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NEW FORMULATIONS ON THE ACCELERATION ENERGY OF SECOND AND THIRD ORDER APPLIED IN ANALYTICAL MECHANICS

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Abstract: *This paper is devoted to the presentation of new formulations on the energies of higher order, that are used in the dynamic study of mechanical systems. Integral part of these mechanical systems is the mechanical robot structures, on which an application will be presented in order to highlight the importance on the energies of higher order regarding the dynamic behavior. In current dynamic studies, the kinetic energy is used as a central function in Lagrange - Euler equations. This paper extends the study by developing the acceleration energies of second and third order and their implementation, it give the possibility of applying the initial motion conditions: in positions, velocities and accelerations.*

Key words:*, robot, dynamics, energies of higher order, multibody systems*

1. INTRODUCTION

This paper will be structure in three main parts. The first one is focused on the acceleration energies of higher order. Known aspects, but also new ones are presented about them. Acceleration energies of second order and of third order are given in explicit form and matrix form. The second part of this paper proposes an application in which the theoretical aspects presented in this paper are used implementation of the motion for any robot's arm with 5R degrees (FANUC).

2. ACCELERATION ENERGIES

The main objective of this paper is oriented towards a few formulations, some of which are new, development by the first author, regarding the energies of higher order used in advanced dynamics of multibody mechanical systems (MBS). Therefore, the definition expressions for acceleration energies of first, second and third order will be presented, in *explicit* and *matrix* form, in which the kinematic parameters will be expressed using matrix exponential functions.

2.1 Acceleration energies of first order

According to the scientific literature, from which the following are mentioned [1], [2] and [4] the acceleration energy of first order is known as Appell's central function, and the presentation form of the differential equations of motion was developed for material point and for discrete system of material points.So, in according to [1] and [4] acceleration energy of first order was developed for a rigid body, and for multibody systems, which will be presented in this paper in a lapidary form.

This presentation is necessary to explain the next development acceleration, respective acceleration of second order.

The starting equation for defining the *acceleration energy of first order* is the following:

$$E_A^{(1)i} = \frac{1}{2}\cdot\int \dot{\bar{v}}_i^T \cdot \dot{\bar{v}}_i \cdot dm = \frac{1}{2}\cdot\int Trace\left(\ddot{\bar{r}}_i \cdot \ddot{\bar{r}}_i^T\right)\cdot dm \tag{1}$$

where $\ddot{\bar{r}}_i = \ddot{\bar{r}}_{C_i} + {}^0_i[\ddot{R}] \cdot {}^i\bar{r}_i^{\,*}$ (2)

and $\ddot{\bar{r}}_i^T = \ddot{\bar{r}}_{C_i}^T + {}^i\bar{r}_i^{\,*T} \cdot {}^0_i[\ddot{R}]^T$

The expression (1) presents the acceleration energy of first order $E_A^{(1)}$ reported of the elementary mass.

In expression (2), $\ddot{\bar{r}}_{C_i}$ represents the absolute acceleration of the mass center, $\ddot{\bar{r}}_i$ represents the absolute acceleration regarding system {i}, and ${}^{0}_{i}[\ddot{R}]$ represents the rotation matrix between the reference system $\{i\}$ and $\{0\}$, and finally their transposes.

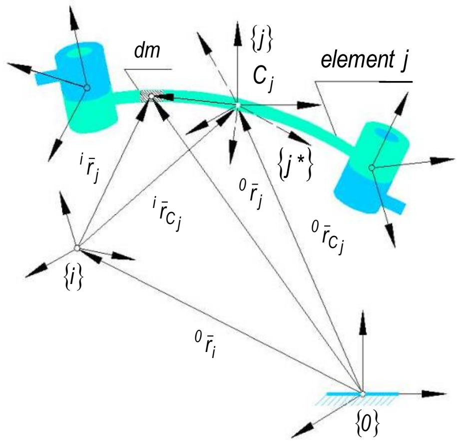

Fig. 1 A kinetic ensemble from MBS.

According to [2], the following two matrix proprieties can be developed:

$$ {}^{0}_{i}[\ddot{R}]\cdot {}^{0}_{i}[R]^T = (\dot{\bar{\omega}}_i\times)+(\bar{\omega}_i\times)\cdot(\bar{\omega}_i\times) \qquad (3)$$

$$ {}^{0}_{i}[R]\cdot {}^{0}_{i}[\ddot{R}]^T = (\dot{\bar{\omega}}_i\times)^T+(\bar{\omega}_i\times)^T\cdot(\bar{\omega}_i\times)^T \qquad (4)$$

After several matrix and differential transformations, the expression for defining the acceleration energy of first order is obtained in the matrix form in equation (5).

The study is extended to the MBS (Fig. 2) made out of n kinetic ensembles like the one in figure 1. It is considered that the MBS is characterized by n degrees of freedom (generalized coordinates) symbolized by: $\bar{\theta}(t)=(q_i(t),\ \dot{\bar{\theta}}(t)=(\dot{q}_i(t),\ \ddot{\bar{\theta}}(t)=(\ddot{q}_i(t),$ representing the generalized coordinates, velocities and accelerations and $\dddot{\bar{\theta}}(t)=(\dddot{q}_i(t),\ for\ i=1\rightarrow n)^T$ are acceleration of the second order of the system.

According to [1] and [3] the acceleration energy of first order for the MBS has the following matrix form:

$$\left\{\begin{array}{l} E_A^{(1)}\left[\bar{\theta}(t);\dot{\bar{\theta}}(t);\ddot{\bar{\theta}}(t)\right]= \\ =\frac{1}{2}\cdot\left\{\ddot{\bar{\theta}}^T(t)\cdot M\left[\bar{\theta}(t)\right]\cdot\ddot{\bar{\theta}}(t)+\ddot{\bar{\theta}}^T(t)\cdot V\left[\bar{\theta}(t);\dot{\bar{\theta}}^2(t)\right]+\right. \\ \left.\left[\dot{\bar{\theta}}^T(t)\cdot D\left[\bar{\theta}(t);\dot{\bar{\theta}}(t)\right]\cdot\ddot{\bar{\theta}}(t)\right]\right\} \end{array}\right\} \qquad (5)$$

