{1}{4} \left[ (r_0^2 + R^2) \log \frac{r_0}{R} + (R^2 - r_0^2) \right] & 1 & R \log \frac{r_0}{R} \\ \frac{1}{2} \left( r_0 - \frac{R^2}{r_0} \right) & -\frac{1}{2} \left( r_0 \log \frac{r_0}{R} + \frac{R^2}{2r_0} - \frac{r_0}{2} \right) & 1 & \frac{R}{r_0} \end{bmatrix} \tag{39}$$

We put the boundary conditions in the relation (38):

- for the interior circumference embedded ( $r=r_0$ ):

$$w_{r_0}=0 \quad (40)$$

$$\omega_{r_0}=0 \quad (41)$$

- for the exterior free circumference ( $r=R$ ):

$$\tau_R=0 \quad (42)$$

and:

$$M_r(R)=0 \quad (43)$$

For the matrix equation (38) we obtain:

$$\begin{Bmatrix} M_{res_0} \\ \tau_0 \\ 0 \\ 0 \end{Bmatrix} = [T]_{r_0} \begin{Bmatrix} 0 \\ 0 \\ w_R \\ \omega_R \end{Bmatrix} + \begin{Bmatrix} 0 \\ -\frac{Fr_0}{D} \\ 0 \\ 0 \end{Bmatrix} \quad (44)$$

with  $[T]_{r_0}$  given by (39) and  $q_i$ ,  $i=1,4$  is given by (34), (35), (36) and (37). We have a linear equations system, with four unknowns. After solving the system we obtain the four solutions for  $w_R$ ,  $\omega_R$ ,  $M_{res_0}$  and  $\tau_0$ . Now, we can calculate in all sections of the tapping plate, between  $r=r_0$  and  $r=R$  the four elements for all the state vectors.

## 6. CONCLUSIONS

This work presents another method for an application of the tapping circular plate calculus, using the Transfer-Matrix Method, embedded at the interior circumference, free at the exterior circumference, loaded with a concentrated vertical charge at the free exterior boundary. The calculus with the Transfer-Matrix Method is very easy to program and so, that is very important for optimization calculus, function of different optimization criteria, with applications in the mechanical, robotic, medical, military and aerospace industries.

## 7. REFERRING

- [1] Gery, M., Calgaro, J.-A., *Les matrices-transfert dans le calcul des structures*, Editions Eyrolles, Paris, 1973.
- [2] Tripa, M., *Reziztenta materialelor*, Editura Didactica si Pedagogica, Bucuresti, 1967.
- [3] Warren, C. Y., *ROARK'S Formulas for Stress & Strain*, 6-th Edition, McGrawHill Book Company, 1989.

**Calculul la incovoiere al placilor circulare gaurite cu forta circumferentiala exterioara concentrata prin  
Metoda Matricelor de Transfer**

**Rezumat:** Calculul la incovoiere al placilor circulare gaurite este foarte important pentru aplicatiile sale in industria mecanica, robotica, medicala, militara si aerospatiale. Se prezinta o aplicatie a acestui calcul: o placa circulara gaurita, incastrata pe circumferinta interioara si libera pe circumferinta exterioara, incarcata cu o forta circumferentiala exterioara concentrata, utilizand Metoda Matricelor de Transfer. Aceasta este o importanta posibilitate de a rezolva calculul placilor circulare, cu oportunitatea de a programa acest algoritm si a obtine un cod simplu pentru acest calcul.

**SUCIU Mihaela**, PhD. Dr.-Eng., Professor, Technical University of Cluj-Napoca, Mechanical Engineering Department, Mechanical Faculty, E-mail: mihaelaica2007@yahoo.fr, Office Phone: 00.40.264.401.752., Home Address: Bd. 21 Decembrie 1989 nr. 23-35, App.14, 400105-Cluj-Napoca, Romania, Home Phone: 00.40.264.591.397., Mobile Phone: 00.40.723.878.112.