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AESTHETIC OPTIMIZATION OF A BASIC SHAPE

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Abstract: *An important aspect in nowadays industries is to obtain end products that are optimal with respect to several criteria. The aesthetic criterion is an essential one and it has been studied since Antiquity by several artists and scientists. In this paper we propose two aesthetic optimization models of a basic shape using evolutionary algorithms. The objective functions are formulated using the Golden Ratio as an aesthetic standard.*
Key words: *aesthetic standards, mathematical mean, Golden Ratio, optimization, evolutionary algorithms.*

1. INTRODUCTION

All industries are situated in a complex environment that constantly challenges the decision-makers to choose the best option. In this context the optimization of processes, technologies, strategies, and products becomes crucial. However, the optimization criteria have to be carefully chosen in order to ensure the success of the end product.

A significant criterion is the aesthetic one due to the fact that aesthetics is a part of every person's live, not just artists'. This criterion has been intensely studied with the purpose of providing a mathematical model that allows quantifying the aesthetic value brought by the aesthetic objects. Several theories of art or aesthetics such as mystical, pedagogic, and hedonistic have been developed during the years. In opposition to these theories have appeared the analytic ones that are focused on identifying the most frequently encountered aesthetic factors and the general laws that govern them [2].

In 1933, Birkhoff [2] defined the aesthetics measure as the ratio between the order and the complexity of the aesthetic object.

Other authors have turned to the famous five sacred geometry constants: π, $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$, $\left(1+\sqrt{5}\right)/2$ and with their help defined some aesthetic standards. The last constant, hereinafter denoted by φ, is known in the literature as the "golden number", "golden section", "golden ratio, "divine proportion" or „extreme and mean ratio". It has been the subject of the research of many authors (see for example [5], [6], and [7]). Some of them even extended the notion by defining generalized golden ratios such as golden *p*-proportions in [8] or the golden ratio defined by power means in [3].

This constant has been found in many forms in nature. In the last decades it has been used to develop beauty standards such as the system for measuring facial attractiveness from [1]. Moreover, it has been proven recently that "our heart has been beating in a ratio of 1.618, obeying the beautiful order of universe settled by God" [10].

We recall the definition of the Golden Ratio; see [4] for more details.

Let [AB] be a segment and C a point belonging to it is such that CB > CA (Fig. 1). If

$$\frac{AC}{CB}=\frac{CB}{AB} \quad \text{or} \quad CB=\sqrt{AC\cdot AB} \quad (1)$$

then the segments CB and CA are said to be in the Golden Ratio. Observe that the Golden Ratio is defined by the *geometric mean.*

The number $\varphi = \frac{CB}{AC} = \frac{1+\sqrt{5}}{2}$ is called the Golden Ratio and it is the positive solution of the following equation:

$$t^2 - t - 1 = 0 \quad (2)$$

Fig. 1. Golden Ratio

One of the most studied problems in aesthetics, starting with the famous Greek vases is the one of the aesthetics of the form of a vase. This problem is considered to be a very difficult one [2]. The approach proposed in this paper for solving this problem consists in obtaining the vase generated by circular arcs that is optimal with respect to the aesthetic criteria defined regarding the vase shape. The optimization model developed for the vase can be easily adjusted to the aesthetic optimization of other shapes. The aesthetic criteria are defined using the Golden Ratio and the optimization is conducted using evolutionary algorithms.

2. MATHEMATICAL MODEL

2.1 Basic shape generated by circular arcs

Let there be two circular arcs: $\overset{\frown}{AB}$ from the circle (C_1) of center O and radius r_1 and $\overset{\frown}{BC}$ from the circle (C_2) of center E and radius r_2. We assume a smooth connection at the point B (the circular arcs have the same tangent at the connection point).

The parametric equations of the first circle are:

$$(C_1)\begin{cases} x = r_1 \cdot \cos t \\ y = r_1 \cdot \sin t \end{cases}, \ t \in [-\alpha, \beta],\ \alpha, \beta \in \left[0, \frac{\pi}{2}\right] \quad (3)$$

Due to the assumption regarding the point B, it follows that the point O, B, and E are collinear. Therefore the coordinates of point E are:

$$(E)\begin{cases} x = \lambda \cdot r_1 \cdot \cos\beta \\ y = \lambda \cdot r_1 \cdot \sin\beta \end{cases}, \ with\ \lambda \in (1, \infty) \quad (4)$$

Moreover:

$$r_2 = OE - OB = \lambda \cdot r_1 - r_1 = (\lambda - 1) \cdot r_1 \quad (5)$$

The parametric equations of the second circle are:

