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NEW FORMULATIONS IN ANALYTICAL DYNAMICS OF SYSTEMS

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Abstract: *In the Newtonian Dynamics of the multibody mechanical systems (MBS), the most general theorem is considered the kinetic energy theorem in the differential form. In the advanced studies on the MBS have led to existence of some higher energy, corresponding to accelerations of higher order. According to the literature the general principles underlying the entire analytical dynamics are: the principle of D`Alembert, the principle of virtual mechanic work, specific to dynamic behavior of mechanical systems, known as D'Alembert-Lagrange principle. The second part of the paper is focused on a few formulations, based on author research on advanced dynamics of multibody systems, when they are characterized by sudden and transitory motions. It demonstrates theoretical and experimental the existing of time variations in accelerations. According to main author research, they are integrated into higher order energies and these in the differential equations of motion in higher order, which will lead to variations in time of generalized forces which they are dominating the mechanical systems with sudden and transitory motions.*

Key words: *analytical dynamics, acceleration energies, differential principles, dynamics equations.*

1. INTRODUCTION

Analytical dynamics of systems is one of the most important parts from Mechanics, as fundamental science. According to various treatises of mechanics, for example [1], [2], [3], [9] and [10], in this paper have been developed differential and integral principles which they will determine the dynamical equations of the mechanical motions for any multibody systems. Among of these, in the currently paper, are mentioned: *D'Alembert principle and D'Alembert-Lagrange principles.* In the first part of the paper, the above principles will be presented in the lapidary form. They will be followed by the vector conditions of static equilibrium. In second part, generalization of the differential principle and dynamical equations of higher order typical to sudden motions and transitory phase will be defined.

2. D'ALEMBERT-LAGRANGE PRINCIPLE

In this section, using the aspects from various treatises [2], [3] and [9], *D'Alembert principle and then D'Alembert-Lagrange principles will be defined.* In this context, first of all a discrete system of (n) material particles located in the mechanical interaction is taken into study. The material system is subjected to $(p \le 3 \cdot n)$ scleronomic and holonomic physical links without frictions (perfect), according to Fig.1.

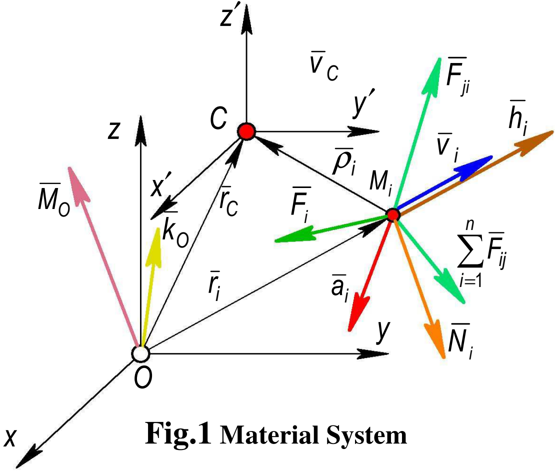

Fig.1 Material System

The material system is characterized by mass (m_i), position vectors $(\bar{r}_i)$, and $(k = 3 \cdot n - m)$ degree of freedom (d.o.f). Every material particle is located under the action: active and external forces $(\bar{F}_i)$, external link forces $(\bar{N}_i)$, and internal link forces $\sum_{j=1}^{n} \bar{F}_{ij}$. Material system performs a mechanical motion defined by velocities $(\bar{v}_i)$ and accelerations $(\bar{a}_i)$.

The above parameters are synthesized thus:

$$\left\{\begin{matrix} m_i;\ \bar{r}_i=\left[q_j(t)\,j=1\to k\right];\ \bar{F}_i;\ \bar{F}_{ji};\ \bar{N}_i \\ \sum\limits_{j=1}^{n}\bar{F}_{ij};\ \bar{v}_i=\dot{\bar{r}}_i;\ \bar{a}_i=\dot{\bar{v}}_i\equiv\ddot{\bar{r}}_i;\ i=1\to n \end{matrix}\right\}. \quad (1)$$

According to D'Alembert artifice, the torsor of the inertia forces is virtual or fictional applied on the material system. As a result, the equations of fictional dynamic equilibrium are:

$$\begin{aligned} &\sum_{i=1}^{n}\bar{F}_i+\sum_{i=1}^{n}\bar{F}_{ji}+\sum_{i=1}^{n}\bar{N}_i+\sum_{i=1}^{n}\sum_{j=1}^{n}\bar{F}_{ij}= \\ &=\sum_{i=1}^{n}\bar{F}_i+\left(-\sum_{i=1}^{n}m_i\cdot\bar{a}_i\right)+\sum_{i=1}^{n}\bar{N}_i+\sum_{i=1}^{n}\sum_{j=1}^{n}\bar{F}_{ij}=0; \end{aligned} \quad (2)$$

$$\begin{aligned} &\sum_{i=1}^{n}\bar{r}_i\times\bar{F}_i+\sum_{i=1}^{n}\bar{r}_i\times\bar{F}_{ji}+\sum_{i=1}^{n}\bar{r}_i\times\bar{N}_i+ \\ &+\sum_{i=1}^{n}\sum_{j=1}^{n}\bar{r}_i\times\bar{F}_{ij}\equiv\sum_{i=1}^{n}\bar{r}_i\times\bar{F}_i+\left(-\sum_{i=1}^{n}\bar{r}_i\times m_i\cdot\bar{a}_i\right)+ \\ &+\sum_{i=1}^{n}\bar{r}_i\times\bar{N}_i+\sum_{i=1}^{n}\sum_{j=1}^{n}\bar{r}_i\times\bar{F}_{ij}=0; \end{aligned} \quad (3)$$

$$\text{where } \sum_{i=1}^{n}\sum_{j=1}^{n}\bar{F}_{ij}=0;\quad \sum_{i=1}^{n}\sum_{j=1}^{n}\bar{r}_i\times\bar{F}_{ij}=0, \quad (4)$$

$$\begin{aligned} &\sum_{i=1}^{n}\bar{F}_i+\sum_{i=1}^{n}\bar{F}_{ji}+\sum_{i=1}^{n}\bar{N}_i= \\ &=\sum_{i=1}^{n}\bar{F}_i+\left(-\sum_{i=1}^{n}m_i\cdot\bar{a}_i\right)+\sum_{i=1}^{n}\bar{N}_i=0; \end{aligned} \quad (5)$$

$$\begin{aligned} &\sum_{i=1}^{n}\bar{r}_i\times\bar{F}_i+\sum_{i=1}^{n}\bar{r}_i\times\bar{F}_{ji}+\sum_{i=1}^{n}\bar{r}_i\times\bar{N}_i= \\ &=\sum_{i=1}^{n}\bar{r}_i\times\bar{F}_i+\left(-\sum_{i=1}^{n}\bar{r}_i\times m_i\cdot\bar{a}_i\right)+\sum_{i=1}^{n}\bar{r}_i\times\bar{N}_i=0. \end{aligned} \quad (6)$$

