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POSITION PROBLEM FOR 3RPR 3-MOBILE PLAN PARALLEL MANIPULATOR

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Abstract: *The paper presents the inverse and direct problem for 3RPR manipulator. The inverse positions problem is analytically solved. The direct problem is solved both analytically and numerically. An original method using a computer, for establishing the possibilities for manipulator assembling is also presented.*
Key words: *parallel manipulator, inverse position problem, direct position problem, mobile platform, Cartesian equations for mobile guiding curves.*

1. INTRODUCTION

Fig. 1 presents the kinematic scheme of 3RRR plane manipulator characterized by 3 degrees of freedom and three identical kinematic chains [1]. The symbolic acronym of this mechanism is connected with his characteristics: 3 – number of degrees of freedom, RPR – the type of successive kinematic joints contained in one kinematic chain, starting from the base till the final element (R – rotation, P- prismatic, R-rotation). The passive rotation joints near the basis are displayed in the Bi vertices of an equilateral triangle (B_1,B_2,B_3), having the edge b and the radius of the circumscribed circle g_i =g, i =1,2,3. In order to study the mechanism a fixed coordinate system OXY has been set having the origin O in the centre of the circumscribed triangle of the basic platform. The OXY plane was established to contain the platform, and the OX axis should be orthogonal on the edge B_1B_3. Another Cartesian mobile system, Pxy connected to the mobile platform, has been considered. Its origin P is in the centre of the circumscribed circle of the mobile triangular platform and the Px axis is orthogonal on the edge A_1A_3.

Generalized coordinates of the mechanism are q_i, the linear displacements from the active joints, (i =1,2,3). Generalized coordinates of the mobile platform are X_P, Y_P - the Cartesian coordinates of the point P in regard with the fixed reference system OXY and ϕ the angle giving the orientation of the mobile system Pxy.

The angles α_i are determined by the kinematic chains having variable length A_iB_i and OX axis.

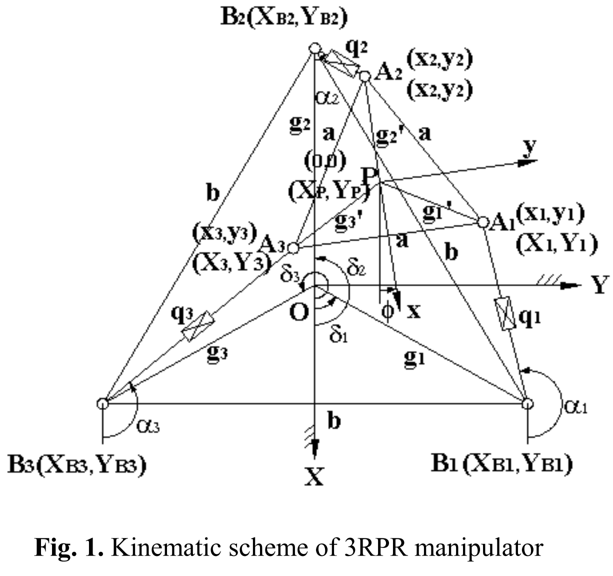

Fig. 1. Kinematic scheme of 3RPR manipulator

The centres of the rotation joints are located in the vertices A_i of the triangular mobile platform having an edge and being inscribed in a circle with radius $g_i'=g'$, i=1,2,3. By varying the coordinates q_i, i=1,2,3, the manipulated object can be positioned in relation with the manipulation operation phases