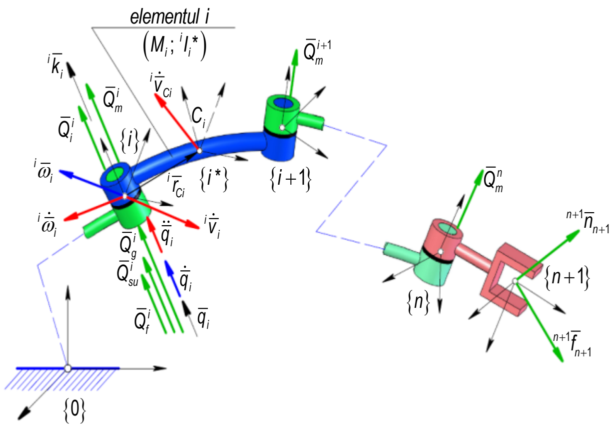

Fig. 2 Symbolical shape of a MBS.

where in the relation (5) the $M(\bar{\theta})$ is the inertial mass matrix, $V(\bar{\theta},\dot{\bar{\theta}},\ddot{\bar{\theta}})$ is the Coriolis matrix, and $D(\bar{\theta},\dot{\bar{\theta}},\ddot{\bar{\theta}})$ is pseudoinertial matrix correspond of the acceleration energy, and they are the follow relations:

$$M_{ij}=\sum_{k=\max(i;j)}^{n} Trace\left[A_{ki}\cdot {}^kI_{psk}\cdot A_{kj}^T\right]$$

$$M(\bar{\theta})=\underset{(nxn)}{Matrix}\left\{\begin{array}{ll} M_{ij}=M_{ji}= & \\ \quad =\sum_{k=1}^{n} Trace\left\{A_{ki}\cdot {}^kI_{psk}\cdot A_{kj}^T\right\} & \begin{array}{l} i=1\rightarrow n \\ j=1\rightarrow n\end{array}\end{array}\right\} \qquad (6)$$

$$\left\{V_{ijm}=V_{imj}\right\}=\sum_{k=\max(i;j;m)}^{n} Trace\left[A_{ki}\cdot {}^kI_{psk}\cdot A_{kjm}^T\right]$$

$$V\left(\bar{\theta};\dot{\bar{\theta}};\ddot{\bar{\theta}}\right)=\underset{(nxn)}{Matrix}\left\{V_i \quad i=1\rightarrow n\right\}$$

$$V_{i=}\left\{\ddot{\bar{\theta}}^T\cdot\left[\begin{array}{l}\left\{V_{ijm}=V_{imj}\right\}\ j=1\rightarrow n \\ \qquad\qquad m=1\rightarrow n\end{array}\right]\cdot\dot{\bar{\theta}}\right\}^T \qquad (7)$$

$$D_{ijlm}=\sum_{k=\max(i;j;l;m)}^{n} Trace\left[A_{kij}\cdot {}^kI_{psk}\cdot A_{klm}^T\right]$$

$$D\left(\bar{\theta};\dot{\bar{\theta}};\ddot{\bar{\theta}}\right)=\underset{(nxn)}{Matrix}\left\{D_{ij}\quad \begin{array}{l} i=1\rightarrow n \\ j=1\rightarrow n\end{array}\right\} \qquad (8)$$

$$D_{ij}=\ddot{\bar{\theta}}^T\cdot\left[\begin{array}{ll} D_{ijlm} & l=1\rightarrow n \\ & m=1\rightarrow n\end{array}\right]\cdot\dot{\bar{\theta}}$$

Equation (9) is known as the pseudo-inertial matrix of acceleration energy.

Their components illustrate on one hand the mass proprieties included in the pseudo-inertial tensor:

$$ {}^{k}I_{psk} = \begin{bmatrix} \int {}^{k}\overline{r}_k \cdot {}^{k}\overline{r}_k^{T} \cdot dm & \int {}^{k}\overline{r}_k \cdot dm \\ \int {}^{k}\overline{r}_k^{T} \cdot dm & \int dm \end{bmatrix} = \left[\begin{array}{c|c} {}^{k}I_{pk} & M_k \cdot {}^{k}\overline{r}_{C_k} \\ \hline M_k \cdot {}^{k}\overline{r}_{C_k}^{T} & M_k \end{array}\right] \quad (9) $$

where ${}^{k}I_{psk}$ is the pseudo inertial tensor of the kinetic assembly k relative to system {i}, and ${}^{k}I_{pk}$ represents the planar centrifugal inertial tensor corresponding to the entire kinetic assembly (k), relative to frame $\{k\}$.

On the other hand, the dynamics matrices components include the so called differential matrices of first and second order, that correspond to the homogeneous transformations (position and orientation transformations) between the reference systems of the MBS, as follows:

$$ A_{ki(j)} = \left[\begin{array}{ccc|c} & A_{ki(j)}(R) & & A_{ki(j)}(\overline{p}) \\ \hline 0 & 0 & 0 & 0 \end{array}\right] \quad (10) $$

This is called the differential matrix of first order of the homogeneous transformations ($\overline{p}$ -position, R-orientation). The sub-matrices from equation (10) are determined using matrix exponential functions:

$$ \begin{Bmatrix} A_{ij}(R) = \dfrac{\partial}{\partial q_j}\{{}^{0}_{i}[R]\} = \\ = \left\{\exp\left\{\sum_{k=0}^{j-1}\left(\overline{k}_k^{(0)}\times\right)\cdot q_k\cdot\Delta_k\right\}\right\}\cdot\left(\overline{k}_j^{(0)}\times\right)\cdot\Delta_j\cdot A_{ij}^{*}(R) \end{Bmatrix} \quad (11) $$

$$ \text{where} \quad A_{ij}^{*}(R) = \exp\left\{\sum_{l=j}^{i}\left(\overline{k}_l^{(0)}\times\right)\cdot q_l\cdot\Delta_l\right\}\cdot R_{i0}^{(0)} $$

$$ \begin{Bmatrix} A_{ij}(\overline{p}) = \left\{\exp\left[\sum_{k=0}^{j-1}\left(\overline{k}_k^{(0)}\times\right)\cdot q_k\cdot\Delta_k\right]\right\}\cdot X_j + \\ +\exp\left[\sum_{l=j}^{i}\left(\overline{k}_l^{(0)}\times\right)\cdot q_l\cdot\Delta_l\right]\cdot\overline{p}_i^{(0)} + A_{ij}^{*}(\overline{p}) \end{Bmatrix} \quad (12) $$

$$ \text{where } X_j = \left(\overline{p}_j^{(0)}\times\right)\cdot\overline{k}_j^{(0)}\cdot\Delta_j + \left(1-\Delta_j\right)\cdot\overline{k}_j^{(0)} \quad (13) $$

$$ A_{ij}^{*}(\overline{p}) = \Delta_j\cdot\exp\left[\sum_{k=0}^{j-1}\left(\overline{k}_k^{(0)}\times\right)\cdot q_k\cdot\Delta_k\right]\cdot A_{ij}^{**}(\overline{p}) \quad (14) $$

$$ A_{ij}^{**}(\overline{p}) = \sum_{l=j}^{i}\left\{\exp\left[\sum_{m=j-1}^{l-1}\left(\overline{k}_m^{(0)}\times\right)\cdot q_m\cdot\Delta_m\cdot\delta_m\right]\right\}\cdot\overline{b}_l \quad (15) $$