The vector and differential equations of second order (5) and (6) are known as *D'Alembert principle,* also named as *kinetostatic method.*

According to [1]-[3], to every material particle is associated a virtual displacement compatible with the physical and perfect links:

$$\delta\bar{r}_i=\sum_{j=1}^{k}\frac{\partial\bar{r}_i}{\partial q_j}\cdot\delta q_j; \quad (7)$$

$$\bar{v}_i=\dot{\bar{r}}_i=\sum_{j=1}^{k}\frac{\partial\bar{r}_i}{\partial q_j}\cdot\dot{q}_j; \quad (8)$$

$$\bar{a}_i=\dot{\bar{v}}_i=\ddot{\bar{r}}_i=\sum_{j=1}^{k}\frac{\partial\bar{r}_i}{\partial q_j}\cdot\ddot{q}_j+\sum_{j=1}^{k}\sum_{m=1}^{k}\frac{\partial^2\bar{r}_i}{\partial q_j\cdot\partial q_m}\cdot\dot{q}_j\cdot\dot{q}_m. \quad (9)$$

Every equation of fictional dynamic equilibrium is dot multiplied with (7) and they are summing: As a result, the virtual and resultant work is:

$$\begin{aligned} &\delta L=\sum_{i=1}^{n}\bar{F}_i\cdot\delta\bar{r}_i+\left(-\sum_{i=1}^{n}m_i\cdot\bar{a}_i\right)\cdot\delta\bar{r}_i+ \\ &+\sum_{i=1}^{n}\bar{N}_i\cdot\delta\bar{r}_i+\sum_{i=1}^{n}\sum_{j=1}^{n}\bar{F}_{ij}\cdot\delta\bar{r}_i=0; \end{aligned} \quad (10)$$

$$\text{where}\quad \delta L_N=\sum_{i=1}^{n}\bar{N}_i\cdot\delta\bar{r}_i=0, \quad (11)$$

$$\text{and}\quad \delta L_{int}=\sum_{i=1}^{n}\sum_{j=1}^{n}\bar{F}_{ij}\cdot\delta\bar{r}_i=0. \quad (12)$$

Considering (11) and (12), the equation (10) is changed in the following:

$$\delta L=\sum_{i=1}^{n}\left(\bar{F}_i-m_i\cdot\bar{a}_i\right)\cdot\delta\bar{r}=0. \quad (13)$$

The above differential equation is, according to [1] and [3], *D'Alembert-Lagrange principle* also known the *principle of virtual mechanic work*, typical to dynamic behavior of systems.

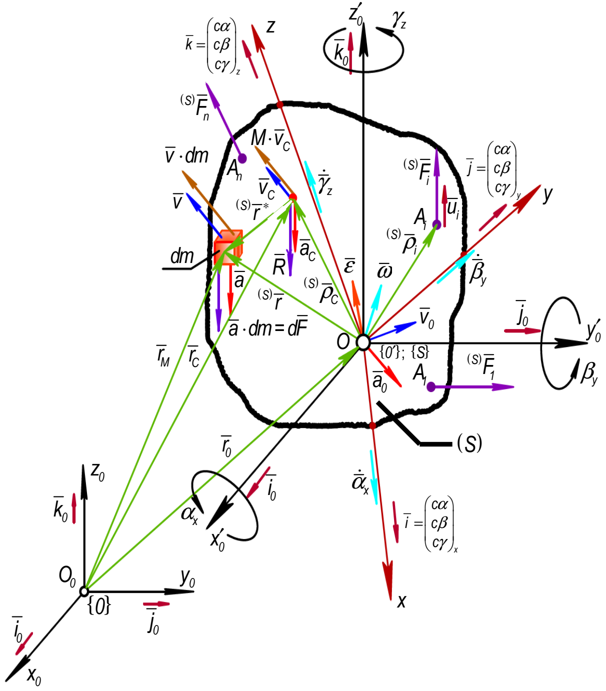

Fig.2 Rigid Body Free in Cartesian Frame

The above studies are extended in the case of the rigid solid (see Fig.2) free in the Cartesian space. In this figure are implemented the following changes: $O=C$, $\bar{\rho}_C=0$, $\bar{r}_0=\bar{r}_C$, and $I_S'=I_S^*$ as the inertial tensor axial and centrifugal of the rigid with respect to frame whose origin is the mass center. According to theory referring to reduction torsor, there are:

$$\bar{R}^{*} = \sum_{i=1}^{n} \bar{F}_i \qquad \bar{M}_C^{*} = \sum_{i=1}^{n} \bar{\rho}_i \times \bar{F}_i \, ; \tag{14}$$

and
$$\left\{ \begin{aligned} \bar{M}_O &= \sum_{i=1}^{n} \bar{r}_i \times \bar{F}_i = \sum_{i=1}^{n} (\bar{r}_C + \bar{\rho}_i) \times \bar{F}_i = \\ &= \bar{r}_C \times \bar{R}^{*} + \bar{M}_C^{*} \end{aligned} \right\}, \tag{15}$$

where $\bar{R}^{*}$ and $\bar{M}_C^{*}$ are the resultant vector and resultant moment of the active forces with respect to frame applied in the mass center.

The motion theorem of the mass center and theorem of the angular momentum are:

$$M \cdot \bar{a}_C = \bar{R}^{*} \, ; \quad I_S^{*} \cdot \bar{\varepsilon} + \bar{\omega} \times I_S^{*} \cdot \bar{\omega} = \bar{M}_C^{*} \, . \tag{16}$$

The rigid body is free. As a result, this is characterized by six independent parameters (generalized coordinates) included in symbol:

$$\left\{ \begin{aligned} &\bar{\bar{X}} = \begin{pmatrix} \bar{r}_C \\ \cdots \\ \bar{\bar{\Omega}} \end{pmatrix} = \begin{bmatrix} \begin{pmatrix} x_C & y_C & z_C \end{pmatrix}^T \\ \cdots\cdots\cdots\cdots\cdots \\ \begin{pmatrix} \alpha_A & \beta_B & \gamma_C \end{pmatrix}^T \end{bmatrix} = \\ &= \begin{bmatrix} \begin{pmatrix} q_1 & q_2 & q_3 \end{pmatrix}^T \\ \cdots\cdots\cdots\cdots\cdots \\ \begin{pmatrix} q_4 & q_5 & q_6 \end{pmatrix}^T \end{bmatrix} = \begin{bmatrix} q_j \, ; \, j = 1 \to 6 \end{bmatrix}^T \end{aligned} \right\} \tag{17}$$

For example, from Figure 2, the angular vector of the orientation is established, [2] and [3], with:

$$\bar{\Omega} = \begin{bmatrix} 1 & 0 & s\beta_y \\ 0 & c\alpha_x & -s\alpha_x \cdot c\beta_y \\ 0 & s\alpha_x & c\alpha_x \cdot c\beta_y \end{bmatrix} \cdot \begin{bmatrix} \alpha_x \\ \beta_y \\ \gamma_z \end{bmatrix} = J_\Omega \cdot \bar{\bar{\Omega}} \tag{18}$$

