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CONTRIBUTIONS TO THE STUDY OF THE MASSES CHAOTIC MOVEMENTS OF A MECHANICAL SYSTEM

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Abstract: *The well known mechanical model of Atwood's machine was used for the experimental determination of gravitational acceleration. Considering the generalized model of this machine, a study was made to assess the chaotic behavior of the mechanical movement of its masses. The theoretical background is presented and also some numerical results, calculated with MatLab software, for different values of the mechanical characteristics of this machine.*
Key words: *swinging Atwood's machine, Lagrange's equations, nonlinear differential equations, Runge-Kutta method, chaos*

1. INTRODUCTION

The swinging Atwood's machine is a mechanical system with two degrees of freedom, derived from the well-known simple Atwood machine. The latter was built in 1784 by George Atwood, to determine the value of Earth gravitational acceleration [1], [3], [11].

In the Atwood's machine, two masses are mechanically linked by an inextensible thread and a pulley, whereas in the swinging Atwood's machine one of the mass (m_2) is allowed to swing in a plane, while the other mass plays the role of a counterweight (m_1), executing vertical movements (Fig.1).

The masses of the swinging Atwood's machine can perform periodic, quasiperiodic and chaotic movements. As known, if the movements are described by nonlinear differential equations, in some cases the trajectories depend on the initial conditions and the chaotic movements can appear.

Research on the swinging Atwood's machine started in 1982 as part of the thesis develop by Nicholas Tufillaro at Reed College, referring to the shape of some trajectories of the system. The chaotic movements were also studied in [2], [5], [6], [7], [8], [9], considering this mechanical model.

2. THEORETICAL BACKGROUND

The mechanical system studied in this paper is composed by two masses: m_1 and m_2, connected by an inextensible thread passed over two pulleys with moments of inertia J_1 and J_2, and radii r_1 respectively r_2.

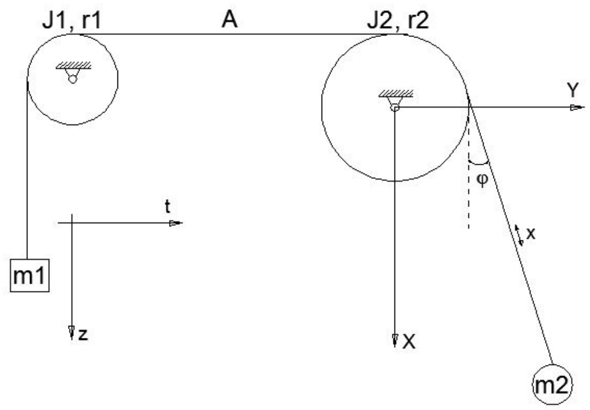

Fig.1 The swinging Atwood's machine

As shown in Fig.1, the generalized coordinates are: the displacement x of the mass m_2 and the angle φ between the thread and the vertical direction. We consider the following initial conditions:

$$t = 0 \ , \ x = x_0 \ , \ \varphi = \varphi_0 \ , \ \dot{x} = \dot{x}_0 \ , \ \dot{\varphi} = \dot{\varphi}_0 \ .$$

From the following relation of calculus for the thread length:

$$\ell = z + \frac{\pi}{2} r_1 + A + \left(\frac{\pi}{2} - \varphi \right) r_2 + x,$$

one may obtain the length of the thread in vertical position, noted with z:

$$z = \underbrace{\ell - \frac{\pi}{2}\left(r_1 + r_2\right) - A}_{c} - x + r_2\,\varphi = \\ = c - x + r_2\,\varphi, \quad \dot{z} = -\dot{x} + r_2\,\dot{\varphi} \tag{1}$$

with the initial value:

$z_0 = c - x_0 + r_2\,\varphi_0$.