2. INPUT-OUTPUT EQUATIONS SYSTEM

The first thing to do in the study of a parallel mechanism is the determination of its positions at a certain moment. One may do it if the *input-output equations* are established [1]. These equations contain two kinds of parameters: *kinematic parameters* – generalized coordinates of the mechanism and manipulated object and *geometric* ones – the constructive dimensions of the mechanism components. The analytic method for establishing the input-output equations system for the 3RPR manipulator consists in the following steps:

a) Obtaining the analytical expressions of Cartesian coordinates of guided points A_i (X_i, Y_i) i=1,2,3, in respect with the fixed reference system (OXY), using the generalized coordinates of the mechanism - q_i, curvilinear coordinates α_i of guided points A_i and constructive geometrical parameters of the manipulator:

$$\begin{cases} X_i(q_i,\alpha_i) = g_i\cos\delta_i + q_i\cos\alpha_i \\ Y_i(q_i,\alpha_i) = g_i\sin\delta_i + q_i\sin\alpha_i \end{cases}, i=1,2,3 \qquad (1)$$

b) Obtaining the analytical expressions of the Cartesian coordinates of the guided points A_i (X_i, Y_i), i=1,2,3 in respect with the same fixed reference system OXY, but using the generalized coordinates of the manipulated object (X_P, Y_P, Φ) and constructive – geometrical parameters of the manipulator(g_i' if δ_i'), base don the relation between the points A_i, P, and O:

$$\begin{cases} X_i(X_P,Y_P,\Phi) = X_P + (x_i - x_P)\alpha' + (y_i - y_P)\alpha'' \\ Y_i(X_P,Y_P,\Phi) = Y_P + (x_i - x_P)\beta' + (y_i - y_P)\beta'' \end{cases} \qquad (2)$$

where x_i, y_i, x_p, y_p, i=1,2,3 are the coordinates of the points A_i şi P in the mobile system and α′,β′,α″,β″ are directional cosines of PXY system axis in respect with OXY system.. Their expressions in function of Φ are presented by relations (3) and (4).

$$x_i = g_i'\cos\delta_i',\ y_i = g_i'\sin\delta_i',\ x_P = 0,\ y_P = 0 \qquad (3)$$

$$\begin{cases} \alpha' = \cos\Phi & \alpha'' = -\sin\Phi \\ \beta' = \sin\Phi & \beta'' = \cos\Phi \end{cases} \qquad (4)$$

Using (3) and (4) in (2), one may obtain finally:

$$\begin{cases} X_i(X_P,Y_P,\Phi) = X_P + g_i'\cos(\delta_i' + \Phi) \\ Y_i(X_P,Y_P,\Phi) = Y_P + g_i'\sin(\delta_i' + \Phi) \end{cases} \qquad (5)$$

c) By equating relations (1) and (5) a system of six equations is obtained containing as variables (q_i ,α_i, i=1,2,3, X_p, Y_p, Φ) defined as the system of position equations for the analysed mechanism.

$$\begin{cases} X_i(q_i,\alpha_i,X_P,\Phi) = 0, \\ Y_i(q_i,\alpha_i,Y_P,\Phi) = 0 \quad i=1,2,3 \end{cases} \quad \text{id est.}$$

$$\begin{cases} g_i\cos\delta_i + q_i\cos\alpha_i - X_P - g_i'\cos(\delta_i' + \Phi) = 0 \\ g_i\sin\delta_i + q_i\sin\alpha_i - Y_P - g_i'\sin(\delta_i' + \Phi) = 0 \end{cases} \qquad (6)$$

This system can be simplified if the equations of *mobile guiding curves* are used. These can be obtained by eliminating the relative curvilinear coordinates α_i from equations (1).

$$F_i = (q_i, X_i, Y_i) = 0 \quad i = 1,2,3 \quad \text{namely}$$

$$(X_i - g_i c\delta_i)^2 + (Y_i - g_i s\delta_i)^2 - q_i^2 = 0, \qquad (7)$$

Imposing the condition that the coordinates of the guided points (A_i) given by the relation (5) to verify the equations of mobile guiding curves namely (7) and obtains finally a system of three equations as the *input – output equations system*.

$$f_i(q_i, X_p, Y_p, \Phi) = 0, \quad i = 1,2,3 \quad \text{namely,}$$

$$[X_p + g_i'\cos(\delta_i'+\Phi) - g_i\cos\delta_i]^2 + \\ [Y_p + g_i'\sin(\delta_i'+\Phi) - g_i\sin\delta_i]^2 - q_i^{\ 2} = 0 \quad (8)$$