In the general case of the orientation (see [2] and [3]), the resultant rotation matrix shows as:

$$ {}_S^0[R](q_j \, ; \, j = 4 \to 6) = R(\alpha_A - \beta_B - \gamma_C) \, , \tag{19}$$

$$\begin{aligned} &R(\alpha_A - \beta_B - \gamma_C) = R(\bar{A};\alpha_A) \cdot R(\bar{B};\beta_B) \cdot R(\bar{C};\gamma_C) \, ; \\ &R^T(\alpha_A - \beta_B - \gamma_C) = \\ &= R^T(\bar{C};\gamma_C) \cdot R^T(\bar{B};\beta_B) \cdot R^T(\bar{A};\alpha_A) \, . \end{aligned} \tag{20}$$

In the above matrix, the unit vectors of the instantaneous rotation axes are included thus:

$$\begin{aligned} &\left\{ \begin{aligned} &{}^{(S)0}\bar{A} = \left\{ {}^{(S)0}\bar{x} \, ; {}^{(S)0}\bar{y} \, ; {}^{(S)0}\bar{z} \right\} ; \\ &\bar{A} = \left\{ \bar{x} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} ; \ \bar{y} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} ; \ \bar{z} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\} \end{aligned} \right\} \\ &\left\{ \begin{aligned} &{}^{(S)0}\bar{B} = \left\{ {}^{(S)0}\bar{y} \, ; {}^{(S)0}\bar{z} \, ; {}^{(S)0}\bar{x} \right\} ; \\ &\bar{B} = \left\{ \bar{y} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} ; \ \bar{z} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} ; \ \bar{x} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\} \neq \bar{A} \end{aligned} \right\} \\ &\left\{ \begin{aligned} &{}^{(S)0}\bar{C} = \left\{ {}^{(S)0}\bar{z} \, ; {}^{(S)0}\bar{x} \, ; {}^{(S)0}\bar{y} \right\} ; \\ &\bar{C} = \left\{ \bar{z} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} ; \ \bar{x} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} ; \ \bar{y} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\} \neq \bar{B} \end{aligned} \right\} \end{aligned} \tag{21}$$

For the general case, the angular vector of the orientation is established, [2] and [3], with:

$$\bar{\Omega} = \begin{bmatrix} {}^0\bar{A} & {}^0\bar{B} & \bar{C}^0 \end{bmatrix} \cdot \bar{\bar{\Omega}} \, , \tag{22}$$

$$\left\{ \begin{aligned} &where \ {}^0\bar{A} = \bar{A} \, ; \ {}^0\bar{B} = R(\bar{A};\alpha_A) \cdot \bar{B} \\ &{}^0\bar{C} = R(\bar{A};\alpha_A) \cdot R(\bar{B};\beta_B) \cdot \bar{C} \end{aligned} \right\} \tag{23}$$

To rigid body, free in Cartesian space, the virtual displacements are associated as follows:

$$\delta\bar{r}_C = \sum_{j=1}^{6} \frac{\partial \bar{r}_C}{\partial q_j} \cdot \delta q_j \, , \ \left\{ \begin{aligned} &\delta\bar{r}_C \neq 0, \, j = 1 \to 3 \\ &\delta\bar{r}_C = 0, \, j = 4 \to 6 \end{aligned} \right\} , \tag{24}$$

$$\delta\bar{\Omega} = \sum_{j=1}^{6} \frac{\partial \bar{\Omega}}{\partial q_j} \cdot \delta q_j \, , \ \left\{ \begin{aligned} &\delta\bar{\Omega}_C = 0, \, j = 1 \to 3 \\ &\delta\bar{\Omega} \neq 0, \, j = 4 \to 6 \end{aligned} \right\} . \tag{25}$$

As a result, the fundamental theorems (16) are dot multiplied with (24) and respectively (25):

$$\left\{ \begin{aligned} &\delta L = \left(\bar{R}^{*} - M \cdot \bar{a}_C \right) \cdot \delta\bar{r}_C + \\ &\quad + \left[\bar{M}_C^{*} - \left(I_S^{*} \cdot \bar{\varepsilon} + \bar{\omega} \times I_S^{*} \cdot \bar{\omega} \right) \right] \cdot \delta\bar{\Omega} = 0 \end{aligned} \right\} , \tag{26}$$

$$\bar{\omega} = \frac{\partial \bar{\Omega}}{\partial t} = J_\Omega \cdot \dot{\bar{\bar{\Omega}}} \, , \qquad \bar{\varepsilon} = \dot{\bar{\omega}} = J_\Omega \cdot \ddot{\bar{\bar{\Omega}}} + \dot{J}_\Omega \cdot \dot{\bar{\bar{\Omega}}} \, . \tag{27}$$

The differential equation (26) is *generalization of D'Alembert-Lagrange principle*, typical to dynamic behavior of the rigid body.

In the case of MBS, the torsor of the inertia forces is substituted in (26) by the equations:

$$\begin{aligned} &{}^{(0)i}\bar{F}_{ji}^{*} = - M_i \cdot {}^{(0)i}\dot{\bar{v}}_{C_i} = \\ &= -M_i \cdot \left[{}^{(0)i}\dot{\bar{v}}_i + {}^{(0)i}\dot{\bar{\omega}}_i \times {}^{(0)i}\bar{r}_{C_i} + {}^{(0)i}\bar{\omega}_i \times {}^{(0)i}\bar{\omega}_i \times {}^{(0)i}\bar{r}_{C_i} \right] ; \end{aligned} \tag{28}$$

$$\begin{aligned} &{}^i\bar{N}_{ji}^{*} = - {}^iI_i^{*} \cdot {}^i\dot{\bar{\omega}}_i + {}^i\bar{\omega}_i \times {}^iI_i^{*} \cdot {}^i\bar{\omega}_i \, ; \\ &{}^i\bar{N}_{ji} = - \left({}^i\bar{N}_i^{*} + {}^iI_{C_i} \cdot {}^i\dot{\bar{\omega}}_i + {}^i\bar{\omega}_i \times {}^iI_{C_i} \cdot {}^i\bar{\omega}_i \right) ; \end{aligned} \tag{29}$$

where the components are shown in the Fig.3.