The expressions of the potential energies, corresponding to the masses m_1 and m_2, are:

$$E_{p1} = m_1 g(z_0 - z) = \\ = m_1 g(x - x_0 - r_2\varphi + r_2\varphi_0) \tag{2}$$

$$E_{p2} = m_2\,g\,(X_0 - X) \tag{3}$$

where the swinging mass abscissa expressions are:

$$X = -r_2 \sin\varphi + x\cos\varphi, \\ X_0 = -r_2 \sin\varphi_0 + x_0 \cos\varphi_0, \tag{4}$$

The expression of the potential energy E_{p2} becomes:

$$E_{p2} = m_2\,g\,(-r_2 \sin\varphi_0 + x_0 \cos\varphi_0 + \\ + r_2 \sin\varphi - x\cos\varphi) \tag{5}$$

By summing the two expressions (2) and (5), the final form of the mechanical system potential energy is:

$$E_p = m_1 g\left[x - x_0 + r_2(\varphi_0 - \varphi)\right] + m_2\,g\cdot \\ \cdot\left[r_2(\sin\varphi - \sin\varphi_0) - x\cos\varphi + x_0\cos\varphi_0\right] \tag{6}$$

The kinetic energies of the mechanical system elements are calculated as follows:

$$E_{c1} = \frac{1}{2} m_1\,\dot{z}^2 = \\ = \frac{1}{2} m_1\left(\dot{x}^2 + r_2^2\dot{\varphi}^2 - 2\,r_2\,\dot{x}\,\dot{\varphi}\right) \tag{7}$$

$$E_{c2} = \frac{1}{2}\left(J_1\,\omega_1^2 + J_2\,\omega_2^2\right) \tag{8}$$

With $\omega_1 = \frac{\dot{z}}{r_1}$, $\omega_2 = \frac{\dot{z}}{r_2}$, E_{c2} becomes:

$$E_{c2} = \frac{1}{2}\left(\frac{J_1}{r_1^2} + \frac{J_2}{r_2^2}\right)\left(\dot{x}^2 + r_2^2\,\dot{\varphi}^2 - 2 r_2\,\dot{x}\,\dot{\varphi}\right) \tag{9}$$

The third expression of the kinetic energy is:

$$E_{c3} = \frac{1}{2} m_2\left(\dot{X}^2 + \dot{Y}^2\right) \tag{10}$$

Considering the expressions of X and Y and their derivatives with respect to the time:

$$X = -r_2 \sin\varphi + x\cos\varphi \\ \dot{X} = -r_2\,\dot{\varphi}\cos\varphi + \dot{x}\cos\varphi - x\,\dot{\varphi}\sin\varphi \tag{11}$$

$$Y = r_2\cos\varphi + x\sin\varphi \\ \dot{Y} = -r_2\,\dot{\varphi}\sin\varphi + \dot{x}\sin\varphi + x\,\dot{\varphi}\cos\varphi \tag{12}$$

equation (10) becomes:

$$E_{c3} = \frac{1}{2} m_2\left(\dot{x}^2 + r_2^2\,\dot{\varphi}^2 - 2\,r_2\dot{x}\dot{\varphi} + x^2\dot{\varphi}^2\right) \tag{13}$$

From (7), (8) and (13), the final expression of the system's kinetic energy becomes:

$$E_c = \frac{1}{2}\left[\left(m_1 + m_2 + \frac{J_1}{r_1^2} + \frac{J_2}{r_2^2}\right)\cdot \\ \cdot\left(\dot{x}^2 + r_2^2\,\dot{\varphi}^2 - 2 r_2\dot{x}\dot{\varphi}\right) + m_2\,x^2\dot{\varphi}^2\right] \tag{14}$$

The differential equations that model the mechanical system movements will be established using the method of Lagrange's equations. The Lagrangian is:

$$L = E_c - E_p =$$
$$= \frac{1}{2}\left[\left(m_1 + m_2 + \frac{J_1}{r_1^2} + \frac{J_2}{r_2^2}\right)\left(\dot{x}^2 + r_2^2\dot{\varphi}^2 - 2r_2\dot{x}\dot{\varphi}\right) + m_2 x^2 \dot{\varphi}^2\right] -$$
$$-m_1\, g\left[x - x_0 + r_2(\varphi_0 - \varphi)\right] -$$
$$-m_2\, g\left[r_2(\sin\varphi - \sin\varphi_0) - x\cos\varphi + x_0\cos\varphi_0\right] \quad (15)$$