For the 3RPR manipulator presented in fig. 1, taken into account its particularities (9):

$$\begin{cases} g_i = g = (\sqrt{3}/3)b, & g_i' = g' = (\sqrt{3}/3)a \\ q_{\min} = \frac{\sqrt{3}}{3}(b-a) - \frac{a}{2} & q_{\max} = \frac{\sqrt{3}}{3}(b-a) + \frac{a}{2} \end{cases} \quad (9)$$

the following system of input – output equations is obtained:

$$[X_p + g'\cos(\delta_i'+\Phi) - g\cos\delta_i]^2 + \\ [Y_p + g'\sin(\delta_i'+\Phi) - g\sin\delta_i]^2 - q_i^2 = 0 \quad (10)$$

3. THE INVERSE POSITION PROBLEM

Solving the inverse position problem of 3RPR manipulator consists in obtaining the generalized coordinates of the mechanism q_i, i=1,2,3 knowing the generalized coordinates of the manipulated object X_P, Y_P, Φ.

For the 3RPR manipulator one may start from the input –output equations, relations (8) written in the form of (11):

$$q_i = \sqrt{\begin{matrix}[X_P + g_i^/\cos(\delta_i^/ + \Phi) - g_i\cos\delta_i]^2 + \\ [Y_P + g_i^/\sin(\delta_i^/ + \Phi) - g_i\sin\delta_i]^2\end{matrix}} \quad (11)$$

Relations (11) represent the solutions of the inverse positions problem and so it is solved analytically.

The curvilinear relative coordinates from (1) are:

$$\tan\alpha_i = \frac{Y_i - Y_{Bi}}{X_i - X_{Bi}}$$

$$\alpha_i = \arctan\frac{Y_P + g_i^/\sin(\delta_i^/ + \Phi) - g_i\sin\delta_i}{X_P + g_i^/\cos(\delta_i^/ + \Phi) - g_i\cos\delta_i} \quad (12)$$

For the 3RPR manipulator having the following geometrical-constructive configuration:

$$a = 200mm \quad b = 400mm \\ \delta_1 = \delta_1^/ = 60^0 \ \delta_2 = \delta_2^/ = 180^0 \ \delta_3 = \delta_3^/ = 300^0 \quad (13)$$

In the Table 1, the results of numerical solving of the inverse position problem are presented.

Table 1

No.	X_P [mm]	Y_P [mm]	Φ [°]	q_i [mm]	
1.	80	50	10	1.	41.69
				2.	199.49
				3.	166.13
2.	-50	-25	30	1.	198.44
				2.	115.74
				3.	134.30
3.	50	-25	-45	1.	200.48
				2.	207.19
				3.	114.54

4. THE DIRECT POSITION PROBLEM

The direct position problem for the parallel manipulators was intensively studied in the last period of time.

The central problem is that of determining of all constructive solutions for a precise mechanism thus the maximum number of assembly modes corresponding to a set of input variables, starting from a non-linear system of closing equations.

The analytical methods are difficult to be applied, but are attractive to researchers due to the facilities offered by symbolic calculus performed with specialized computer software [2].

In any configuration of a plane-parallel mechanism, solving the direct positional problem consists in obtaining the generalized coordinates of manipulated object X_P, Y_P, Φ knowing the generalized coordinates of the mechanism - q_i ,i=1,2,3.

4.1. The numerical method

In case of the 3RPR manipulator the method applies the implicit Cartesian equations of the mobile guiding curves (7) to which a rigid constraint written in OXY system, between points A_i of the mobile platform is added.

$$\overline{A_i A_{i+1}} = \overline{cst.}, \quad i = 1,2; \quad i = 3 \rightarrow i + 1 = 1 \qquad (14)$$

The system (15) is obtained:

$$\begin{cases} (X_i - g_i \cos\delta_i)^2 + (Y_i - g_i \sin\delta_i)^2 - q_i^2 = 0 \\ (X_{i+1} - X_i)^2 + (Y_{i+1} - Y_i)^2 = a^2 \end{cases} \qquad (15)$$

This system with six equations and six unknown (X_i, Y_i, i=1, 2, 3) is non-linear and by solving it with numerical methods, the instantaneous values of unknown variables may be found.