3. STATIC EQUILIBRIUM OF THE RIGID

When the rigid body is located in the static balance, in the equations (13) and (26), the following kinematical conditions are substituted:

$$\bar{a}_i = 0 \, , \ \bar{\omega} = 0 \text{ and } \bar{\varepsilon} = 0 \, .$$

So, *D'Alembert-Lagrange principle* is changed in *the virtual work principle* corresponding to absolute conditions of *static equilibrium*:

$$\delta L = \sum_{i=1}^{n} \bar{F}_i \cdot \delta\bar{r}_i = 0 \, ; \tag{30}$$

$$\delta L = \bar{R}^{*} \cdot \delta\bar{r}_C + \bar{M}_C^{*} \cdot \delta\bar{\Omega} = 0 \, . \tag{31}$$

Substituting (7) in (30), as well (24) and (25) in (31), the above equations are changed thus:

$$\delta L=\sum_{i=1}^{n}\bar{F}_i\cdot\delta\bar{r}_i=\sum_{j=1}^{k}Q_j\cdot\delta q_j=0\,, \tag{32}$$

where $$Q_j=\sum_{i=1}^{n}\bar{F}_i\cdot\frac{\partial\bar{r}_i}{\partial q_j}\,, \tag{33}$$

and $$\left\{\begin{matrix}\delta L=\bar{R}^{*}\cdot\delta\bar{r}_C+\bar{M}_C^{*}\cdot\delta\bar{\Omega}=\sum\limits_{j=1}^{6}Q_j\cdot\delta q_j=\\ =\sum\limits_{j=1}^{6}\left(\bar{R}^{*}\cdot\dfrac{\partial\bar{r}_C}{\partial q_j}+\bar{M}_C^{*}\cdot\dfrac{\partial\bar{\Omega}}{\partial q_j}\right)\cdot\delta q_j=0\end{matrix}\right\}, \tag{34}$$

where $$Q_j=\bar{R}^{*}\cdot\frac{\partial\bar{r}_C}{\partial q_j}+\bar{M}_C^{*}\cdot\frac{\partial\bar{\Omega}}{\partial q_j}\,. \tag{35}$$

The expressions (33) and (35) are known (see [1] and [3]) *generalized forces*. It observes form (35) that it can be force or moment of force in function of generalized coordinate type.

In the following, the vector conditions of static equilibrium of the rigid body, free in the Cartesian space, are determined. The starting equation is the virtual work principle (34). In the case of the holonomic systems the superposition effect principle is applied, according to:

$$\begin{pmatrix}q_j\neq 0;\ \delta q_j\neq 0,\ j=1\rightarrow 3\\ q_i=0;\ \delta q_i=0,\ i=4\rightarrow 6\end{pmatrix} \tag{36}$$

The principle (34), using (35), is changed as:

$$Q_j=\bar{R}^{*}\cdot\frac{\partial\bar{r}_C}{\partial q_j}=0,\ \textit{where}\ j=1\rightarrow 3\,, \tag{37}$$

$$\left\{\begin{matrix}\dfrac{\partial\bar{r}_C}{\partial q_1}=\dfrac{\partial\bar{r}_C}{\partial x_C}=\dfrac{\partial}{\partial x_C}(\bar{r}_i-\bar{\rho}_i)=(1\quad 0\quad 0)^T\\ \dfrac{\partial\bar{r}_C}{\partial q_2}=\dfrac{\partial\bar{r}_C}{\partial y_C}=\dfrac{\partial}{\partial y_C}(\bar{r}_i-\bar{\rho}_i)=(0\quad 1\quad 0)^T\\ \dfrac{\partial\bar{r}_C}{\partial q_3}=\dfrac{\partial\bar{r}_C}{\partial z_C}=\dfrac{\partial}{\partial z_C}(\bar{r}_i-\bar{\rho}_i)=(0\quad 0\quad 1)^T\end{matrix}\right\} \tag{38}$$

Substituting (38) in (37), the generalized forces become the Cartesian components of the resultant vector of the active forces, as follows:

$$\left\{\begin{matrix}Q_1=\bar{R}^{*}\cdot\dfrac{\partial\bar{r}_C}{\partial q_1}=\sum\limits_{i=1}^{n}\bar{F}_i\cdot\dfrac{\partial\bar{r}_i}{\partial q_1}=\sum\limits_{i=1}^{n}F_{ix}=R_x=0\\ Q_2=\bar{R}^{*}\cdot\dfrac{\partial\bar{r}_C}{\partial q_2}=\sum\limits_{i=1}^{n}\bar{F}_i\cdot\dfrac{\partial\bar{r}_i}{\partial q_2}=\sum\limits_{i=1}^{n}F_{iy}=R_y=0\\ Q_3=\bar{R}^{*}\cdot\dfrac{\partial\bar{r}_C}{\partial q_3}=\sum\limits_{i=1}^{n}\bar{F}_i\cdot\dfrac{\partial\bar{r}_i}{\partial q_3}=\sum\limits_{i=1}^{n}F_{iz}=R_z=0\end{matrix}\right. \tag{40}$$

Superposition effect principle (36) is changed:

$$\begin{pmatrix}q_j=0;\ \delta q_j=0,\ j=1\rightarrow 3\\ q_i\neq 0;\ \delta q_i\neq 0,\ i=4\rightarrow 6\end{pmatrix} \tag{41}$$

Considering (35), the principle (34) becomes:

$$Q_i=\bar{M}_C^{*}\cdot\frac{\partial\bar{\Omega}}{\partial q_i}=0,\ \textit{where}\ i=4\rightarrow 6\,, \tag{42}$$

$$\left\{\begin{matrix}\dfrac{\partial\bar{\Omega}}{\partial q_4}=\dfrac{\partial\bar{\Omega}}{\partial\alpha_A}={}^{0}\bar{A}\\ \dfrac{\partial\bar{\Omega}}{\partial q_5}=\dfrac{\partial\bar{\Omega}}{\partial\beta_B}={}^{0}\bar{B}=R\left(\bar{A};\alpha_A\right)\cdot\bar{B}\\ \dfrac{\partial\bar{\Omega}}{\partial q_5}=\dfrac{\partial\bar{\Omega}}{\partial\gamma_C}={}^{0}\bar{C}=R\left(\bar{A};\alpha_A\right)\cdot R\left(\bar{B};\beta_B\right)\cdot\bar{C}\end{matrix}\right\}. \tag{43}$$

Substituting (43) in (42), the generalized forces become the Cartesian components of the resultant moment of the active forces, as:

$$\left\{\begin{matrix}Q_4=\bar{M}_C^{*}\cdot\dfrac{\partial\bar{\Omega}}{\partial q_4}=\left(\sum\limits_{i=1}^{n}\bar{\rho}_i\times\bar{F}_i\right)\cdot\dfrac{\partial\bar{\Omega}}{\partial\alpha_A}=M_A^{*}=0\\ Q_5=\bar{M}_C^{*}\cdot\dfrac{\partial\bar{\Omega}}{\partial q_5}=\left(\sum\limits_{i=1}^{n}\bar{\rho}_i\times\bar{F}_i\right)\cdot\dfrac{\partial\bar{\Omega}}{\partial\beta_B}=M_B^{*}=0\\ Q_6=\bar{M}_C^{*}\cdot\dfrac{\partial\bar{\Omega}}{\partial q_6}=\left(\sum\limits_{i=1}^{n}\bar{\rho}_i\times\bar{F}_i\right)\cdot\dfrac{\partial\bar{\Omega}}{\partial\gamma_C}=M_C^{*}=0\end{matrix}\right. \tag{44}$$

The above conditions can be also obtained, applying (41) in (33), according to following:

$$Q_j=\sum_{i=1}^{n}\bar{F}_i\cdot\frac{\partial\bar{r}_i}{\partial q_j}=0,\ \textit{where}\ j=4\rightarrow 6\,, \tag{45}$$

$$Q_j=\sum_{i=1}^{n}\bar{F}_i\cdot\frac{\partial}{\partial q_j}\left[\bar{r}_i(q_1;q_2;\ldots;q_6)\right]=0,\ \ j=4\rightarrow 6$$

and $\bar{r}_i=\bar{r}_C+\bar{\rho}_i=R(\alpha_A-\beta_B-\gamma_C)\cdot\left({}^{S}\bar{r}_C+{}^{S}\bar{\rho}_i\right)$.