The terms from the matrix exponentials expressions mentioned above have the following meanings:

$\overline{k}_{k(j)}^{(0)}$ - represents the unit vector, in the initial configuration, of the axis corresponding to the generalized coordinate $q_{k(j)}$, while $\Delta_{k(j)}$ is an operator that is 1 if $q_{k(j)}$ is an angular coordinates and is 0 otherwise. The terms $\overline{p}_i^{(0)}$ and $R_{i0}^{(0)}$ represent the position vector, respectively the orientation matrix of the system $\{i\}$ in relation to $\{0\}$, in the same initial configuration.

$A_{ij}(\overline{p}), A_{ij}(R)$ - represent the matrix of the first order regarding position and rotation in the system.

$$ A_{ijk}(R) = \left[\begin{array}{ccc|c} & A_{ijk}(R) & & A_{ijk}(\overline{p}) \\ \hline 0 & 0 & 0 & 0 \end{array}\right] \quad (16) $$

$$ \begin{Bmatrix} A_{ijk}(R) = \dfrac{\partial^2}{\partial q_j\cdot\partial q_k}\{{}^{0}_{i}[R]\} = \\ = \left\{\exp\left\{\sum_{l=0}^{k-1}\left(\overline{k}_l^{(0)}\times\right)\cdot q_l\cdot\Delta_l\right\}\right\}\cdot A_{ijk}^{*}(R) \end{Bmatrix} $$

where

$$ A_{ijk}^{*}(R) = \left(\overline{k}_k^{(0)}\times\right)\cdot\Delta_k\cdot A_{ijk}^{**}(R) $$

$$ \begin{Bmatrix} A_{ijk}^{**}(R) = \\ = \left\{\exp\left\{\sum_{m=k}^{j-1}\left(\overline{k}_m^{(0)}\times\right)\cdot q_m\cdot\Delta_m\right\}\right\}\cdot A_{ijk}^{***}(R) \end{Bmatrix} $$

$$ \begin{Bmatrix} A_{ijk}^{***}(R) = \\ = \left(\overline{k}_m^{(0)}\times\right)\cdot A_{ijk}^{****}(R) \\ \text{where} \\ A_{ijk}^{****}(R) = \Delta_m\cdot\left\{\exp\left\{\sum_{p=m}^{i}\left(\overline{k}_p^{(0)}\times\right)\cdot q_p\cdot\Delta_p\right\}\right\}\cdot R_{i0}^{(0)} \end{Bmatrix} \quad (17) $$

and $A_{ijk}(\overline{p}) = \dfrac{\partial}{\partial q_k}\left(A_{ij}(\overline{p})\right)$ and $A_{ij}(\overline{p})$ is given by (12).

2.2 Acceleration energies of second order

The dynamic study of MBS can be extended when the active exterior forces system is characterized by a certain law of variation in relation to time (Fig.3).

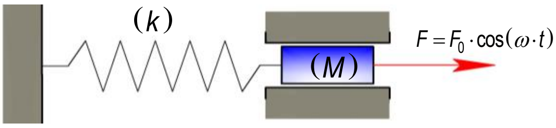

Fig. 3 Simplified mechanical system.

A simple example in agreement with this statement is the simplified mechanical system shown in figure 3. Therefore, the MBS dominated by such physical condition has acceleration energies of higher order. In the following, it is presented, according to [4], in a simplified form, the acceleration energy of second order $E_A^{(2)}$, whose starting equation is the following:

$$E_A^{(2)}=\frac{1}{2}\cdot\int {}^i\dddot{\overline{v}}_i^T\cdot{}^i\dddot{\overline{v}}_i\cdot dm=\frac{1}{2}\cdot Trace\left(\dddot{\overline{r}}\cdot\dddot{\overline{r}}^T\right)\cdot dm \qquad (18)$$

$$where\ \ \dddot{\overline{r}}_i=\dddot{\overline{r}}_{C_i}+{}_i^0[\dddot{R}]\cdot{}^i\overline{r}_i^*$$

$$\dddot{\overline{r}}_i^T=\dddot{\overline{r}}_{C_i}^T+{}^i\overline{r}_i^{*T}\cdot{}_i^0[\dddot{R}]^T \qquad (19)$$

The relation (18) represents the energy of acceleration of second order of the elementary mass *dm*.

And $\ddot{v}_i$, $\ddot{v}_i^T$ - represent the absolute acceleration and their transpose of the element (i).

According to the same paper of the main author, the following matrix property can be developed:

$$\begin{aligned}&{}_i^0[\dddot{R}]\cdot{}_i^0[R]^T=-{}_i^0[\ddot{R}]\cdot{}_i^0[\dot{R}]^T+\\&+\left(\ddot{\overline{\omega}}_i\times\right)+\left(\dot{\overline{\omega}}_i\times\right)\cdot\left(\overline{\omega}_i\times\right)+\left(\overline{\omega}_i\times\right)\cdot\left(\dot{\overline{\omega}}_i\times\right)=\\&=\left(\ddot{\overline{\omega}}_i\times\right)+2\cdot\left(\dot{\overline{\omega}}_i\times\right)\cdot\left(\overline{\omega}_i\times\right)+\left(\overline{\omega}_i\times\right)\cdot\left(\dot{\overline{\omega}}_i\times\right)+\\&+\left(\overline{\omega}_i\times\right)\cdot\left(\overline{\omega}_i\times\right)\cdot\left(\overline{\omega}_i\times\right)\end{aligned} \qquad (20)$$

where $\overline{\omega}_i$, $\dot{\overline{\omega}}_i$ and $\ddot{\overline{\omega}}_i$ represent the angular velocity, acceleration of first order and acceleration of second order according to the rotation of the kinetic ensemble (*i*) and $\dddot{\overline{r}}_{C_i}=\ddot{\overline{v}}_{C_i}$ expresses the absolute acceleration of second order of the mass center C_i.