The coordinates of the point $P(X_P,Y_P)$ in the OXY system can be computed in relation with the coordinates of guided points in respect with the same system using relations (16):

$$X_P = \frac{\sum_{i=1}^{n} X_i}{3}, \quad Y_P = \frac{\sum_{i=1}^{n} Y_i}{3}, \qquad (16)$$

From relation (5) equation (17) for calculating angle Φ can be obtained:

$$\Phi = \arctan\frac{Y_i - Y_P}{X_i - X_P} - \delta_i \qquad (17)$$

Angular movements within the passive joints B_i i=1,2,3 can be computed using relations (12). In case of 3RPR manipulators having the configuration given by (13) the results of numerical solving the direct problem are displayed in table 2.

The values presented in Table 1 – generalized coordinates of the mechanism - q_i, i=1,2,3 obtained in inverse position problem were considered as input data.

One may observe from the table that the values for generalized coordinates for the manipulated object (or mobile platform) X_P, Y_P, Φ are identical with those from table 1.

Table 2

No.	i	q_i [mm]	X_i [mm]	Y_i [mm]	X_P [mm]	Y_P [mm]	Φ [^{0}]
1	1	41.69	119.46	158.49	80	50	10
	2	199.49	-33.7	29.89			
	3	166.13	154.25	-38.45			
2	1	198.44	-50	90.46	-50	-25	30
	2	115.74	-150	-82.73			
	3	134.3	49.99	-82.73			
3	1	200.48	161.53	4.87	50	-25	-45
	2	207.19	-31.67	56.64			
	3	114.54	20.11	-136.53			

4.2. The analytical method (polynomial)

The procedure to carry on the mathematical model to solve the positions direct problem for the 3RPR manipulator is presented here:

The variable length of the three segments $A_iB_i = q_i$ are expressed in relation with the unknown variables X_3, Y_3 namely the coordinates of the point A_3. The coordinates of the points A_1 and A_2 are also expressed in relation with variables X_3 and Y_3 as follows:

$$\begin{cases} X_1 = X_3 - a\sin\phi \\ Y_1 = Y_3 + a\cos\phi \end{cases} \quad \begin{cases} X_2 = X_3 - a\sin(\phi + 60) \\ Y_2 = Y_3 + a\cos(\phi + 60) \end{cases} \qquad (18)$$

$$q_3^2 = X_3^2 + Y_3^2 - 2g_3\cos\delta_3 X_3 - 2g_3\sin\delta_3 Y_3 + g_3^2$$

$$q_3^2 = X_3^2 + Y_3^2 + AX_3 + BY_3 + C \qquad (19)$$

$$q_1^2 = X_3^2 + Y_3^2 - 2(a\sin\Phi + q_1\cos\delta_1)X_3 + 2(a\cos\Phi + q_1\sin\delta_1)Y_3 + a^2 + g_1^2 + 2ag_1\sin(\Phi - \delta_1)$$

$$q_1{}^2 = X_3{}^2 + Y_3{}^2 + DX_3 + EY_3 + F \qquad (20)$$

$$q_2^2 = -2(a\sin\Phi\cos 60 + a\cos\Phi\sin 60 + g_2\cos\delta_2)X_3 + 2(a\cos\Phi\cos 60 - a\sin 60\sin\Phi - g_2\sin\delta_2)Y_3 + 2ag_2\cos\delta_2(\sin\Phi\cos 60 + \sin 60\cos\Phi) + a^2 + g_2^2 - 2ag_2\sin\delta_2 + X_3^2 + Y_3^2$$