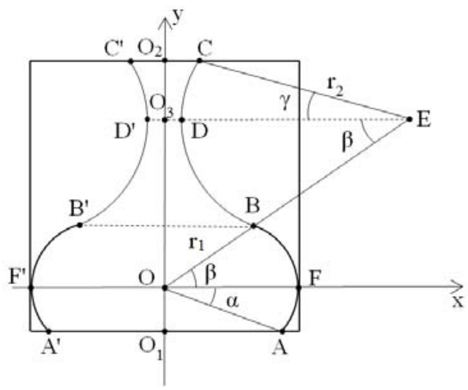


Fig. 2. The vertical section of a vase generated by two circular arcs

$$(C_2)\begin{cases} x = x_1 + r_2 \cdot \cos s \\ y = y_1 + r_2 \cdot \sin s \end{cases}, \quad (6)$$

where $s \in [\pi - \gamma, \pi + \beta]$, $\gamma \in \left[0, \frac{\pi}{2}\right]$. They can also be written in the following form:

$$(C_2)\begin{cases} x = \lambda \cdot r_1 \cdot \cos\beta + (\lambda - 1) \cdot r_1 \cdot \cos s \\ y = \lambda \cdot r_1 \cdot \sin\beta + (\lambda - 1) \cdot r_1 \cdot \sin s \end{cases} \quad (7)$$

The basic shape of the vase is generated by symmetry with respect to the axis Oy. In order to have a functional vase the circular arc $\overset{\frown}{BC}$ cannot intersect the vertical axis. Thus, the following condition should be satisfied:

$$\lambda r_1 \cos\beta - r_2 > 0 \quad (8)$$

which can be rewritten as:

$$1 - \cos\beta - \frac{1}{\lambda} < 0 \quad (9)$$

The section of the basic shape is fitted in a rectangle (see Fig. 2) of height $h = O_1O_2$ and width $l = \max\{FF', CC'\}$. Therefore:

$$\begin{aligned} h &= r_1(\sin\alpha + \sin\beta) + r_2(\sin\beta + \sin\gamma) \\ &= r_1[\sin\alpha + \sin\beta + (\lambda - 1)(\sin\beta + \sin\gamma)] \quad (10) \\ &= r_1[\sin\alpha + \lambda\sin\beta + (\lambda - 1)\sin\gamma] \end{aligned}$$

and

$$\begin{aligned} l &= 2\max\{r_1; (r_1 + r_2)\cos\beta - r_2\cos\gamma\} \\ &= 2\max\{r_1; \lambda r_1\cos\beta - (\lambda - 1) r_1\cos\gamma\} \end{aligned} \quad (11)$$

The total surface of the basic shape is obtained by summing the two surfaces obtained by rotating the circular arcs.

Let there be a circular arc (see Fig. 3) of equation:

$$x^2 + y^2 = r^2,\ x \in [x_1, x_2],\ -r \le x_1 \le x_2 \le r \quad (12)$$

Considering its explicit form:

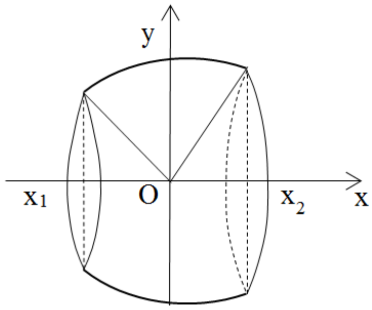


Fig.3. The rotation of a circular arc around the Ox axis

$$y = \sqrt{r^2 - x^2},\ x \in [x_1, x_2] \quad (13)$$

the area of the surface obtained by rotating around the axes Ox is:

$$A_1 = 2\pi \int_{x_1}^{x_2} f(x)\sqrt{1 + \left(f'(x)\right)^2} = 2\pi\left(x_2 - x_1\right) \quad (14)$$

The area of the surface obtained by rotating a circular arc from a circle that does not have its center in the origin can be similarly obtained. The total surface of the vase generated using circular arcs is:

$$A = 2\pi r_1^2 \{\sin\alpha + \lambda\sin\beta + (\lambda-1)\sin\gamma + \lambda\sin\beta\left[\arcsin((\lambda-1)\sin\gamma) - \arcsin((1-\lambda)\sin\beta)\right]\} \quad (15)$$

2.2 Problem formulation

The general purpose of the two optimizations conducted in this paper is to obtain a vase made out of basic shapes that is aesthetically optimal. In order to do this several qualitative aspects have been defined. The first optimization encompasses three aesthetic standards.