The partial derivatives from (45) are based on the property referring to differentials of the rotation matrices [2]-[8]. The expressions are:

$$\left\{\begin{matrix}\dfrac{\partial\bar{r}_i}{\partial q_4}=\dfrac{\partial\bar{r}_i}{\partial\alpha_A}=\dfrac{\partial}{\partial\alpha_A}\left[R(\alpha_A-\beta_B-\gamma_C)\right]\cdot\bar{r}_i\\ \dfrac{\partial\bar{r}_i}{\partial q_5}=\dfrac{\partial\bar{r}_i}{\partial\beta_B}=\dfrac{\partial}{\partial\beta_B}\left[R(\alpha_A-\beta_B-\gamma_C)\right]\cdot\bar{r}_i\\ \dfrac{\partial\bar{r}_i}{\partial q_6}=\dfrac{\partial\bar{r}_i}{\partial\gamma_C}=\dfrac{\partial}{\partial\gamma_C}\left[R(\alpha_A-\beta_B-\gamma_C)\right]\cdot\bar{r}_i\end{matrix}\right\} \tag{46}$$

The first partial derivative from (46) shows as:

$$\left\{\begin{matrix}\dfrac{\partial}{\partial\alpha_A}\left[R(\alpha_A-\beta_B-\gamma_C)\right]=\dfrac{\partial}{\partial\alpha_A}\left\{{}_{S}^{0}[R]\right\}=\\ =\dfrac{\partial}{\partial\alpha_A}\left[R(\alpha_A-\beta_B-\gamma_C)\right]\cdot{}_{S}^{0}[R]^T\cdot{}_{S}^{0}[R]=\\ =\left\{\dfrac{\partial}{\partial\alpha_A}\left[R\left(\bar{A};\alpha_A\right)\right]\cdot R^T\left(\bar{A};\alpha_A\right)\right\}\cdot{}_{S}^{0}[R]=\\ =\left(\bar{A}\times\right)\cdot R(\alpha_A-\beta_B-\gamma_C)\end{matrix}\right\}; \tag{47}$$

$$ {}^{0}\overline{A} = vect\left\{ \frac{\partial}{\partial \alpha_{A}} \left[R\left(\overline{A};\alpha_{A} \right) \right] \cdot R^{T}\left(\overline{A};\alpha_{A} \right) \right\} = \begin{pmatrix} \alpha_{SA} \\ \beta_{SA} \\ \gamma_{SA} \end{pmatrix} . $$

The second partial derivative from (46) is:

$$ \left\{ \begin{array}{c} \frac{\partial}{\partial \beta_{B}} \left[R\left(\alpha_{A} - \beta_{B} - \gamma_{C} \right) \right] = \frac{\partial}{\partial \beta_{B}} \left\{ {}_{S}^{0}[R] \right\} = \\ = \frac{\partial}{\partial \beta_{B}} \left[R\left(\alpha_{A} - \beta_{B} - \gamma_{C} \right) \right] \cdot {}_{S}^{0}[R]^{T} \cdot {}_{S}^{0}[R] = \\ = \left\{ R\left(\overline{A};\alpha_{A} \right) \cdot \left(\overline{B} \times \right) \cdot R^{T}\left(\overline{A};\alpha_{A} \right) \right\} \cdot {}_{S}^{0}[R] \\ = \left\{ R\left(\overline{A};\alpha_{A} \right) \cdot \overline{B} \times \right\} \cdot {}_{S}^{0}[R] \end{array} \right\} ; \quad (48) $$

$$ \left(\overline{B} \times \right) = \left\{ \frac{\partial}{\partial \beta_{B}} \left[R\left(\overline{B};\beta_{B} \right) \right] \cdot R^{T}\left(\overline{B};\beta_{B} \right) \right\} ; $$

$$ {}^{0}\overline{B} = vect\left\{ R\left(\overline{A};\alpha_{A} \right) \cdot \overline{B} \times \right\} = \left(\alpha_{SB} \quad \beta_{SB} \quad \gamma_{SB} \right)^{T} . $$

The third partial derivative from (46) shows as:

$$ \left\{ \begin{array}{c} \frac{\partial}{\partial \gamma_{C}} \left[R\left(\alpha_{A} - \beta_{B} - \gamma_{C} \right) \right] = \frac{\partial}{\partial \gamma_{C}} \left\{ {}_{S}^{0}[R] \right\} = \\ = \frac{\partial}{\partial \gamma_{C}} \left[R\left(\alpha_{A} - \beta_{B} - \gamma_{C} \right) \right] \cdot {}_{S}^{0}[R]_{T} \cdot {}_{S}^{0}[R] = \\ = \left\{ R\left(\overline{A};\alpha_{A} \right) \cdot R\left(\overline{B};\beta_{B} \right) \cdot \overline{C} \times \right\} \cdot R\left(\alpha_{A} - \beta_{B} - \gamma_{C} \right) \end{array} \right\} \quad (49) $$

$$ \left(\overline{C} \times \right) = \frac{\partial}{\partial \gamma_{C}} \left[R\left(\overline{C};\gamma_{C} \right) \right] \cdot R^{T}\left(\overline{C};\gamma_{C} \right) ; $$

$$ {}^{0}\overline{C} = vect\left\{ R\left(\overline{A};\alpha_{A} \right) \cdot R\left(\overline{B};\beta_{B} \right) \cdot \overline{C} \times \right\} . $$

Substituting (47) – (49) in (45), it obtains:

$$ \left\{ \begin{array}{c} Q_{4} = \sum_{i=1}^{n} \overline{F}_{i} \cdot \left[{}^{0}\overline{A} \times \left(\overline{r}_{C} + \overline{\rho}_{i} \right) \right] = \\ = \sum_{i=1}^{n} \left[\left(\overline{r}_{C} + \overline{\rho}_{i} \right) \times \overline{F}_{i} \right]^{T} \cdot {}^{0}\overline{A} = \\ = \left[\overline{r}_{C} \times \left(\sum_{i=1}^{n} \overline{F}_{i} \right) + \sum_{i=1}^{n} \overline{\rho}_{i} \times \overline{F}_{i} \right]^{T} \cdot {}^{0}\overline{A} = \\ = \left(\overline{r}_{C} \times \overline{R}^{*} + \overline{M}_{C}^{*} \right)^{T} \cdot \overline{A} = \overline{M}_{O}^{T} \cdot {}^{0}\overline{A} = M_{A} = 0 \end{array} \right\} ; \quad (50) $$