After several differential transformations, *the explicit form* $E_A^{(2)}$ of the acceleration energy of second order is obtained for the whole MBS system (rel. 21). Expression (21) contains only terms which are a function of $\dddot{\overline{\theta}}=(\dddot{q}_i,\ for\ i=1\to n)^T$, representing the generalized accelerations of second order. The other terms from equation (27) have the meaning from paragraph 2.1

$$\left\{\begin{aligned}&E_A^{(2)}\left[\overline{\theta}(t);\dot{\overline{\theta}}(t);\ddot{\overline{\theta}}(t);\dddot{\overline{\theta}}(t)\right]=\\&=(-1)^{\Delta_m}\cdot\frac{1-\Delta_m}{1+3\cdot\Delta_m}\cdot\sum_{i=1}^{n}\left\{\frac{1}{2}\cdot M_i\cdot{}^i\ddot{\overline{v}}_{C_i}^T\cdot{}^i\ddot{\overline{v}}_{C_i}\right\}+\\&+\Delta_m^2\cdot\sum_{i=1}^{n}\left\{\frac{1}{2}\cdot{}^i\ddot{\overline{\omega}}_i^T\cdot{}^iI_i^*\cdot{}^i\ddot{\overline{\omega}}_i+2\cdot{}^i\dot{\overline{\omega}}_i^T\cdot\left({}^i\ddot{\overline{\omega}}_i\times{}^iI_{pi}^*\cdot{}^i\dot{\overline{\omega}}_i\right)+\right.\\&+{}^i\dot{\overline{\omega}}_i^T\cdot\left({}^i\ddot{\overline{\omega}}_i\times{}^iI_{pi}^*\cdot{}^i\overline{\omega}_i\right)-{}^i\overline{\omega}_i^T\cdot\left({}^i\ddot{\overline{\omega}}_i^T\cdot{}^iI_i^*\cdot{}^i\overline{\omega}_i\right)\cdot{}^i\overline{\omega}_i+\\&+2\cdot{}^i\dot{\overline{\omega}}_i^T\cdot\left({}^i\overline{\omega}_i^T\cdot{}^iI_i^*\cdot{}^i\dot{\overline{\omega}}_i\right)\cdot{}^i\overline{\omega}_i+\\&+2\cdot{}^i\overline{\omega}_i^T\cdot\left[{}^i\dot{\overline{\omega}}_i^T\cdot{}^iI_{pi}^*\cdot{}^i\dot{\overline{\omega}}_i\right]\cdot{}^i\overline{\omega}_i-\\&-5\cdot\left({}^i\dot{\overline{\omega}}_i^T\cdot{}^iI_{pi}^*\right)\cdot\left({}^i\dot{\overline{\omega}}_i^T\cdot{}^i\overline{\omega}_i\right)\cdot{}^i\overline{\omega}_i+\\&+\frac{5}{2}\cdot\left({}^i\dot{\overline{\omega}}_i^T\cdot{}^i\overline{\omega}_i\right)\cdot Trace\left({}^iI_{pi}^*\right)\cdot\left({}^i\dot{\overline{\omega}}_i^T\cdot{}^i\overline{\omega}_i\right)+\\&+\frac{1}{2}\cdot{}^i\dot{\overline{\omega}}_i^T\cdot\left[{}^i\overline{\omega}_i^T\cdot{}^iI_{pi}^*\cdot{}^i\overline{\omega}_i\right]\cdot{}^i\dot{\overline{\omega}}_i+\\&+{}^i\overline{\omega}_i^T\cdot\left[{}^i\overline{\omega}_i^T\cdot\left({}^i\dot{\overline{\omega}}_i\times{}^iI_{pi}^*\cdot{}^i\overline{\omega}_i\right)\right]\cdot{}^i\overline{\omega}_i+\\&+\left.\frac{1}{2}\cdot{}^i\overline{\omega}_i^T\cdot\left[{}^i\overline{\omega}_i^T\cdot\left({}^i\overline{\omega}_i^T\cdot{}^iI_i^*\cdot{}^i\overline{\omega}_i\right)\cdot{}^i\overline{\omega}_i\right]\cdot{}^i\overline{\omega}_i\right\}\end{aligned}\right\} \qquad (21)$$

Applying a series of matrix and differential transformations, the matrix expression of the acceleration energy of second order is obtained, according to [4].

To the dynamic matrices presented in paragraph 2.1, two more matrices are added to the components of the acceleration energy of second order:

The acceleration energy of the second order $E_A^{(2)}$ in *the matrix form* is given us by:

$$\left\{\begin{aligned}&E_A^{(2)}\left[\overline{\theta}(t);\dot{\overline{\theta}}(t);\ddot{\overline{\theta}}(t);\dddot{\overline{\theta}}(t)\right]=\frac{1}{2}\cdot\dddot{\overline{\theta}}^T(t)\cdot M\left[\overline{\theta}(t)\right]\cdot\dddot{\overline{\theta}}(t)+\\&+3\cdot\dddot{\overline{\theta}}^T(t)\cdot V\left[\overline{\theta}(t);\dot{\overline{\theta}}(t);\ddot{\overline{\theta}}(t)\right]+\\&+\dddot{\overline{\theta}}^T(t)\cdot H\left[\overline{\theta}(t);\dot{\overline{\theta}}(t)\right]\cdot\ddot{\overline{\theta}}(t)+\\&+3\cdot\ddot{\overline{\theta}}^T(t)\cdot K\left[\overline{\theta}(t);\dot{\overline{\theta}}(t)\right]+\\&+\frac{9}{2}\cdot\ddot{\overline{\theta}}^T(t)\cdot D\left[\overline{\theta}(t);\dot{\overline{\theta}}(t);\ddot{\overline{\theta}}(t)\right]\cdot\ddot{\overline{\theta}}(t)+\\&+\frac{1}{2}\cdot\dot{\overline{\theta}}^T(t)\cdot N\left[\overline{\theta}(t);\dot{\overline{\theta}}(t)\right]\cdot\dot{\overline{\theta}}(t)\end{aligned}\right\} \qquad (22)$$

where $M(\bar{\theta})$ is the inertial mass matrix, $V(\bar{\theta},\dot{\bar{\theta}},\ddot{\bar{\theta}})$ is the Coriolis matrix, and $D(\bar{\theta},\dot{\bar{\theta}},\ddot{\bar{\theta}})$ is pseudoinertial matrix correspond of the acceleration energy, and they are the follow relations:

$$M_{ij}=\sum_{k=\max(i;j)}^{n} Trace\left[A_{ki}\cdot {}^{k}I_{psk}\cdot A_{kj}^{T}\right]$$

$$M(\bar{\theta})=\underset{(nxn)}{Matrix}\left\{\begin{matrix} M_{ij}=M_{ji}= & \\ =\sum_{k=1}^{n} Trace\left\{A_{ki}\cdot {}^{k}I_{psk}\cdot A_{kj}^{T}\right\} & \begin{matrix} i=1\to n \\ j=1\to n\end{matrix}\end{matrix}\right\} \quad (23)$$

$$\left\{V_{ijm}=V_{imj}\right\}=\sum_{k=\max(i;j;m)}^{n} Trace\left[A_{ki}\cdot {}^{k}I_{psk}\cdot A_{kjm}^{T}\right]$$

$$V\left(\bar{\theta};\dot{\bar{\theta}};\ddot{\bar{\theta}}\right)=\underset{(nxn)}{Matrix}\left\{V_{i}\quad i=1\to n\right\}$$

$$V_{i=}\left\{\ddot{\bar{\theta}}^{T}\cdot\begin{bmatrix}\left\{V_{ijm}=V_{imj}\right\} & j=1\to n\\ & m=1\to n\end{bmatrix}\cdot\dot{\bar{\theta}}\right\}^{T} \quad (24)$$

$$D_{ijlm}=\sum_{k=\max(i;j;l;m)}^{n} Trace\left[A_{kij}\cdot {}^{k}I_{psk}\cdot A_{klm}^{T}\right]$$

$$D\left(\bar{\theta};\dot{\bar{\theta}};\ddot{\bar{\theta}}\right)=\underset{(nxn)}{Matrix}\left\{D_{ij}\quad \begin{matrix} i=1\to n\\ j=1\to n\end{matrix}\right\}$$

$$D_{ij}=\ddot{\bar{\theta}}^{T}\cdot\begin{bmatrix}D_{ijlm} & l=1\to n\\ & m=1\to n\end{bmatrix}\cdot\dot{\bar{\theta}} \quad (25)$$

and $H(\bar{\theta},\dot{\bar{\theta}}^{2})$, $K(\bar{\theta},\dot{\bar{\theta}}^{4})$, $N(\bar{\theta},\dot{\bar{\theta}}^{4})$ have the follow expressions:

$$\left\{H_{ijlm}=H_{imlj}\right\}=\sum_{k=\max(i;j;l;m)}^{n} Trace\left[A_{ki}\cdot {}^{k}I_{psk}\cdot A_{kjlm}^{T}\right]$$

$$H\left(\bar{\theta};\dot{\bar{\theta}}^{2}\right)=\underset{(nxn)}{Matrix}\left\{H_{ij}\quad \begin{matrix} i=1\to n\\ j=1\to n\end{matrix}\right\}$$

$$H_{ij}=\dot{\bar{\theta}}^{T}\cdot\begin{bmatrix}H_{ijlm} & l=1\to n\\ & m=1\to n\end{bmatrix}\cdot\dot{\bar{\theta}} \quad (26)$$

$$K_{ijlmp}=\sum_{k=\max(i;j;l;m;p)}^{n} Trace\left[A_{kij}\cdot {}^{k}I_{psk}\cdot A_{klmp}^{T}\right]$$

$$K\left(\bar{\theta};\dot{\bar{\theta}}^{4}\right)=\underset{(nx1)}{Matrix}\left\{K_{i}\quad i=1\to n\right\}$$

$$K_{i}=\dot{\bar{\theta}}^{T}\cdot\left\{\dot{\bar{\theta}}^{T}\cdot\begin{bmatrix}K_{ijlmp} & m=1\to n\\ & p=1\to n\end{bmatrix}\cdot\dot{\bar{\theta}};\quad \begin{matrix} j=1\to n\\ l=1\to n\end{matrix}\right\}\dot{\bar{\theta}} \quad (27)$$

$$N_{ijlmpr}=\sum_{k=\max(i;j;l;m;p;r)}^{n} Trace\left[A_{kijl}\cdot {}^{k}I_{psk}\cdot A_{kmpr}^{T}\right]$$

$$N\left(\bar{\theta};\dot{\bar{\theta}}^{4}\right)=\underset{(nxn)}{Matrix}\left\{N_{ij}\quad \begin{matrix} i=1\to n\\ j=1\to n\end{matrix}\right\}$$

$$N_{ij}=\dot{\bar{\theta}}^{T}\cdot\left\{\dot{\bar{\theta}}^{T}\cdot\begin{bmatrix}N_{ijlmpr} & p=1\to n\\ & r=1\to n\end{bmatrix}\cdot\dot{\bar{\theta}};\quad \begin{matrix} l=1\to n\\ m=1\to n\end{matrix}\right\}\cdot\dot{\bar{\theta}} \quad (28)$$

Equations (23) and (25) are differential matrices of second order with mass and inertial properties. The differential matrix of third order, component of the dynamics matrix given by equation (22), has the following form:

$$A_{ijkm}=\left[\begin{array}{ccc|c} & A_{ijkm}(R) & & A_{ijkm}(\bar{p})\\ \hline 0 & 0 & 0 & 0\end{array}\right] \quad (29)$$

The sub-matrices included in equation (29) can be expressed with exponential functions as:

$$A_{ijkm}(R)=\left\{\exp\left\{\sum_{l=0}^{m-1}\left(\bar{k}_{l}^{(0)}\times\right)\cdot q_{l}\cdot\Delta_{l}\right\}\right\}\cdot A_{ijkm}^{*}(R)$$

$$where: A_{ijkm}^{*}(R)=\left(\bar{k}_{m}^{(0)}\times\right)\cdot\Delta_{m}\cdot A_{ijkm}^{**}(R)$$

$$A_{ijkm}^{**}(R)=\left\{\exp\left\{\sum_{p=m}^{k-1}\left(\bar{k}_{p}^{(0)}\times\right)\cdot q_{p}\cdot\Delta_{p}\right\}\right\}\cdot A_{ijkm}^{***}(R)$$

$$A_{ijkm}^{***}(R)=\left(\bar{k}_{p}^{(0)}\times\right)\cdot\Delta_{p}\cdot A_{ijkm}^{****}(R) \quad (30)$$