and the two Lagrange's equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0\,, \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\varphi}}\right) - \frac{\partial L}{\partial \varphi} = 0 \quad (16)$$

will be:

$$\begin{cases} \left(m_1 + m_2 + \frac{J_1}{r_1^2} + \frac{J_2}{r_2^2}\right)\ddot{x} - r_2\left(m_1 + m_2 + \frac{J_1}{r_1^2} + \frac{J_2}{r_2^2}\right)\ddot{\varphi} = \\ -m_1\, g + m_2\, g\, \cos\varphi, \\ \\ -r_2\left(m_1 + m_2 + \frac{J_1}{r_1^2} + \frac{J_2}{r_2^2}\right)\ddot{x} + \\ + \left[m_2 x^2 + \left(m_1 + m_2 + \frac{J_1}{r_1^2} + \frac{J_2}{r_2^2}\right) r_2^2\right]\ddot{\varphi} \\ = -2m_2\, x\dot{x}\dot{\varphi} + m_1\, g\, r_2 - m_2\, g\,(r_2\cos\varphi + x\sin\varphi) \end{cases} \quad (17)$$

with the previously specified initial conditions.

Each equation contains the second derivatives of generalized coordinates, so the system can be written as follows:

$$\begin{cases} a_{11}\ \ddot{x} + a_{12}\ \ddot{\varphi} = b_1 \\ a_{21}\ \ddot{x} + a_{22}\ \ddot{\varphi} = b_2 \end{cases}, \Delta = a_{11}\ a_{22} - a_{12}\ a_{21} \quad (18)$$

where

$$a_{11} = m_1 + m_2 + \frac{J_1}{r_1^2} + \frac{J_2}{r_2^2}\ ,$$
$$a_{12} = -\, r_2\left(m_1 + m_2 + \frac{J_1}{r_1^2} + \frac{J_2}{r_2^2}\right) \quad (19)$$

$$a_{21} = a_{12}\ ,$$
$$a_{22} = m_2\, x^2 + \left(m_1 + m_2 + \frac{J_1}{r_1^2} + \frac{J_2}{r_2^2}\right) r_2^2 \quad (20)$$

$$b_1 = -\, m_1\, g + m_2\, g\, \cos\varphi \quad (21)$$

$$b_2 = -2\, m_2\, x\dot{x}\dot{\varphi} + m_1\, g\, r_2 -$$
$$-\, m_2\, g\,(r_2\cos\varphi + x\sin\varphi) \quad (22)$$

By making the following notations in the linear system (18): $y_1 = x$, $y_2 = \varphi$, $y_3 = \dot{x}$, $y_4 = \dot{\varphi}$, one obtains a system of four first order differential equations:

$$\begin{cases} \dot{y}_1 = y_3 \\ \dot{y}_2 = y_4 \\ \dot{y}_3 = \frac{b_1\ a_{22} - b_2\ a_{12}}{\Delta} \\ \dot{y}_4 = \frac{b_2\ a_{11} - b_1\ a_{21}}{\Delta} \end{cases} \quad (23)$$

with the initial conditions $t = 0$, $x = x_0$, $\varphi = \varphi_0$, $\dot{x} = \dot{x}_0$, $\dot{\varphi} = \dot{\varphi}_0$. This system was numerically solved using MatLab software [4].

Four particular cases were considered:
- The moments of inertia J_1 and J_2 are null;
- J_1 and J_2 are different from zero;
- J_1 is equal to zero and J_2 is greater than zero;
- J_1 is greater than zero and J_2 is equal to zero.

In all four cases, it was considered that $m_1 \neq m_2$.

3. NUMERICAL RESULTS

The system of differential equations (23) has been repeatedly integrated considering different values for the mechanical system elements (values of masses, pulley radii and moments of inertia) and also for the initial conditions.

In order to obtain the numerical solutions, we used MatLab software with "ode45" function that integrates differential equation systems automatically choosing the iteration step value.