$$ \left\{ \begin{array}{c} Q_{5} = \sum_{i=1}^{n} \overline{F}_{i} \cdot \left[\overline{B} \times \left(\overline{r}_{C} + \overline{\rho}_{i} \right) \right] = \\ = \sum_{i=1}^{n} \left[\left(\overline{r}_{C} + \overline{\rho}_{i} \right) \times \overline{F}_{i} \right]^{T} \cdot \overline{B} = \\ = \left[\overline{r}_{C} \times \left(\sum_{i=1}^{n} \overline{F}_{i} \right) + \sum_{i=1}^{n} \overline{\rho}_{i} \times \overline{F}_{i} \right]^{T} \cdot \overline{B} = \\ = \left(\overline{r}_{C} \times \overline{R}^{*} + \overline{M}_{C}^{*} \right)^{T} \cdot \overline{B} = \overline{M}_{O}^{T} \cdot \overline{B} = M_{B} = 0 \end{array} \right\} ; \quad (51) $$

$$ \left\{ \begin{array}{c} Q_{6} = \sum_{i=1}^{n} \overline{F}_{i} \cdot \left[\overline{C} \times \left(\overline{r}_{C} + \overline{\rho}_{i} \right) \right] = \\ = \sum_{i=1}^{n} \left[\left(\overline{r}_{C} + \overline{\rho}_{i} \right) \times \overline{F}_{i} \right]^{T} \cdot \overline{C} = \\ = \left[\overline{r}_{C} \times \left(\sum_{i=1}^{n} \overline{F}_{i} \right) + \sum_{i=1}^{n} \overline{\rho}_{i} \times \overline{F}_{i} \right]^{T} \cdot \overline{C} = \\ = \left(\overline{r}_{C} \times \overline{R}^{*} + \overline{M}_{C}^{*} \right)^{T} \cdot \overline{C} = \overline{M}_{O}^{T} \cdot \overline{C} = M_{C} = 0 \end{array} \right\} . \quad (52) $$

Considering (50 – (52), the resultant moment of the active forces, with respect to fixe frame, for any rigid body free in the Cartesian frame and located in the static equilibrium is following:

$$ \left\{ \begin{array}{c} \overline{M}_{O} = \left[\begin{pmatrix} \alpha_{SA} \\ \beta_{SA} \\ \gamma_{SA} \end{pmatrix} \begin{pmatrix} \alpha_{SB} \\ \beta_{SB} \\ \gamma_{SB} \end{pmatrix} \begin{pmatrix} \alpha_{SC} \\ \beta_{SC} \\ \gamma_{SC} \end{pmatrix} \right] \cdot \begin{pmatrix} M_{A} \\ M_{B} \\ M_{C} \end{pmatrix} = \\ = \overline{r}_{C} \times \left(\sum_{i=1}^{n} \overline{F}_{i} \right) + \sum_{i=1}^{n} \overline{\rho}_{i} \times \overline{F}_{i} = \overline{r}_{0} \times \overline{R}^{*} + \overline{M}_{C}^{*} = \\ = \left[{}^{0}\overline{A} \quad {}^{0}\overline{B} \quad {}^{0}\overline{C} \right] \cdot \begin{pmatrix} Q_{4} \\ Q_{5} \\ Q_{6} \end{pmatrix} = \begin{pmatrix} M_{x} \\ M_{y} \\ M_{z} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \end{array} \right\} . (53) $$

Remark: Therefore, applying the virtual work principle, one the on hand the scalar equations of static equilibrium for any rigid free in the Cartesian frame (40) and (53) have been demonstrated, on the other hand these equations validate that the symbol Q_j is generalized force.

4. ADVANCED DYNAMIC PRINCIPLES

In keeping with researches of the author [4], [5], [6], [7] and [8], for multibody systems, MBS, symbolically represented in Figure 3, *the generalization of the D'Alembert - Lagrange principle* and *the differential equations of higher order*, will be presented in this section.

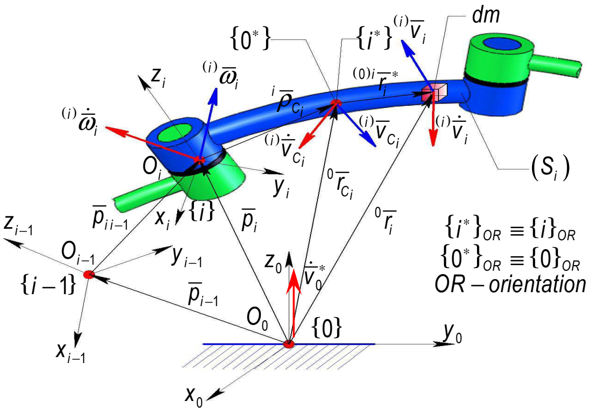

Fig. 3 Rigid Body from Multibody System

Considering (26), for any MBS, *generalization of D'Alembert - Lagrange principle* shows as:

$$ \left\{ \begin{array}{c} \sum_{i=1}^{n} \left(\overline{F}_{i}^{*} - M_{i} \cdot \overline{a}_{C_{i}} \right) \cdot \delta \overline{r}_{C_{i}} + \\ + \sum_{i=1}^{n} \left(\overline{N}_{i}^{*} - I_{i}^{*} \cdot \overline{\varepsilon}_{i} - \overline{\omega}_{i} \times I_{i}^{*} \cdot \overline{\omega}_{i} \right) \cdot \delta \overline{\psi}_{i} = 0 \end{array} \right\} , \quad (54) $$

where $\overline{F}_i^*$ and $\overline{N}_i^*$ are the active forces and theirs moments with respect to mass center, and I_i^* is the axial and centrifugal inertia tensor. The expression (54) contains the angular parameters: $\overline{\omega}_i$ and $\overline{\varepsilon}_i$, named the angular velocity and acceleration, corresponding to resultant rotational movements of each body belonging to the system. As a result, from (54), a *generalization of the Lagrange`s equations of first kind*, according to [4], [5], [6], [7] and [8], for a multibody system is established thus:

$$\left\{\begin{array}{c} \sum_{i=1}^{n} M_i\cdot\overline{a}_{C_i}^{T}\cdot\frac{\partial\overline{r}_{C_i}}{\partial q_j}+ \\ +\sum_{i=1}^{n}\left(I_i^*\cdot\overline{\varepsilon}_i+\overline{\omega}_i\times I_i^*\cdot\overline{\omega}_i\right)^T\cdot\frac{\partial\overline{\psi}_i}{\partial q_j}\cdot\Delta_j= \\ =\sum_{i=1}^{n}\overline{F}_i^{*T}\cdot\frac{\partial\overline{r}_{C_i}}{\partial q_j}+\sum_{i=1}^{n}\overline{N}_i^{*T}\cdot\frac{\partial\overline{\psi}_i}{\partial q_j}\cdot\Delta_j \end{array}\right\} \qquad (55)$$