The first considered aesthetic standard refers to the height-width ratio which is desired to be the Golden Ratio:

$$\frac{h}{l} = \varphi,\ \varphi = \frac{1+\sqrt{5}}{2} \quad (16)$$

which can be rewritten in the following form:

$$\frac{\sin\alpha + \sin\beta + (\lambda-1)(\sin\beta + \sin\gamma)}{2\max\{1; \lambda\cos\beta - (\lambda-1)\cos\gamma\}} = \varphi \quad (17)$$

If $CC' \le FF'$ then the ratio becomes:

$$\frac{\sin\alpha + \sin\beta + (\lambda-1)(\sin\beta + \sin\gamma)}{2} = \varphi \quad (18)$$

A similar definition of the height-width ratio can be formulated for the case when $CC' > FF'$.

The second qualitative aspect that can be imposed is the ratio between the base diameter and the opening diameter to be equal to the Golden Ratio:

$$\frac{AA'}{CC'} = \varphi \Leftrightarrow \frac{O_1A}{O_2C} = \varphi \quad (19)$$

which can be rewritten:

$$\frac{\cos\alpha}{\lambda\cos\beta - (\lambda-1)\cos\gamma} = \varphi \quad (20)$$

The third aesthetic standard defined for the vase is the condition that the opening-neck ratio has to be equal to the Golden Ration:

$$\frac{CC'}{DD'} = \varphi \Leftrightarrow \frac{O_2C}{O_3D} = \varphi \quad (21)$$

which can be rewritten:

$$\frac{\lambda\cos\beta - (\lambda-1)\cos\gamma}{\lambda\cos\beta - \lambda + 1} = \varphi \quad (22)$$

The second optimization aims not only to obtain a basic shape that is aesthetically optimal but also to ensure that the obtained shape is economical. Consequently, the criterion of minimum surface has been defined:

$$\min_{\alpha,\beta,\gamma,\lambda} A \quad (23)$$

where A is given by the equation (15).

3. AESTHETIC OPTIMIZATION

3.1 Optimization using evolutionary algorithms

The advantage of the evolutionary algorithms is given by the fact that they mimic the best elements met in nature, especially in the biological systems that have evolved over so many years through natural selection. Two main features that stand out are environmental adaptation and survival of the best. One of these algorithms is Cuckoo Search developed in 2009 by Xin-She Yang and Suash Deb [9] and based on the so-called Lévy flights. This algorithm was used to conduct the optimizations proposed in this paper.

Cuckoo Search based on Lévy flights was implemented in MATLAB R2009b and can be described using the following three rules [9]:

1. Each cuckoo lays one egg at a time, and dumps it in a randomly chosen nest using Lévy flights;
2. The best nests with high-quality eggs will be carried over to the next generations;
3. The number of available host nests is fixed, and the egg laid by a cuckoo is discovered by the host bird with a probability $pa \in (0, 1)$. In this case, the host bird can either get rid of the egg, or simply abandon the nest and build a completely new nest.

The program starts with generating an initial population of n random cuckoos within the limits of the search space representing the possible configurations of vases. A cuckoo is a vector of four variables whose values are necessary for drawing the vase (see 3.2). For every generation each cuckoo is evaluated using the objective function and the best cuckoo is found.

The migration of the cuckoos towards the area with the best basic shapes with respect to the chosen aesthetic criteria is implemented in a repetitive structure and it is based on the three already mentioned rules. It has to be mentioned that each new cuckoo chick which was obtained by laying an egg in a nest, replaces its parent in the population only if it is better than its parent.

The migration ends when the maximum allowed number of evaluations of the objective function $Nmax$ is reached and the optimal vase with respect to the chosen aesthetic criteria is returned as solution.

3.2 Objective functions

Due to the fact that the vase is generated by symmetry the constraint that restricts the circular arcs from intersecting the Oy axis given by the equation (9) has to be considered in both optimizations.

For the first optimization that is focused on achieving aesthetic standards, the objective function that is minimized is given by:

$$f_{01}(\alpha,\beta,\gamma,\lambda)=\begin{cases}\left(f_1(\alpha,\beta,\gamma,\lambda)-\varphi\right)^2+\left(f_2(\alpha,\beta,\gamma,\lambda)-\varphi\right)^2 & \text{if } f_3(\beta,\lambda)\le 0\\ \left(f_1(\alpha,\beta,\gamma,\lambda)-\varphi\right)^2+\left(f_2(\alpha,\beta,\gamma,\lambda)-\varphi\right)^2+\Lambda\cdot f_3(\beta,\lambda) & \text{if } f_3(\beta,\lambda)>0\end{cases} \quad (24)$$

where

$$f_1(\alpha,\beta,\gamma,\lambda)=\frac{\sin\alpha+\sin\beta+(\lambda-1)(\sin\beta+\sin\gamma)}{2\max\{1;\lambda\cos\beta-(\lambda-1)\cos\gamma\}} \quad (25)$$

$$f_2(\alpha,\beta,\gamma,\lambda)=\frac{\cos\alpha}{\lambda\cos\beta-(\lambda-1)\cos\gamma} \quad (26)$$

$$f_3(\beta,\lambda)=1-\cos\beta-\frac{1}{\lambda} \quad (27)$$

The symbol Λ is used for the penalty factor that directs the search of the optimal solution towards the shapes that do not intersect the vertical axis.