$$A_{ijkm}^{****}(R)=\left\{\exp\left\{\sum_{r=p}^{j-1}\left(\bar{k}_{r}^{(0)}\times\right)\cdot q_{r}\cdot\Delta_{r}\right\}\right\}\cdot A_{ijkm}^{*****}(R)$$

$$A_{ijkm}^{*****}(R)=\left(\bar{k}_{r}^{(0)}\times\right)\cdot\Delta_{r}\cdot A_{ijkm}^{******}(R)\cdot R_{i0}^{(0)}$$

$$A_{ijkm}^{******}(R)=\exp\left\{\sum_{s=r}^{i}\left(\bar{k}_{s}^{(0)}\times\right)\cdot q_{s}\cdot\Delta_{s}\right\}$$

$$A_{ijkm}(\bar{p})=\frac{\partial}{\partial q_{k}}\left[A_{ijk}(\bar{p})\right]=\frac{\partial^{2}}{\partial q_{k}\cdot\partial q_{m}}\left[A_{ij}(\bar{p})\right] \quad (31)$$

where $A_{ij}(\bar{p})$ is given by equation (12).

$A_{ij}(\bar{p}),A_{ijk}(\bar{p})$- represented the differential matrices of first and second order regarding position of system.

In the explicit and matrix formulations of the acceleration energies of higher order, it can be noticed that the usage of matrix exponential functions is an important advantage for the kinematic study of MBS. This are given by the fact, that the terms included in these functions correspond to the initial configuration.

2.3 Acceleration Energy of Third Order

For more precise modeling on the transitory motion phases of the MBS, in which the robots are included, the dynamic study is extended on the acceleration energies of third order (see above aspects and example from the paper developed by the first author). So, using the above aspects referring to acceleration energy of first and second order, in this case, the first author proposes the equation of the acceleration energy of third order $E_A^{(3)}$ as follows:

$$\left\{\begin{array}{l} E_A^{(3)}(t)=\frac{1}{2}\sum_{i=1}^{n}\int \dddot{\bar{v}}_i^T\cdot\dddot{\bar{v}}_i\cdot dm= \\ =\frac{1}{2}\sum_{i=1}^{n}\int Trace\left(\dddot{\bar{r}}_i\cdot\dddot{\bar{r}}_i^T\right)\cdot dm= \\ =\frac{1}{2}\cdot\sum_{i=1}^{n}Trace\left\{{}_i^0[\dddot{R}]\cdot\left[{}^iI_{pi}^*+M_i\cdot{}^i\bar{r}_{C_i}\cdot{}^i\bar{r}_{C_i}^T\right]\cdot{}_i^0[\dddot{R}]^T\right\}+ \\ +\frac{1}{2}\cdot\sum_{i=1}^{n}Trace\left[\dddot{\bar{p}}_i\cdot\dddot{\bar{p}}_i^T\right]\cdot M_i \end{array}\right\} \quad (32)$$

where $\dddot{\bar{r}}_i$ represents the absolute acceleration of third order of the elementary mass dm (see Fig.1), and $\dddot{\bar{p}}_i$ expresses the absolute acceleration of third order of the origin $O_i\in\{i\}$ (see Fig. 2). Applying the differential transformations in (32), the expression of the acceleration energy of third order, containing only terms which are a function of the generalized accelerations of third order, shows as:

The relation (32) it could write of *the explicit form* such as:

$$\left\{\begin{array}{l} E_A^{(3)i}\left[\dddot{q}_i(t)\right]=E_A^{(3)i}\left[q_i(t);\dot{q}_i(t);\ddot{q}_i(t);\dddot{q}_i(t);\dddot{q}_i(t)\right]= \\ =\frac{1}{2}\cdot M_i\cdot{}^i\dddot{\bar{v}}_{C_i}^T\cdot{}^i\dddot{\bar{v}}_{C_i}+\frac{1}{2}\cdot{}^i\dddot{\bar{\omega}}_i^T\cdot{}^iI_i^*\cdot{}^i\dddot{\bar{\omega}}_i+ \\ +3\cdot\bar{\omega}_i^T\cdot\left(\dddot{\bar{\omega}}_i\times I_{pi}^*\cdot\ddot{\bar{\omega}}_i\right)+3\cdot\dot{\bar{\omega}}_i^T\cdot\left(\dddot{\bar{\omega}}_i\times I_{pi}^*\cdot\dot{\bar{\omega}}_i\right)+ \\ +3\cdot\ddot{\bar{\omega}}_i^T\cdot\left(\dddot{\bar{\omega}}_i\times I_{pi}^*\cdot\bar{\omega}_i\right)+2\cdot\left(\bar{\omega}_i\times\dddot{\bar{\omega}}_i\right)^T\cdot I_{pi}^*\cdot\left(\dot{\bar{\omega}}_i\times\bar{\omega}_i\right)- \\ -5\cdot\dot{\bar{\omega}}_i^T\cdot\left[\bar{\omega}_i^T\cdot I_i^*\cdot\dddot{\bar{\omega}}_i\right]\cdot\bar{\omega}_i- \\ -\bar{\omega}_i^T\cdot\left[\dot{\bar{\omega}}_i^T\cdot I_i^*\cdot\dddot{\bar{\omega}}_i\right]\cdot\bar{\omega}_i+\bar{\omega}_i^T\cdot\left[\bar{\omega}_i^T\cdot I_{pi}^*\cdot\left(\dddot{\bar{\omega}}_i\times\bar{\omega}_i\right)\right]\cdot\bar{\omega}_i \end{array}\right\} \quad (33)$$

And the relation (32) could write in *the matrix form* such as:

$$\left\{\begin{array}{l} E_A^{(3)}\left[\bar{\theta}(t);\dot{\bar{\theta}}(t);\ddot{\bar{\theta}}(t);\dddot{\bar{\theta}}(t);\ddddot{\bar{\theta}}(t)\right]= \\ =\frac{1}{2}\cdot\ddddot{\bar{\theta}}^T(t)\cdot M[\bar{\theta}(t)]\cdot\ddddot{\bar{\theta}}(t)+ \\ +4\cdot\ddddot{\bar{\theta}}^T(t)\cdot V\left[\bar{\theta}(t);\dot{\bar{\theta}}(t);\dddot{\bar{\theta}}(t)\right]+ \\ +3\cdot\ddddot{\bar{\theta}}^T(t)\cdot V^*\left[\bar{\theta}(t);\ddot{\bar{\theta}}(t)\right]+ \\ +6\cdot\ddddot{\bar{\theta}}^T(t)\cdot H\left[\bar{\theta}(t);\dot{\bar{\theta}}^2(t)\right]\cdot\ddot{\bar{\theta}}(t)+ \\ +\ddddot{\bar{\theta}}^T(t)\cdot K\left[\bar{\theta}(t);\dot{\bar{\theta}}^4(t)\right] \end{array}\right\} \quad (34)$$