$$q_2^2 = X_3^2 + Y_3^2 + GX_3 + HY_3 + I \qquad (21)$$

Subtracting relation (19) from (20) and (21) a linear system with two equations and two unknown variables X_3 and Y_3 is obtained:

$$q_1^2 - q_3^2 = (D - A)X_3 + (E - B)Y_3 + (F - C) = KX_3 + LY_3 + M \qquad (22)$$

$$q_2^2 - q_3^2 = (G - A)X_3 + (H - B)Y_3 + (I - C) = NX_3 + RY_3 + S \qquad (23)$$

If the system determinant is Δ=KR-LN, then the system solutions will be given by the relation (24).

$$X_3 = \frac{A_1 R - A_2 L}{\Delta} \qquad Y_3 = \frac{K A_2 - N A_1}{\Delta} \tag{24}$$

$$\begin{cases} A_1 = q_1^2 - q_3^2 - M \\ A_2 = q_2^2 - q_3^2 - S \end{cases}$$

Changing X_3 and Y_3 in relation (19) one obtains the equation (25) with the variable Φ.

$$\begin{aligned} &(A_1 R - A_2 L)^2 + (K A_2 - N A_1)^2 + \\ &(C - q_3^2)(KR - LN)^2 + \\ &A(A_1 R - A_2 L)(KR - LN) + \\ &B(K A_2 - N A_1)(KR - LN) = 0 \end{aligned} \tag{25}$$

The expressions used in equation (25) are as follows:

$$\begin{aligned} &K = -2a\sin\Phi - 2g_1\cos\delta_3 + 2g_3\cos\delta_3 \\ &L = 2a\cos\Phi - 2g_1\sin\delta_1 + 2g_3\sin\delta_3 \\ &M = a_2 + g_1^{\,2} - g_3^{\,2} + 2ag_1\sin\Phi\cos\delta_1 - \\ &\qquad 2ag_1\sin\delta\cos\Phi \\ &N = -2a\cos 60\sin\Phi - 2a\sin 60\cos\Phi - \\ &\qquad 2g_2\cos\delta_2 + 2g_3\cos\delta_3 \\ &R = 2a\cos 60\cos\Phi - 2a\sin 60\cos\Phi - \\ &\qquad 2g_2\cos\delta_2 + 2g_3\cos\delta_3 \\ &S = 2ag_2\cos\delta_2\cos 60\sin\Phi + \\ &\qquad 2ag_2\cos\delta_2\cos 60\cos\Phi - g_3^{\,2} + a_2 + g_2^{\,2} \\ &A_1 = q_1^{\,2} - q_3^{\,2} - a^2 - g_1^{\,2} + g_3^{\,2} - \\ &\qquad 2ag_1\cos\delta_1\sin\Phi + 2ag_1\sin\delta_1\cos\Phi \\ &A_2 = q_2^{\,2} - q_3^{\,2} - 2ag_2\cos\delta_2\cos 60\sin\Phi - \\ &\qquad 2ag_2\sin\delta_2\cos 60\cos\Phi - a^2 \end{aligned}$$

Introducing the above expressions in relation (25) after imposing to the kinematic chains, the length of the assembly position q1 = q2 = q3 = 115.47, one obtains relation (26).

$$\begin{aligned} &-57.344\sin^2\Phi + 16.38\sin^2\Phi\cos\Phi \\ &-82.944\cos\Phi + 82.944 = 0 \end{aligned} \tag{26}$$

Using also the following relations:

$$\sin\Phi = \frac{2t}{1+t^2}, \quad \cos\Phi = \frac{1-t^2}{1+t^2}, \quad \text{where } t = tg\frac{\Phi}{2},$$

equation (26) becomes a six-degree equation with variable *t*.

$$41.472\, t^6 + 9.22\, t^4 + 0.508\, t^2 = 0 \tag{27}$$

The variables X_3, and Y_3 can be calculated with relation (24), after determining the coordinates of the platform centre with equations (28)

$$\begin{cases} X_P = X_3 - g_3^{/}\cos(\delta_3^{/} + \Phi) \\ Y_P = Y_3 - g_3^{/}\sin(\delta_3^{/} + \Phi) \end{cases} \tag{28}$$

Analysing equation (27) one may observe that it has only one real double solution for $t = 0$ namely $\Phi = 0$. Consequently there is only one possibility for assembling the 3RPR manipulator, for which $\Phi = 0^0$ and the length of kinematic chains A_iB_i having the 115.47mm value.