From the obtained results, some were selected and presented in figures from 2 to 9. Two diagrams are contained in each figure: the first one represents the variation in time of the vertical displacement of mass m1, and the second one the trajectory in vertical plane of the mass m2.

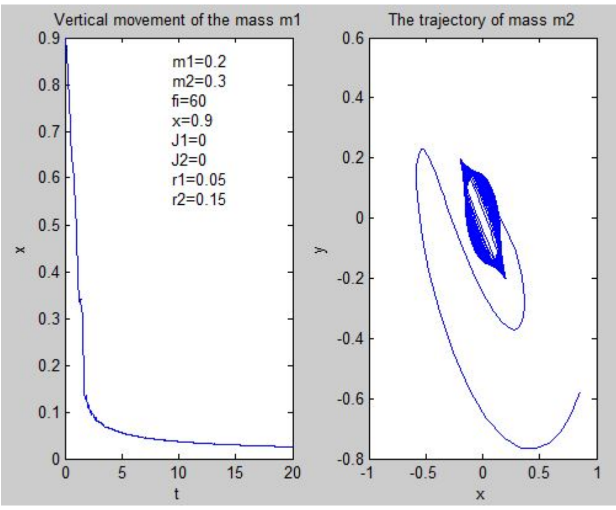

Fig.2 Numerical trajectories obtained by solving the system of equations (23). The displacement x = 0.9m and the angle φ = 60°

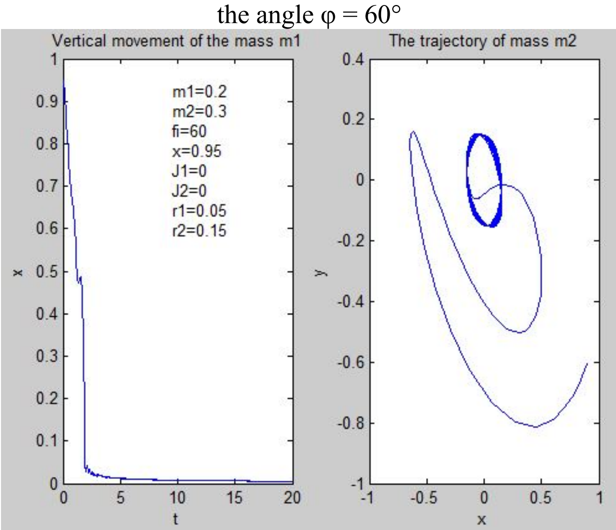

Fig.3 Numerical trajectories obtained by solving the system of equations (23). The displacement x = 0.95m and the angle φ = 60°

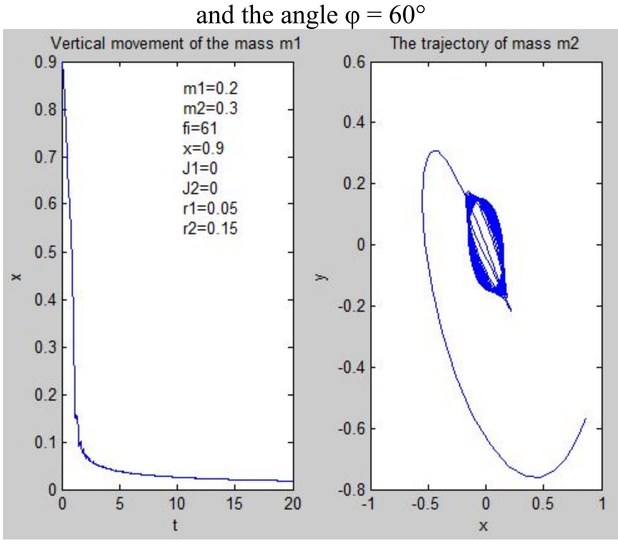

Fig.4 Numerical trajectories obtained by solving the system of equations (23). The displacement x = 0.9m and the angle φ = 61°