The components of the differential matrix of fourth order, included in (34) have the following form:

$$\left\{\begin{array}{l} A_{ijkmp}(R)=\frac{\partial^4}{\partial q_j\cdot\partial q_k\cdot\partial q_m\cdot\partial q_p}\left\{{}_i^0[R]\right\}= \\ =\left\{\exp\left\{\sum_{l=0}^{p-1}\left(\bar{k}_l^{(0)}\times\right)\cdot q_l\cdot\Delta_l\right\}\right\}\cdot A_{ijkmp}^*(R) \end{array}\right\}$$

where

$$A_{ijkmp}^*(R)=\left(\bar{k}_p^{(0)}\times\right)\cdot\Delta_p\cdot\left\{\exp\left\{\sum_{r=p}^{m-1}\left(\bar{k}_p^{(0)}\times\right)\cdot q_r\cdot\Delta_r\right\}\right\}\cdot A_{ijkmp}^{**}(R)$$

where $A_{ijkmp}^{**}(R)=\left(\bar{k}_r^{(0)}\times\right)\cdot\Delta_r\cdot A_{ijkmp}^{***}(R)$ (35)

$$A_{ijkmp}^{***}(R)=\left\{\exp\left\{\sum_{s=r}^{k-1}\left(\bar{k}_r^{(0)}\times\right)\cdot q_s\cdot\Delta_s\right\}\right\}\cdot A_{ijkmp}^{****}(R)$$

where $A_{ijkmp}^{****}(R)=\left(\bar{k}_s^{(0)}\times\right)\cdot\Delta_s\cdot A_{ijkmp}^{*****}(R)\cdot R_{i0}^{(0)}$

$$A_{ijkmp}^{*****}(R)=\left\{\exp\left\{\sum_{u=s}^{j-1}\left(\bar{k}_u^{(0)}\times\right)\cdot q_u\cdot\Delta_u\right\}\right\}\cdot A_{ijkmp}^{******}(R)$$

$$A_{ijkmp}^{******}(R)=\left(\bar{k}_u^{(0)}\times\right)\cdot\Delta_u\cdot\exp\left\{\sum_{v=u}^{i}\left(\bar{k}_v^{(0)}\times\right)\cdot q_v\cdot\Delta_v\right\}\cdot R_{i0}^{(0)}$$

$$\text{and}\ \left\{ \begin{aligned} & A_{ijkmp}(\bar{p}) = \frac{\partial A_{ijkm}(\bar{p})}{\partial q_p} = \frac{\partial^2 A_{ijkm}(\bar{p})}{\partial q_m \cdot \partial q_p} = \\ & = \frac{\partial^3 A_{ij}(\bar{p})}{\partial q_k \cdot \partial q_m \cdot \partial q_p} = \frac{\partial^4 \bar{p}_i}{\partial q_j \cdot \partial q_k \cdot \partial q_m \cdot \partial q_p} \end{aligned} \right\} \quad (36)$$

where $\left\{ \bar{k}_i^{(0)} \times \right\}$ is the skew-symmetric matrix associated to the unit vector belonging to every kinematical axis. In the position study based on matrix exponentials, a new column vector is established, according to [1] and [2]; $\Delta = \{(1, \text{if } i = R); (0, \text{if } i = T)\}$ is an operator which marks out the type of joint: (R-rotation; T-prismatic joint), according to figure 1 and $q_i\ \ i = 1 \to n$ is generalize coordinate regarding the element of the system.

The above terms have been determined by means of the matrix exponentials that are not presented in this paper.

4. EXPERIMENTAL METHOD

In the following step, it will be presents an experimental study regarding the demonstration of this acceleration energy of second order and third order (relation (38)) on the moving arm of the FANUC Robot (Fig. 4).

In this way, it will present a correlation between the experimental and theoretical study, to validate the research studies. The mono-axial accelerometer (Fig. 4) measured the tangential acceleration of arm robot's moving (0-π rad.), and, it was fixed on the gripper with a magnet.

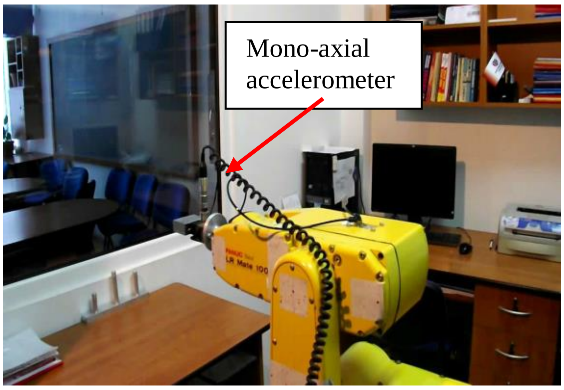

Fig. 4 The Kinematical Structure for the FANUC (5R) Robot.

The generalized variable:

$$\left\{ q_3(\tau);\ \dot{q}_3(\tau);\ \ddot{q}_3(\tau);\ \dddot{q}_3(\tau);\ \ddddot{q}_3(\tau) \right\}$$

The particular acceleration energies of second and third order expression, corresponding arm of robot by type Fanuc, there are determinate using inter-polar function of 5-th order with general coordinates:

$$\ddddot{q}_{jik}(\tau), \dddot{q}_{jik}(\tau), \ddot{q}_{jik}(\tau), \dot{q}_{jik}(\tau), q_{jik}(\tau)$$

These general coordinate used in the inter-polar function for determinate the acceleration energy of second and third order.