4.3 Computer added method for appraisal the possibilities for assembling the manipulator

A software in Delphi environment was designed. It was meant to determine the number of assembly possibilities of 3RPR manipulator having a precise geometrical-constructive configuration given by the relation (13).

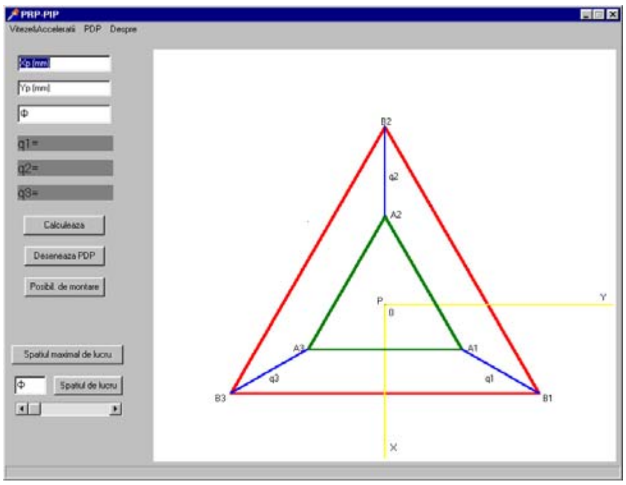

Fig. 2. Graphical interface of the software program manipulator 3RPR

P point belonging to the mobile platform scans the surface of a rectangle covering the robot maximal working space with a certain step on X and Y axis. For each point (X_P, Y_P) within the rectangle and Φ varying between Φ_{min} and Φ_{max}, with a certain step, the length values A_iB_i are calculated. The points corresponding to $q_1 = q_2 = q_3 = 115.47$ are selected. In this way all assembly possibilities of the mechanism are obtained and represented graphically as in fig. 2. As one may observe a single possibility was obtained due to the symmetry of mobile platform on one side , and due to symmetry of kinematic chains A_iB_i as length and orientation, on the other side.

5. CONCLUSIONS

The paper presents the calculus procedure for the inverse and direct problem of the positions of the 3RPR manipulator, based on input and output equations. Finally the following conclusions are envisaged:

- The inverse problem is relatively simple and the analytic solution can be found easily;
- The direct problem can be simplified to solving a nonlinear system of equations using numerical methods;
- The problem of determining all assembly possibilities for 3RPR manipulator, when a set of input values was adopted, has been reduced to solving a six degree equation with variable Φ (the orientation angle of the mobile platform in respect with the fix platform). The procedure for obtaining the equation used the closing equations for the kinematic chains and other manipulations that are presented in detail above.

6. REFERENCES

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Problema pozițiilor manipulatorului paralel plan trimobil 3RPR

Abstract: *În acest articol se studiază problema inversă şi directă a pozițiilor manipulatorului 3RPR, se rezolvă analitic problema inversă a pozițiilor iar problema directă a pozițiilor se rezolvă atât analitic cât şi numeric. De asemenea, în articol se prezintă o metodă originală de determinare a posibilităților de asamblare a manipulatorului cu ajutorul calculatorului.* ***Cuvinte cheie:*** *manipulator paralel, problema inversă a pozițiilor manipulatorului, problema directă a pozițiilor manipulatorului, platformă mobilă, ecuații de intrare – ieşire, ecuațiile carteziene ale curbelor mobile de ghidare.*

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