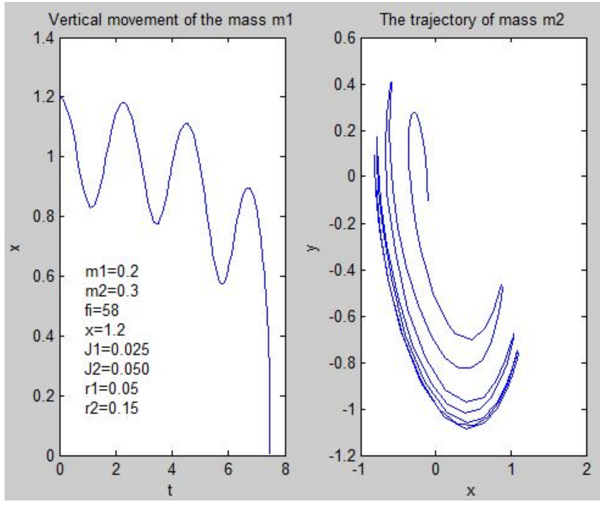

Fig.5 Numerical trajectories obtained by solving the system of equations (23). The displacement x = 1.2m and the angle φ = 58°

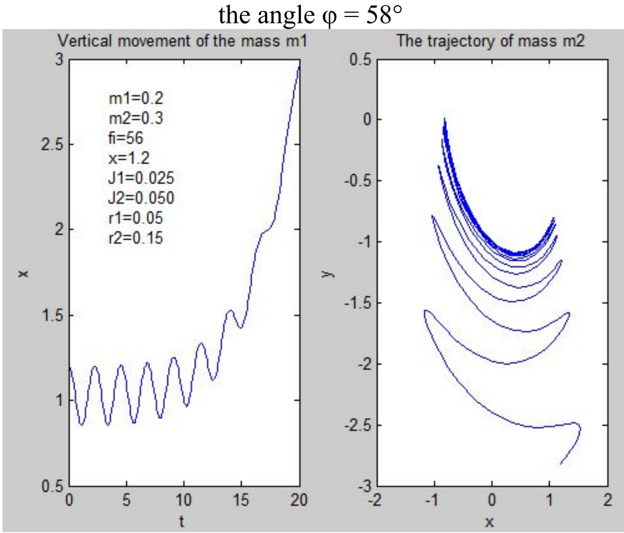

Fig.6 Numerical trajectories obtained by solving the system of equations (23). The displacement x = 1.2m and the angle φ = 56°

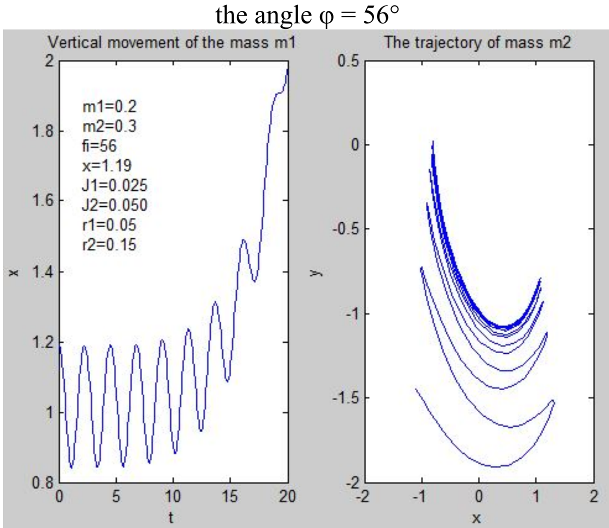

Fig.7 Numerical trajectories obtained by solving the system of equations (23). The displacement x = 1.19m and the angle φ = 56°