In base of interpolation function are determinate the acceleration energies left the acceleration energy of second order and arrive of acceleration energy of third order and their graphical representations.

$$\left\{ \begin{aligned} & E_{Aik}^{(2)}(\tau) = \\ & \quad = \frac{1}{2} \cdot \left(M_1 \cdot x_{c1}^2 + M_1 \cdot z_{c1}^2 + {}^3 I_y \right) \cdot \left(\dddot{q}_{3ik}^2(\tau) - \right. \\ & \quad - 2 \cdot \dot{q}_{3ik}^3(\tau) \cdot \dddot{q}_{3ik}(\tau) + \\ & \quad \left. + 9 \cdot \dot{q}_{3ik}^2(\tau) \cdot \ddot{q}_{3ik}^2(\tau) + \dot{q}_{3ik}^6(\tau) \right) \end{aligned} \right\}$$

$$\left\{ \begin{aligned} & E_{Aik}^{(3)}(\tau) = \\ & = \frac{1}{2} \left[\left(M_1 \cdot x_{c1}^2 + M_1 \cdot z_{c1}^2 + {}^3 I_y \right) \cdot \left[\dot{q}_{3ik}^8(\tau) - \right. \right. \\ & - 8 \cdot \dot{q}_{3ik}^5(\tau) \cdot \dddot{q}_{3ik(\tau)} + 30 \cdot \dot{q}_{3ik}^4(\tau) \cdot \ddot{q}_{3ik}^2(\tau) - \\ & - 12 \cdot \dot{q}_{3ik}^2(\tau) \cdot \ddot{q}_{3ik}(\tau) \cdot \ddddot{q}_{3ik}(\tau) + \\ & + 16 \cdot \dot{q}_{3ik}^2(\tau) \cdot \dddot{q}_{3ik}^2(\tau) + 24 \cdot \dot{q}_{3ik}(\tau) \cdot \ddot{q}_{3ik}^2(\tau) \cdot \dddot{q}_{3ik}(\tau) + \\ & \left. \left. + 9 \cdot \ddot{q}_{3ik}^4(\tau) + \ddddot{q}_{3ik}^2(\tau) \right) \right] \end{aligned} \right\} \quad (37)$$

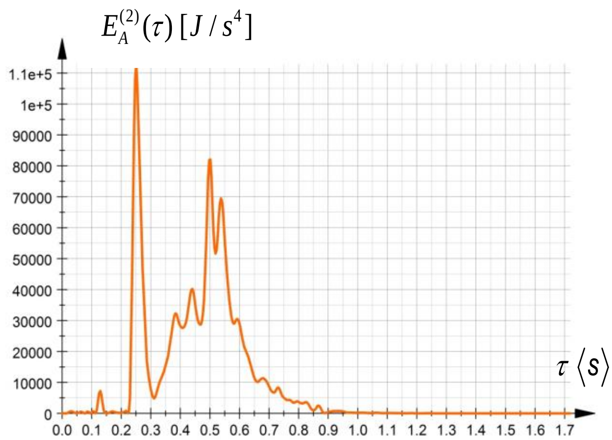

Fig. 5 Time Variation Law of the Acceleration Energy of Second Order

The figure 5 presents the time variation law of acceleration energy of second order, corresponding arm moving of the angular position of 0-π rad. It observes that motion start with zero value and finally, it arrived in zero, again (stop moving).

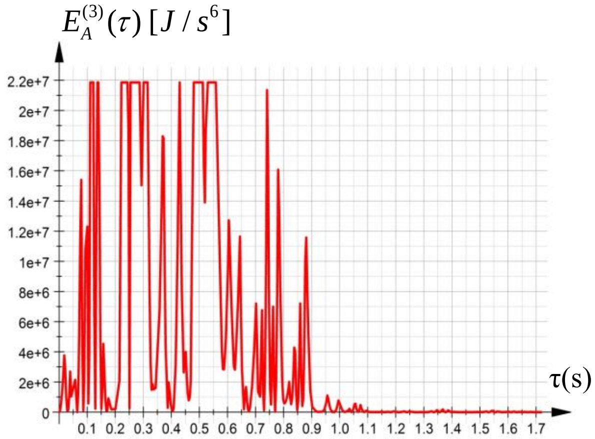

Fig. 6 Time Variation Law of the Acceleration Energy of Third Order

The figures 5 and 6 present a comparison between experimental curves, regarding the variation laws of acceleration energy of second and third order.

It observes that between acceleration of second order an third order the values of these are significant, respective the acceleration of third order arrived with values till 2.2e+7 [J/s^6] in comparison with acceleration of second order with values till 1.1e+5 [J/s^4]. The last conclusion show us, that these superior energies are significant values in the start of moving, and they are not neglected.

5. CONCLUSIONS

The paper is based on new formulations in advanced dynamics of multibody systems, regarding the higher order energies.

These equations have been developed using matrix exponentials that have undeniable advantages in the matrix study of any complex mechanical system. Within the paper there are proposed approaches based on differential principles by using some important dynamics notions, regarding the acceleration energy of first, but especially second and third order, all these development by the first author of paper. The study could be extended upon the equations of higher order. They give the possibility of applying the initial motion conditions in positions, velocities and accelerations of first and second order. These lead to an accuracy control on the transitory motion phases, regarding the dynamic behavior for any multibody system, in which the robot structures are included.

6. REFERENCES

[1] I. Negrean, D. C. Negrean, , "Matrix exponentials to robot kinematics", 17th International Conference on CAD/CAM, Robotics and Factories of the Future, Vol.2, pp. 1250-1257, Durban, South Africa, (2001).

[2] I. Negrean, A. Duca, , Negrean, D.C., Kacso, K., "Mecanică avansată în robotică", ISBN 978-973-662-420-9, UT Press, Cluj-Napoca, (2008), http://users.utcluj.ro/~inegrean.

[3] I. Negrean, C. Schonstein, K. Kacso, A. Duca, "Mecanica. Teorie şi aplicaţii", Editura UT PRESS, ISBN 978-973-662-523-7, Cluj – Napoca, 2012.

[4] Negrean, I., Kacso, K., Schonstein, C., Duca, A., *Energies of Accelerations in Advanced Robotics Dynamics*, The VI-th International Conference on Robotics (ROBOTICS 2014), Applied Mechanics and Materials, ISSN: 1662-7482.

Formulări privind energia acceleraţiilor de ordinul 2 şi 3 aplicate în mecanica analitică

Rezumat: Lucrarea este dedicată prezentării unor noi formulări despre energii de ordin superior, care sunt utilizate în studiul dinamic pe sistemele mecanice. Parte integrantă din aceste sisteme mecanice sunt structuri mecanice, roboţi şi se va evidenţia importanţa dinamici mişcării privind influenţă energiilor de ordin superior. Această lucrare extinde studiul, prin dezvoltarea energiilor superioare ale acceleraţilor de ordinul al doilea şi al treilea şi implementarea lor în practică, oferind posibilitatea de a studia la mişcarea iniţială, condiţiile de poziţie, viteză şi acceleraţie.

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