Applying the time derivatives of (k) order on (55) this is changed in the following:

$$\left\{\begin{array}{c} \frac{d^k}{dt^k}\left[\sum_{i=1}^{n}\left(\overline{F}_i^*-M_i\cdot\overline{a}_{C_i}\right)\cdot\frac{\partial\overline{r}_{C_i}}{\partial q_j}\right]+ \\ +\frac{d^k}{dt^k}\left[\sum_{i=1}^{n}\left(\overline{N}_i^*-I_i^*\cdot\overline{\varepsilon}_i-\overline{\omega}_i\times I_i^*\cdot\overline{\omega}_i\right)\cdot\frac{\partial\overline{\psi}_i}{\partial q_j}\cdot\Delta_j\right]=0 \end{array}\right\}$$

As a result, the above differential expressions are written under the generalized form thus:

$$\left\{\frac{d^{k-1}}{dt^{k-1}}\left(\frac{\partial\overline{r}_{C_i}}{\partial q_j}\right)=\frac{(k-1)!\cdot m!}{(m+k-1)!}\cdot\frac{\partial\overset{(m+k-1)}{\overline{r}_{C_i}}}{\partial\overset{(m)}{q_j}}\right\}; \qquad (56)$$

$$\left\{\begin{array}{c} \frac{d^{k-1}}{dt^{k-1}}\left(\frac{\partial\overline{\psi}_i}{\partial q_j}\cdot\Delta_j\right)=\frac{(k-1)!\cdot m!}{(m+k-1)!}\cdot\frac{\partial\overset{(m+k-3)}{\overline{\varepsilon}_i}}{\partial\overset{(m)}{q_j}}\cdot\Delta_j= \\ =\frac{(k-1)!\cdot m!}{(m+k-1)!}\cdot\frac{\partial\overset{(m+k-1)}{\overline{\psi}_i}}{\partial\overset{(m)}{q_j}}\cdot\Delta_j \end{array}\right\} \qquad (57)$$

$$\left\{\begin{array}{c} k\geq 1;\quad k=\{1;2;3;4;5;.....\} \\ m\geq(k+1);\quad m=\{2;3;4;5;.....\} \end{array}\right\}$$

The above differential equations of position and orientation are substituted in the differential principle under the generalized form (55). So, the motion equations of higher order will be obtained for anything MBS.

Applying the differential transformations in (55), (56) and (57), *the author has proposed* in the paper [8], *the generalized differential equations of higher order* in the case of the mechanical systems (MBS), dynamically characterized by sudden and transitory motions:

$$\left\{\begin{array}{c} \frac{(k-1)!\cdot m!}{(m+k-1)!}\cdot\frac{\partial}{\partial\overset{(m)}{q_j}}\left\{\left(\sum_{p=1}^{k}\Delta_p\right)\cdot\overset{(m+k)-(2\cdot p+1)}{E_A^{(p)}}\right\}= \\ =\overset{(k-1)}{Q_{iö}^{j}}\left[\overline{\theta}(t);\dot{\overline{\theta}}(t);\cdots;\overset{(m)}{\overline{\theta}}(t)\right] \end{array}\right\} \qquad (58)$$

$$\left\{\begin{array}{c} \text{where}\quad E_A^{(p)}=E_A^{(p)}\left[\overline{\theta}(t);\dot{\overline{\theta}}(t);\cdots;\overset{(p+1)}{\overline{\theta}}(t)\right] \\ k\geq 1;\qquad k=\{1;2;3;4;5;.....\} \\ m\geq(k+1);\quad m=\{2;3;4;5;.....\} \\ \text{and}\quad \Delta_p=\left[\frac{p\cdot(p+1)}{2}-\delta_p\right] \\ p=1\rightarrow k\ ;\ \ \delta_p=\{\{0\,;p=1\};\{1\,;p>1\}\} \end{array}\right\}. \qquad (59)$$

The equations (58) contain the acceleration energies of the order $(p=1\rightarrow k)$. Using the aspects from Fig.3, The starting equation is:

$$\left\{\begin{array}{c} E_A^{(p)}\left[\overline{\theta}(t);\dot{\overline{\theta}}(t);\cdots;\overset{(p+1)}{\overline{\theta}}(t)\right]= \\ =\frac{1}{2}\cdot\sum_{i=1}^{n}Trace\left\{{}_{i}^{0}\overset{(p+1)}{[R]}\cdot\left[\int{}^{i}\overline{r}_i^{*}\cdot{}^{i}\overline{r}_i^{*T}\cdot dm+\right.\right. \\ \left.\left.+{}^{i}\overline{r}_{C_i}\cdot{}^{i}\overline{r}_{C_i}^{T}\cdot\int dm\right]\cdot{}_{i}^{0}\overset{(p+1)}{[R]}{}^{T}\right\}+ \\ +\frac{1}{2}\cdot\sum_{i=1}^{n}Trace\left[\overset{(p+1)}{\overline{p}_i}\cdot\overset{(p+1)}{\overline{p}_i^{T}}\right]\cdot\int dm= \\ =\frac{1}{2}\cdot\sum_{i=1}^{n}Trace\left\{{}_{i}^{0}\overset{(p+1)}{[R]}\cdot\left[{}^{i}I_{pi}^{*}+M_i\cdot{}^{i}\overline{r}_{C_i}\cdot{}^{i}\overline{r}_{C_i}^{T}\right]\cdot{}_{i}^{0}\overset{(p+1)}{[R]}{}^{T}\right\}+ \\ +\frac{1}{2}\cdot\sum_{i=1}^{n}Trace\left[\overset{(p+1)}{\overline{p}_i}\cdot\overset{(p+1)}{\overline{p}_i^{T}}\right]\cdot M_i \end{array}\right\} \qquad (60)$$

The author has demonstrated in various papers, as example [4]-[8], the expressions of definition in explicit and matrix form, for acceleration energies of first, second, third and fourth order.

A particular case of the equations (58), corresponding to analytical mechanics [10], is the another research of author, under the form:

$$Q_{iö}^{j}=\frac{\partial\overset{(m-2)}{E_A^{(1)}}}{\partial\overset{(m)}{q_j}};\quad \text{where}\quad \overset{(0)}{E_A^{(1)}}=E_A^{(1)}, \qquad (61)$$

$$\left\{\begin{array}{c} \text{where}\quad k=1,\ m\geq\left[(k+1)=2\right] \\ (m)\ \text{is time deriving operator},\ \text{and}\ j=1\rightarrow n \end{array}\right\}. \qquad (62)$$

The equations (61), with conditions (62), are *generalization of the Gibbs – Appell equations.*

According to [4] – [8], the acceleration energy of first order is defined in the matrix form as:

$$\left\{\begin{gathered} E_A^{(1)}\left[\bar{\theta}(t);\dot{\bar{\theta}}(t);\ddot{\bar{\theta}}(t)\right]= \\ =\frac{1}{2}\cdot\ddot{\bar{\theta}}^T(t)\cdot M[\bar{\theta}(t)]\cdot\ddot{\bar{\theta}}(t) \\ +\frac{1}{2}\cdot\ddot{\bar{\theta}}^T(t)\cdot V\left[\bar{\theta}(t);\dot{\bar{\theta}}^2(t)\right] \end{gathered}\right\}, \tag{63}$$

where the dynamic matrices are well defined in the same papers, [4] – [8], of the author.