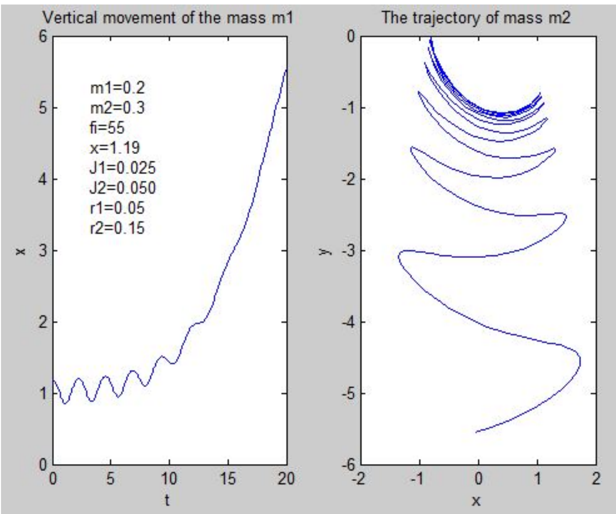

Fig.8 Numerical trajectories obtained by solving the system of equations (23). The displacement x = 1.19m and the angle φ = 55°

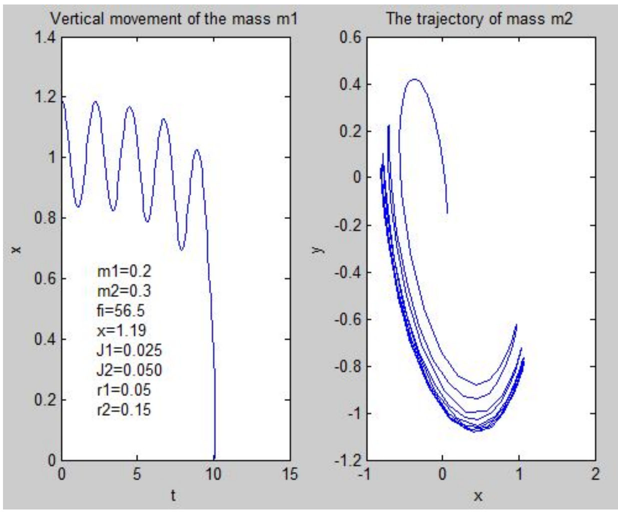

Fig.9 Numerical trajectories obtained by solving the system of equations (23). The displacement x = 1.19m and the angle φ = 56.5°

Analyzing the trajectory shapes of mass m_2 and the displacement of mass m_1, we can notice their dependence on the initial conditions and substantial changes of trajectories and laws of motion based on small variations of mechanical element values.

4. CONCLUSIONS

The swinging Atwood's differs from the simple one by the fact that one of the masses is allowed to swing in a two-dimensional plane, producing a chaotic dynamical movement for some system parameters and initial conditions.

In all the diagrams presented we have taken into account that $m_1 < m_2$ and that the mass ratio m_2/m_1 is 1.5. The rays of the two pulleys remain unchanged for all presented cases: r_1 = 0.05 m respectively r_2 = 0.15 m.

In figures 2, 3 and 4, the displacement x and the angle φ between the thread and the vertical direction vary between 0.9m and 0.95m respectively 60° - 61° resulting the trajectory of mass m_2 with limit cycles. In these cases moments of inertia, J_1 and J_2, are equal to zero.

In figures from 5 to 9, the displacement x takes values between 1.19m and 1.2m, and the angle φ takes values between 55° and 58°. The obtained diagrams show periodic and non-periodic oscillations of the mass m_2. In these cases moments of inertia are: J_1 = 0.025 kg·m^2 and J_2 = 0.050 kg·m^2.

Analyzing the diagrams representing the movement of the swinging Atwood's machine's masses, one can notice that the initial conditions play a very important role regarding the shape of the trajectory for both masses, this being characteristic to the chaotic movements of the mechanical systems.

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Contribuţii la studiul mişcărilor haotice ale corpurilor unui sistem mecanic

Abstract: Modelul mecanic, denumit "Maşina Atwood" a fost folosit pentru determinarea experimentală a acceleraţiei gravitaţionale. Dacă se ia în considerare modelul generalizat al acestui mecanism, se poate determina comportamentul haotic al mişcării celor două mase ale sale. Sunt prezentate atât partea teoretică cât şi rezultate numerice obţinute prin rezolvare cu ajutorul soft-ului Matlab, pentru diferite valori ale caracteristicilor mecanice ale mecanismului.

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