Besides the three aesthetic aspects, the second optimization encompasses the minimization of the total surface of the vase. Consequently, the objective function that is minimized becomes:

$$f_{02}(\alpha,\beta,\gamma,\lambda)=\mu\cdot f_{01}(\alpha,\beta,\gamma,\lambda)+f_4(\alpha,\beta,\gamma,\lambda) \quad (28)$$

where

$$f_4(\alpha,\beta,\gamma,\lambda)=\sin\alpha+\lambda\sin\beta+(\lambda-1)\sin\gamma+\lambda\sin\beta\left[\arcsin\left((\lambda-1)\sin\gamma\right)-\arcsin\left((1-\lambda)\sin\beta\right)\right] \quad (29)$$

and μ is a weight factor whose purpose is to ensure that only the shapes that are optimal with respect to the three aesthetic criteria are found and from these shapes the one with the minimum surface is chosen.

3.3 Optimization results

Both optimization problems were formulated as minimization problems. The parameters required for the optimizations are given in Table 1 and the search domains for each variable are provided in Table 2.

Table 1

Optimization parameters

Parameter	Significance	Value
r_1	The radius of the first circle	0.5
Λ	Penalty factor	10^{10}
μ	Weight factor	10^5
n	The number of cuckoos	25
$Nmax$	The maximum allowed number of objective function evaluations	300,000
pa	The discovery probability	0.25

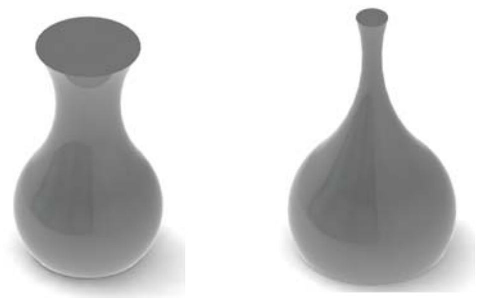
The optimization based on the three aesthetic standards defined using the Golden

Ratio finds an infinity of basic shapes that are aesthetically optimal. Their shape varies from vases with a wide neck and opening as it can be seen in Fig. 4a) to vases with quite a narrow neck and opening as shown in Fig. 4b). However, all the obtained vases have the height-width ratio, the base diameter-opening diameter ratio, and the opening-neck ratio equal to the Golden Ratio. Choosing just one vase from the infinity of optimal vases can be made using the human factor.

Table 2

Search domains for optimization variables

Variable	Search domain
α	$\left[\frac{\pi}{8},\frac{\pi}{4}\right]$
β	$\left[0,\frac{\pi}{3}\right]$
γ	$\left[0,\frac{\pi}{2}\right]$
λ	$[1,5]$



a) b)

Fig. 4. The results of the first optimization

Fig. 5. The result of the second optimization

The second optimization can be seen as a search for the vase between the optimal vases with respect to the aesthetic standards that is more economical to be manufactured. The solution is presented in Fig. 5. It has to be mentioned here that different aesthetic standards even if they are defined using the same Golden Ratio lead to different basic shapes.

4. CONCLUSION

The Golden Ratio, defined by a proportion corresponding to the geometric mean, has been used to define aesthetic standards by many artists and scientists. Using the Golden Ratio the problem of obtaining the aesthetically optimal basic shape is formulated in this paper as a minimization problem of four variables and then solved. The proposed mathematical model allows formulating easily other aesthetic standards for the given basic shape using the Golden Ratio or other sacred geometry constant.

The optimization model can be adapted without difficulty for a different shape or for using more aesthetic criteria. Actually if one wants to obtain a single optimal vase with respect only to the aesthetic criteria and not let the human factor decide, then other standards have to be formulated. However, in this paper, we proposed a different approach for choosing one optimal vase. It did not consist in letting the human factor decide, nor in defining a new aesthetic standard, but in formulating an economical criterion which is suitable for nowadays industrial trend.

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OPTIMIZAREA ESTETICĂ A UNEI FORME DE BAZĂ

Abstract: *Un aspect important în industria zilelor noastre este obținerea unor produse finite care sunt optime în raport cu diferite criterii. Criteriul estetic este unul esential care a fost studiat încă din Antichitate de diferiți artiști și oameni de știință. În această lucrare propunem două modele de optimizare estetică a unei forme de bază folosind algoritmi evolutivi. Funcțiile obiectiv sunt formulate folosind numărul de aur drept standard estetic.*

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