After important differential transformations on the acceleration energy of first order (63), its time derivative of order $(m\geq 0)$ shows thus:

$$\overset{(m)}{E_A^{(1)}}\left[\bar{\theta}(t);\dot{\bar{\theta}}(t);\ddot{\bar{\theta}}(t)\right]= \tag{64}$$

$$\left\{\begin{gathered} =\tau_m\cdot\ddot{\bar{\theta}}^T(t)\cdot M[\bar{\theta}(t)]\cdot\overset{(m+2)}{\bar{\theta}}(t)+ \\ +\lambda_m\cdot m\cdot\overset{(3)}{\bar{\theta}}{}^T(t)\cdot M[\bar{\theta}(t)]\cdot\overset{(m+1)}{\bar{\theta}}(t)+ \\ +m\cdot\ddot{\bar{\theta}}^T(t)\cdot M[\overset{(1)}{\bar{\theta}}(t)]\cdot\overset{(m+1)}{\bar{\theta}}(t)+ \\ +3\cdot\lambda_m\cdot\ddot{\bar{\theta}}^T(t)\cdot M[\overset{(m-1)}{\bar{\theta}}(t)]\cdot\overset{(3)}{\bar{\theta}}(t)+ \\ +\frac{1}{2}\cdot\ddot{\bar{\theta}}^T(t)\cdot M[\overset{(m)}{\bar{\theta}}(t)]\cdot\ddot{\bar{\theta}}(t)+ \\ +3\cdot\lambda_m\cdot\ddot{\bar{\theta}}^T(t)\cdot M[\overset{(2)}{\bar{\theta}}(t)]\cdot\overset{(m)}{\bar{\theta}}(t)+ \\ +3\cdot(m-2)\cdot\lambda_m\cdot\overset{(3)}{\bar{\theta}}{}^T(t)\cdot M[\overset{(m-2)}{\bar{\theta}}(t)]\cdot\overset{(3)}{\bar{\theta}}(t)+ \\ +3\cdot m\cdot\lambda_m\cdot(m-3)\cdot\overset{(3)}{\bar{\theta}}{}^T(t)\cdot M[\overset{(m-3)}{\bar{\theta}}(t)]\cdot\overset{(m)}{\bar{\theta}}(t)+ \\ +\Delta_m\cdot(m-1)\cdot\overset{(m+1-\lambda_m)}{\bar{\theta}}{}^T(t)\cdot M[\bar{\theta}(t)]\cdot\overset{(m+1-\lambda_m)}{\bar{\theta}}(t)+ \\ +\frac{1}{2}\cdot\overset{(m+2)}{\bar{\theta}}{}^T(t)\cdot V\left[\bar{\theta}(t);\dot{\bar{\theta}}^2(t)\right]+ \\ +\frac{1}{2}\cdot\ddot{\bar{\theta}}^T(t)\cdot V\left[\overset{(m)}{\bar{\theta}}(t);\dot{\bar{\theta}}^2(t)\right] \end{gathered}\right\}$$

$$\left\{\begin{gathered} m=0\to 4,\ \overset{(0)}{E_A^{(1)}}=E_A^{(1)},\ \overset{(0)}{\bar{\theta}}(t)=\bar{\theta}(t) \\ \overset{(0)}{M}[\bar{\theta}(t)]=M[\bar{\theta}(t)] \\ \overset{(0)}{V}\left[\bar{\theta}(t);\dot{\bar{\theta}}^2(t)\right]=V\left[\bar{\theta}(t);\dot{\bar{\theta}}^2(t)\right] \\ \delta_m=\left\{\begin{gathered}\{1;\ m=2\cdot k+1\}; \\ \{-1;\ m=2\cdot k\},\ k=0,1,2,\ldots\end{gathered}\right\} \\ \lambda_m=\{1,\ m\geq 3\};\{0,\ m<3\} \\ \tau_m=\{0,\ m=0\};\{1,\ m\geq 1\} \\ \Delta_m=(-1)^{\delta_m}\cdot\frac{1-\delta_m}{1+3\cdot\delta_m}\cdot\tau_m \end{gathered}\right\}. \tag{65}$$

The above conditions (65) are compulsory for development (64) which is the time derivative of higher order applied on the acceleration energy of first order, as component in (61) equations.

6. CONCLUSIONS

The currently paper was devoted to the presentation of a few essential formulations regarding the analytical dynamics of the MBS. Analytical dynamics of systems is one of the most important parts from Mechanics, as fundamental science. According to various treatises of mechanics, within of the paper the differential and integral principles have been developed. They will determine the dynamical equations of the mechanical motions for any multibody systems. Among of these, in the currently paper: *D'Alembert principle and D'Alembert-Lagrange principles* have been mentioned. In the first part of the paper, the above principles have been presented in the lapidary form. In the following they have been applied for establish the conditions of static equilibrium for any rigid body taken into study.

In second part of the paper a generalization of the differential principle and dynamical equations of higher order typical to sudden motions and transitory phase. Therefore, this paper is also devoted to presentation new formulations on differential motions equations that are used in the advanced dynamic study of multibody mechanical systems, when they are characterized by sudden and transitory motions. This states, are characterized by existence of the higher order variations with respect to time of linear and angular accelerations. According to author researches, they are integrated into higher order acceleration energies and the differential equations of motion leading to variations in time of generalized forces which dominating some types of mechanical systems. The dynamics equations of motion have been generally presented for any type of multibody mechanical system. The establishing of these differential equations of motion was based on generalization of the principle form analytical mechanics, known as the D`Alembert – Lagrange Principle. Therefore, all these studies were developed by the author, they leading to a more accuracy control on the sudden and transitory motions of the mutibody systems.

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Formulări Noi în Dinamica Analitică a Sistemelor

Rezumat: *În dinamica newtoniană a sistemelor mecanice multicorp (MBS) teorema generală este teorema energiei cinetice în formă diferenţială. În dinamica avansată se dezvoltă energii de ordin superior, în care sunt incluse acceleraţii de ordin superior. Conform tratatelor de dinamică analitică există: principiul lui D'Alembert, principiul lucrului mecanic virtual într-o problemă dinamică, cunoscut ca principiul lui D'Alembert – Lagrange. În partea a doua, lucrarea se axează pe câteva formulări noi, bazate pe cercetările autorului asupra dinamicii avansate a sistemelor caracterizate prin mişcări rapide şi tranzitorii. Se demonstrează teoretic şi experimental existenţa variaţiilor în raport cu timpul a acceleraţiilor. Conform cercetărilor autorului, variaţiile în timp a acestor parametri, sunt integrate în energiile de acceleraţii, iar acestea în ecuaţiile diferenţiale de mişcare de ordin superior, care vor conduce la variaţia în raport cu timpul a forţelor generalizate pentru sistemele mecanice cu mişcări rapide şi pe regimurile tranzitorii demişcare.*

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